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CALCULATION OF STRESSES IN A WATERED LAYER

A. R. DZHANDIGULOV[1], A. L. KARCHEVSKY[2], §

Abstract. In the paper the analitical expressions for computing stresses in a watered layer have been obtained. It is not required to solve endless systems of equations.

Keywords: stress, plane problem of the elasticity theory, biharmonic equation, watered layer

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1. Introduction

The paper considers the plane problem of the deformation of the horizontal fluid-saturated porous formation under the action of the overlying rocks. The novelty of the approach lies in the fact that the analytical solution of the problem of the evolution of the stress field in the formation is obtain, taking into account the fluid filtration from it, which begins immediately after the opening of the watered layer.

Substantiation of optimum schemes of additional recovery of remaining reserves of hydrocarbons by flooding deposits [20], the implementation of measures to prevent the sudden emission of coal mining [26, 28], forecast for the rock mass in the vicinity of underground disposal of liquid waste products [7] – this is not an exhaustive list of problems to solve that require mathematical modeling of deformation and mass transfer in fluid-saturated porous formations. A number of models poro-plastic and poroelastic have been developed [3, 4, 6, 21, 23, 33], the implementation of which was carried out exclusively by numerical methods [5, 8, 29]. Meanwhile, in the operation of space systems for monitoring geomechanical mineral deposits [24, 31, 32] may be situations requiring a decision almost instantly. In such cases, it is the analytical solutions which can provide a rapid assessment of the state of the object.

2. Problem statement

It is necessary to calculate the stress field in the showdown watered formation. Initially, the layer is opened, causing diffusion of water in the layer, which entails a change in stresse.

[1] L. Gumilyov Eurasian National University, St. Satpayev, 2, 010000, Astana, Republic of Kazakhstan.
e-mail: abeked@mail.ru; ORCID: http://orcid.org/0000-0001-7620-8025.

[2] Sobolev Institute of Mathamatics SO RAN, pr. Koptyga, 4, 630090 Novosibirsk, Russia.
e-mail: karchevs@math.nsc.ru; ORCID: http://orcid.org/0000-0003-1338-5723.

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Due to a large extent of the layer in comparison with its power and length we suppose that the model of plane strain is applicable [25]. In this case, the Navier balance can be written as follows:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} &= \varkappa \frac{\partial p}{\partial x}, \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} &= \varkappa \frac{\partial p}{\partial z}, \end{aligned} \tag{1}$$

where $\varkappa$ is the Biot coefficient, p – fluid pressure in the layer, which satisfies the diffusion equation

$$\frac{\partial p}{\partial t} = D\Delta p, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \tag{2}$$

here D is fluid diffusion coefficient of the layer [6, 24].

The stress state in the formation of deformation is described by the Saint-Venant continuity equation

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \tag{3}$$

and by the Hooke's law

$$\varepsilon_x = \frac{1}{E'}(\sigma_{xx} - \nu' \sigma_{zz}), \quad \varepsilon_z = \frac{1}{E'}(\sigma_{zz} - \nu' \sigma_{xx}), \quad \gamma_{xz} = \frac{2(1+\nu')}{E'}\sigma_{xz}. \tag{4}$$

Boundary conditions:

$$\begin{aligned} &\sigma_{zz}\Big|_{z=\pm l_z} = \left\{ \begin{matrix} f^1(x) \\ f^2(x) \end{matrix} \right\}, \quad \sigma_{xz}\Big|_{z=\pm l_z} = \left\{ \begin{matrix} g^1(x) \\ g^2(x) \end{matrix} \right\}, \quad \frac{\partial p}{\partial z}\Big|_{z=\pm l_z} = 0, \\ &\sigma_{xx}|_{x=\pm l_x} = 0, \qquad \sigma_{xz}|_{x=\pm l_x} = 0, \qquad p|_{x=\pm l_x} = 0, \end{aligned} \tag{5}$$

and the compatibility conditions at the corners: $g^j(\pm l_x) = 0$ $(j = 1, 2)$.

Initial conditions:

$$p(x, z, 0) = p_0(x, z). \tag{6}$$

From physical considerations we suppose that the function $p_0(x, z)$ is even with respect on the variable z. Compatibility conditions: $p_0(\pm l_x, z) = 0$.

Remark. *Parity requirement is not restrictive, since it is possible to obtain formulas for the general case. However, taking into account the physical setting, we did not set such goals.*

We assume that the boundary conditions (5) satisfy the conditions of equilibrium: the torque and the sum of forces acting on a layer are zero. That is, the following equalities

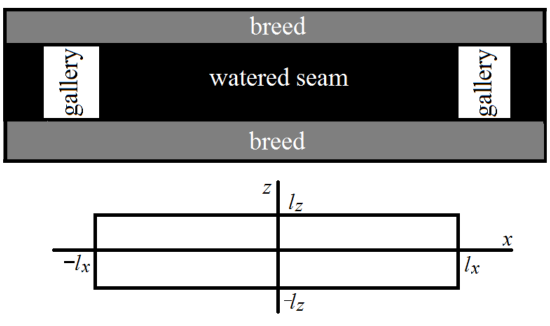

FIGURE 1. Watered layer and coordinate system.

hold [25]:

$$\begin{gathered}
\int\limits_{-l_x}^{l_x} \big[x\sigma_{zz}(x,l_z) - l_z\sigma_{xz}(x,l_z)\big]\,dx + \int\limits_{l_x}^{-l_x} \big[x\sigma_{zz}(x,-l_z) + l_z\sigma_{xz}(x,-l_z)\big]\,dx = 0, \\
\int\limits_{-l_x}^{l_x} \sigma_{zz}(x,l_z)dx = \int\limits_{-l_x}^{l_x} \sigma_{zz}(x,-l_z)dx, \qquad \int\limits_{-l_x}^{l_x} \sigma_{xz}(x,l_z)dx = \int\limits_{-l_x}^{l_x} \sigma_{xz}(x,-l_z)dx
\end{gathered} \tag{7}$$

or, taking into account (5),

$$\begin{gathered}
\frac{1}{2}\left(\int\limits_{-l_x}^{l_x} x\,f_1(x)\,dx - \int\limits_{-l_x}^{l_x} x\,f_2(x)\,dx\right) = l_z \int\limits_{-l_x}^{l_x} g_1(x)\,dx, \\
\int\limits_{-l_x}^{l_x} f_1(x)\,dx = \int\limits_{-l_x}^{l_x} f_2(x)\,dx, \qquad \int\limits_{-l_x}^{l_x} g_1(x)\,dx = \int\limits_{-l_x}^{l_x} g_2(x)\,dx.
\end{gathered} \tag{8}$$

3. Construction of analytical expressions for the stresses

The consequence of the relations (3)-(4) is the equation:

$$\frac{\partial^2}{\partial z^2}(\sigma_{xx} - \nu'\sigma_{zz}) + \frac{\partial^2}{\partial x^2}(\sigma_{zz} - \nu'\sigma_{xx}) = 2(1+\nu')\frac{\partial^2}{\partial x \partial z}\sigma_{xz}. \tag{9}$$

From (1) and (9) the relation is followed

$$\Delta(\sigma_{xx} + \sigma_{zz}) = \varkappa(1+\nu')\Delta p. \tag{10}$$

Input the Airy's function so that the equation (1) automatically satisfied:

$$\sigma_{xx} - \varkappa p = \frac{\partial^2 \Phi}{\partial z^2}, \quad \sigma_{zz} - \varkappa p = \frac{\partial^2 \Phi}{\partial x^2}, \quad \sigma_{xz} = -\frac{\partial^2 \Phi}{\partial x \partial z}, \tag{11}$$

then, from (10) it follows an inhomogeneous biharmonic equation

$$\Delta^2 \Phi = -\frac{\varkappa(1-\nu')}{D}\frac{\partial p}{\partial t}. \tag{12}$$

Thus, to calculate the stresses, it is necessary to find a solution of the equation (12), satisfying the boundary conditions (5) [9]. Function $p(x,z,t)$ is found in Appendix A.

First of all, let us introduce some notation. Let functions $f^j(x)$ and $g^j(x)$ $(j = 1,2)$ be represented in the form

$$\begin{aligned}
f^j(x) &= f^j_{-2}\cdot x + f^j_{-1} + \sum_{m=0}^{\infty} f^j_m X''_m(x;l_x), \\
&\qquad f^j_{-1} = \frac{1}{2l_x}\int\limits_{-l_x}^{l_x} f_j(s)ds, \quad f^j_{-2} = -\frac{3}{2l_x^3}\int\limits_{-l_x}^{l_x} sf_j(s)ds, \\
g^j(x) &= g^j_{-1}\cdot\left(1 - \frac{x^2}{l_x^2}\right) + \sum_{m=0}^{\infty} g^j_m X'_m(x;l_x), \qquad g^j_{-1} = \frac{3}{4l_x}\int\limits_{-l_x}^{l_x} g_j(s)ds
\end{aligned}$$

(see Appendix B).

From the condition (8) follows:

$$f^1_{-1} = f^2_{-1} \equiv \mathcal{F}, \quad g^1_{-1} = g^2_{-1} \equiv \mathcal{G}, \quad \frac{1}{2}(f^1_{-2} - f^2_{-2}) = -2\frac{l_z}{l_x^2}\mathcal{G}.$$

We seek a solution of the equation (12), satisfying the boundary conditions (5), as

$$\Phi(x,z) = \sum_{j=1}^{4} \Phi_j(x,z).$$

Each function $\Phi_j(x,z)$ is the solution of the biharmonic equation and satisfies part of the boundary conditions, and their sum is the solution of our problem, i.e. the function $\Phi(x,z)$ solves the differential equation (12) and satisfies (see (5) and (11)) the following boundary conditions:

$$\begin{aligned} \left.\frac{\partial^2\Phi}{\partial x^2}\right|_{z=\pm l_z} &= \left\{ \begin{matrix} f^1(x) \\ f^2(x) \end{matrix} \right\} - \varkappa p(x, \pm l_z, t), \quad \left.\frac{\partial^2\Phi}{\partial x \partial z}\right|_{z=\pm l_z} = -\left\{ \begin{matrix} g^1(x) \\ g^2(x) \end{matrix} \right\}, \\ \left.\frac{\partial^2\Phi}{\partial z^2}\right|_{x=\pm l_x} &= 0, \quad \left.\frac{\partial^2\Phi}{\partial x \partial z}\right|_{x=\pm l_x} = 0. \end{aligned}$$

A search of solution as sum is a well-known mathematical technique, but in each case can be non-obvious and time-consuming, may require a heuristic approach. In this case, to obtain the expression of solutions of the biharmonic equations (12) two systems of basis functions are used. This approach allows us to obtain the solution in simple form and without solving the infinite algebraic systems that is main result of this work.

The variable t is included in some ratio as a parameter, for simplicity, obvious dependence on t in the arguments of some variables we will omitted.

The function $\Phi_1(x,z)$ is a solution to the homogeneous biharmonic equation that satisfies the boundary conditions:

$$\begin{aligned} \left.\frac{\partial^2\Phi_1}{\partial x^2}\right|_{z=\pm l_z} &= \left\{ \begin{matrix} f^1_{-2} \\ f^2_{-2} \end{matrix} \right\} x + \mathcal{F}, \quad \left.\frac{\partial^2\Phi_1}{\partial x \partial z}\right|_{z=\pm l_z} = \mathcal{G}\left(1 - \frac{x^2}{l_x^2}\right), \\ \left.\frac{\partial^2\Phi_1}{\partial z^2}\right|_{x=\pm l_x} &= 0, \quad \left.\frac{\partial^2\Phi_1}{\partial x \partial z}\right|_{x=\pm l_x} = 0. \end{aligned}$$

Due to the relations (8), function $\Phi_1(x,z)$ can be found in the polynomial form

$$\Phi_1(x,z) = \mathcal{F}_+\frac{x^3}{6} + \mathcal{F}\frac{x^2}{2} - \frac{\mathcal{G}}{3l_x^2}x^3 z + \mathcal{G}xz, \qquad \mathcal{F}_+ = \frac{1}{2}(f^1_{-2} + f^2_{-2}).$$

The function $\Phi_2(x,z)$ can be found as solution of the equation (12), satisfying the boundary conditions

$$\left.\frac{\partial^2\Phi_2}{\partial x^2}\right|_{z=\pm l_z} = -\varkappa p(x, l_z, t), \quad \left.\frac{\partial^2\Phi_2}{\partial x \partial z}\right|_{z=\pm l_z} = 0. \tag{13}$$

The function $\Phi_2(x,z)$ can be represented as

$$\Phi_2(x,z) = \sum_{k=1}^{\infty} Y_k(z)\cos(\alpha_k x) + \sum_{k=1}^{\infty} U_k(z)\sin(\gamma_k x). \tag{14}$$

This is the standard periodic Filon-Ribier solution for the biharmonic equation [27]. Their use satisfies the right side of equation (12) and some boundary conditions but introduces extra values at the boundary, which will be offset by the following function $\Phi_3(x,z)$.

Substitute the expression (14) in the equation (12) and the boundary conditions (13). To determine the functions $Y_k(z)$ and $U_k(z)$ we obtain the following problems

$$Y_k'''' - 2\alpha_k^2 Y_k'' + \alpha_k^4 Y_k = \acute{Y}(z), \quad Y_k'(\pm l_z) = 0, \quad Y_k(\pm l_z) = \varkappa \hat{\rho}_k(l_z, t)/\alpha_k^2,$$

$$U_k'''' - 2\gamma_k^2 U_k'' + \gamma_k^4 U_k = \acute{U}(z), \quad U_k'(\pm l_z) = 0, \quad U_k(\pm l_z) = \varkappa \tilde{\rho}_k(l_z, t)/\gamma_k^2,$$

$$\acute{Y}(z) = \varkappa(1-\nu')\Big[\alpha_k^2 \hat{p}_{0,k0}\,\mathrm{e}^{-D\alpha_k^2 t} + \sum_{n=1}^{\infty}(\alpha_k^2+\beta_n^2)\hat{p}_{0,kn}\,\mathrm{e}^{-D(\alpha_k^2+\beta_n^2)t}\cos(\beta_n z)\Big],$$

$$\acute{U}(z) = \varkappa(1-\nu')\Big[\gamma_k^2 \tilde{p}_{0,k0}\,\mathrm{e}^{-D\gamma_k^2 t} + \sum_{n=1}^{\infty}(\gamma_k^2+\beta_n^2)\tilde{p}_{0,kn}\,\mathrm{e}^{-D(\gamma_k^2+\beta_n^2)t}\cos(\beta_n z)\Big]$$

whose solutions can be written as

$$\begin{aligned} Y_k(z) &= \hat{A}_k\cosh(\alpha_k z) + \hat{B}_k z\sinh(\alpha_k z) + \tilde{A}_k\sinh(\alpha_k z) + \tilde{B}_k z\cosh(\alpha_k z) + \bar{Y}_k(z), \\ U_k(z) &= \hat{C}_k\cosh(\gamma_k z) + \hat{D}_k z\sinh(\gamma_k z) + \tilde{C}_k\sinh(\gamma_k z) + \tilde{D}_k z\cosh(\gamma_k z) + \bar{U}_k(z), \end{aligned}$$

here $\bar{Z}_k(z)$ and $\bar{U}_k(z)$ are partial solutions:

$$\bar{Y}_k(z) = \varkappa(1-\nu')\hat{r}_k(z,t)/\alpha_k^2, \quad \bar{U}_k(z) = \varkappa(1-\nu')\tilde{r}_k(z,t)/\gamma_k^2,$$

where

$$\begin{aligned} \hat{r}_k(z) &= \hat{p}_{0,k0}\,\mathrm{e}^{-D\alpha_k^2 t} + \sum_{n=1}^{\infty}\frac{\alpha_k^2}{\alpha_k^2+\beta_n^2}\hat{p}_{0,kn}\,\mathrm{e}^{-D(\alpha_k^2+\beta_n^2)t}\cos(\beta_n z), \\ \tilde{r}_k(z) &= \tilde{p}_{0,k0}\,\mathrm{e}^{-D\gamma_k^2 t} + \sum_{n=1}^{\infty}\frac{\gamma_k^2}{\gamma_k^2+\beta_n^2}\tilde{p}_{0,kn}\,\mathrm{e}^{-D(\gamma_k^2+\beta_n^2)t}\cos(\beta_n z), \end{aligned}$$

and for constants we obtain

$$\begin{aligned} &\hat{A}_k = \frac{\sinh(\alpha_k l_z) + \alpha_k l_z\cosh(\alpha_k l_z)}{\sinh(\alpha_k l_z)\cosh(\alpha_k l_z) + \alpha_k l_z}\hat{b}_k, && \hat{B}_k = \frac{-\alpha_k\sinh(\alpha_k l_z)}{\sinh(\alpha_k l_z)\cosh(\alpha_k l_z) + \alpha_k l_z}\hat{b}_k, \\ &\tilde{A}_k = 0, && \tilde{B}_k = 0, \\ &\hat{C}_k = \frac{\sinh(\gamma_k l_z) + l_z\gamma_k\cosh(\gamma_k l_z)}{\sinh(\gamma_k l_z)\cosh(\gamma_k l_z) + \gamma_k l_z}\tilde{b}_k, && \hat{D}_k = \frac{-\gamma_k\sinh(\gamma_k l_z)}{\sinh(\gamma_k l_z)\cosh(\gamma_k l_z) + \gamma_k l_z}\tilde{b}_k, \\ &\tilde{C}_k = 0, && \tilde{D}_k = 0, \end{aligned}$$

where

$$\hat{b}_k = \varkappa\big[\hat{\rho}_k(l_z,t) - (1-\nu')\hat{r}_k(l_z)\big]/\alpha_k^2, \quad \tilde{b}_k = \varkappa\big[\tilde{\rho}_k(l_z,t) - (1-\nu')\tilde{r}_k(l_z)\big]/\gamma_k^2.$$

It is easy to see that each coefficient behaves like $\mathrm{e}^{-\alpha_k l_z}$ and $\mathrm{e}^{-\gamma_k l_z}$, i.e. series in (14) converges.

We compute

$$\frac{\partial^2\Phi_2}{\partial z^2}\bigg|_{x=\pm l_x} = 0, \quad \frac{\partial^2\Phi_2}{\partial x\partial z}\bigg|_{x=\pm l_x} = \pm\sum_{k=1}^{\infty}(-1)^k\alpha_k Y_k'(z) + \sum_{k=1}^{\infty}(-1)^k\gamma_k U_k'(z).$$

Consider the functions

$$y(z) = \sum_{k=1}^{\infty}(-1)^k\alpha_k Y_k(z), \qquad u(z) = \sum_{k=1}^{\infty}(-1)^k\gamma_k U_k(z).$$

Calculate the values of the functions $y(z)$ and $u(z)$ and their derivatives at the extreme points:

$$\frac{\partial}{\partial x}y(\pm l_z)=0,\quad y(\pm l_z)=\varkappa\sum_{k=1}^{\infty}(-1)^k\hat{\rho}_k(l_z,t)/\alpha_k\equiv y^0,$$
$$\frac{\partial}{\partial x}u(\pm l_z)=0,\quad u(\pm l_z)=\varkappa\sum_{k=1}^{\infty}(-1)^k\tilde{\rho}_k(l_z,t)/\gamma_k\equiv u^0.$$

The functions $y(z)-y^0$ and $u(z)-u^0$ on the interval $[-l_z,l_z]$ can be expanded in a uniformly convergent Fourier series for functions $X_m(z;l_z)$ (see Appendix B).

Let

$$y(z)-y^0=\sum_{m=0}^{\infty}y_mX_m(z;l_z),\quad u(z)-u^0=\sum_{m=0}^{\infty}u_mX_m(z;l_z),$$

from which obtain

$$y'(z)=\sum_{m=0}^{\infty}y_mX_m'(z;l_z),\quad u'(z)=\sum_{m=0}^{\infty}u_mX_m'(z;l_z).$$

Note that the function $y(z)$ and $u(z)$ are even, which means that y_m and u_m with odd numbers are zero.

The function $\Phi_3(x,z)$ can be found as a solution of homogeneous biharmonic equation, satisfying the following boundary conditions:

$$\begin{aligned}\left.\frac{\partial^2\Phi_3}{\partial x^2}\right|_{z=\pm l_z}&=0,\quad \left.\frac{\partial^2\Phi_3}{\partial x\partial z}\right|_{z=\pm l_z}=0,\\ \left.\frac{\partial^2\Phi_3}{\partial z^2}\right|_{x=\pm l_x}&=0,\quad \left.\frac{\partial^2\Phi_3}{\partial x\partial z}\right|_{x=\pm l_x}=\mp\sum_{m=0}^{\infty}y_mX_m'(z;l_z)-\sum_{m=0}^{\infty}u_mX_m'(z;l_z),\end{aligned}$$

Approximate analytical solution of $\Phi_3(x,z)$ can be found in the form [15]

$$\Phi_3(x,z)=\sum_{m=0}^{\infty}Q_m(x)X_m(z;l_z).$$

Bubnov-Galerkin procedure is applied to the solution of homogeneous biharmonic equation that leads to an infinite system of ordinary differential equations

$$\sum_{m=0}^{\infty}\left[Q_m\langle X_m'',X_s''\rangle-2Q_m''\langle X_m',X_s'\rangle+Q_m''''\delta_{ms}\right]=0,\qquad s=0,1,2...$$

Here δ_{ms} is the Kronecker's symbol.

Use the property of quasi-orthogonal first and second derivatives of functions (see Appendix B). In this case, we get the problem ($m=0,1,...$)

$$Q_m''''-2a_m^2Q_m''+b_m^4Q_m=0,\quad Q_m(\pm l_x)=0,\quad Q_m'(\pm l_x)=\mp y_m-u_m$$

where $a_m^2=\|X_m'(\cdot\,;l_z)\|^2$, $b_m^4=\|X_m''(\cdot\,;l_z)\|^2$. Since $b_m>a_m$ for all m [13], four roots of the characteristic equation can be found: $\pm b_m\mathrm{e}^{\pm i\theta_m}$, where $2\theta_m=\mathrm{arctg}\sqrt{b_m^4/a_m^4-1}$. Consequently, the solutions of the above problems are as follows

$$\begin{aligned}Q_m(x)&=\hat{M}_m\sin(\theta_mx)\sinh(b_mx)+\hat{N}_m\cos(\theta_mx)\cosh(b_mx)\\&+\tilde{M}_m\sin(\theta_mx)\cosh(b_mx)+\tilde{N}_m\cos(\theta_mx)\sinh(b_mx),\end{aligned}$$

where

$$\hat{M}_m = \frac{-2y_m \cos(\theta_m l_x)\cosh(b_m l_x)}{\theta_m \sinh(2b_m l_x) + b_m \sin(2\theta_m l_x)}, \quad \hat{N}_m = \frac{2y_m \sin(\theta_m l_x)\sinh(b_m l_x)}{\theta_m \sinh(2b_m l_x) + b_m \sin(2\theta_m l_x)},$$

$$\tilde{M}_m = \frac{-2u_m \cos(\theta_m l_x)\sinh(b_m l_x)}{\theta_m \sinh(2b_m l_x) - b_m \sin(2\theta_m l_x)}, \quad \tilde{N}_m = \frac{2u_m \sin(\theta_m l_x)\cosh(b_m l_x)}{\theta_m \sinh(2b_m l_x) - b_m \sin(2\theta_m l_x)}.$$

It is easy to see, first, the denominators of each factor will never vanish, and secondly, each coefficient behaves like $\mathrm{e}^{-b_m l_x}$, i.e. series for the $\Phi_3(x,z)$ converges.

The function $\Phi_4(x,z)$ can be found as the solution of homogeneous biharmonic equation that satisfies the boundary conditions

$$\begin{aligned}
\left.\frac{\partial^2 \Phi_4}{\partial x^2}\right|_{z=\pm l_z} &= \sum_{m=0}^{\infty} \left\{ \begin{matrix} f_m^1 \\ f_m^2 \end{matrix} \right\} X_m''(x;l_x), \quad \left.\frac{\partial^2 \Phi_4}{\partial x \partial z}\right|_{z=\pm l_z} = -\sum_{m=0}^{\infty} \left\{ \begin{matrix} g_m^1 \\ g_m^2 \end{matrix} \right\} X_m'(x;l_x), \\
\left.\frac{\partial^2 \Phi_4}{\partial z^2}\right|_{x=\pm l_x} &= 0, \quad \left.\frac{\partial^2 \Phi_4}{\partial x \partial z}\right|_{x=\pm l_x} = 0.
\end{aligned}$$

The expression for the function $\Phi_4(x,z)$ is sought similar to the expression for the function $\Phi_3(x,z)$, i.e.

$$\Phi_4(x,z) = \sum_{m=0}^{\infty} R_m(z) X_m(x;l_x),$$

where

$$\begin{aligned}
R_m(z) &= \hat{L}_m \sin(\vartheta_m z)\sinh(d_m z) + \hat{K}_m \cos(\vartheta_m z)\cosh(d_m z) \\
&+ \tilde{L}_m \sin(\vartheta_m z)\cosh(d_m z) + \tilde{K}_m \cos(\vartheta_m z)\sinh(d_m z),
\end{aligned}$$

where

$$\begin{aligned}
\hat{L}_m &= \frac{\vartheta_m \sin(\vartheta_m l_z)\cosh(d_m l_z) - d_m \cos(\vartheta_m l_z)\sinh(d_m l_z)}{\Delta_1}(f_m^1 + f_m^2) \\
&- \frac{\cos(\vartheta_m l_z)\cosh(d_m l_z)}{\Delta_1}(g_m^1 - g_m^2), \\
\hat{K}_m &= \frac{\vartheta_m \cos(\vartheta_m l_z)\sinh(d_m l_z) + d_m \sin(\vartheta_m l_z)\cosh(d_m l_z)}{\Delta_1}(f_m^1 + f_m^2) \\
&+ \frac{\sin(\vartheta_m l_z)\sinh(d_m l_z)}{\Delta_1}(g_m^1 - g_m^2), \\
\tilde{L}_m &= \frac{\vartheta_m \sin(\vartheta_m l_z)\sinh(d_m l_z) - d_m \cos(\vartheta_m l_z)\cosh(d_m l_z)}{\Delta_2}(f_m^1 - f_m^2) \\
&- \frac{\cos(\vartheta_m l_z)\sinh(d_m l_z)}{\Delta_2}(g_m^1 + g_m^2), \\
\tilde{K}_m &= \frac{\vartheta_m \cos(\vartheta_m l_z)\cosh(d_m l_z) + d_m \sin(\vartheta_m l_z)\sinh(d_m l_z)}{\Delta_2}(f_m^1 - f_m^2) \\
&+ \frac{\sin(\vartheta_m l_z)\cosh(d_m l_z)}{\Delta_2}(g_m^1 + g_m^2), \\
\Delta_1 &= \vartheta_m \sinh(2d_m l_z) + d_m \sin(2\vartheta_m l_z), \\
\Delta_2 &= \vartheta_m \sinh(2d_m l_z) - d_m \sin(2\vartheta_m l_z), \\
c_m^2 &= \|X_m'(\cdot\,;l_x)\|^2, \quad d_m^4 = \|X_m''(\cdot\,;l_x)\|^2, \quad 2\vartheta_m = \mathrm{arctg}\sqrt{d_m^4/c_m^4 - 1}.
\end{aligned}$$

Summarizing the above expressions, we obtain the following relationship:

$$\begin{aligned}\Phi(x,z) &= \frac{1}{6}\mathcal{F}_+ x^3 + \frac{1}{2}\mathcal{F}x^2 - \frac{\mathcal{G}}{3l_x^2}x^3 z + \mathcal{G}xz \\ &+ \sum_{k=1}^{\infty} Y_k(z)\cos(\alpha_k x) + \sum_{k=1}^{\infty} U_k(z)\sin(\gamma_k x) \\ &+ \sum_{m=0}^{\infty} Q_m(x)X_m(z;l_z) + \sum_{m=0}^{\infty} R_m(z)X_m(x;l_x),\end{aligned}$$

where all the values obtained above.

From the definition of (11) to stress we get

$$\begin{aligned}\sigma_{xx}(x,z) &= \varkappa p(x,z,t) \\ &+ \sum_{k=1}^{\infty} Y_k''(z)\cos(\alpha_k x) + \sum_{k=1}^{\infty} U_k''(z)\sin(\gamma_k x) \\ &+ \sum_{m=0}^{\infty} Q_m(x)X_m''(z;l_z) + \sum_{m=0}^{\infty} R_m''(z)X_m(x;l_x), \\ \sigma_{zz}(x,z) &= \varkappa p(x,z,t) + \mathcal{F}_+ x + \mathcal{F} - 2\frac{\mathcal{G}}{l_x^2}xz \\ &- \sum_{k=1}^{\infty} Y_k(z)\alpha_k^2\cos(\alpha_k x) - \sum_{k=1}^{\infty} U_k(z)\gamma_k^2\sin(\gamma_k x) \\ &+ \sum_{m=0}^{\infty} Q_m''(x)X_m(z;l_z) + \sum_{m=0}^{\infty} R_m(z)X_m''(x;l_x), \\ \sigma_{xz}(x,z) &= -\mathcal{G}\left(1-\frac{x^2}{l_x^2}\right) \\ &+ \sum_{k=1}^{\infty} Y_k'(z)\alpha_k\sin(\alpha_k x) - \sum_{k=1}^{\infty} U_k'(z)\gamma_k\cos(\gamma_k x) \\ &- \sum_{m=0}^{\infty} Q_m'(x)X_m'(z;l_z) - \sum_{m=0}^{\infty} R_m'(z)X_m'(x;l_x).\end{aligned}$$

Appendix A. The solution of the diffusion equation

Consider the function $p(x,z,t)$, which satisfies the problem:

$$\frac{\partial p}{\partial t} = D\Delta p,$$
$$\left.\frac{\partial p}{\partial z}\right|_{z=\pm l_z} = 0, \quad p|_{x=\pm l_x} = 0, \quad p|_{t=0} = p_0(x,z).$$

We represent $p_0(x,z)$ as the sum of an even $\hat{p}_0(x,z)$ and odd $\tilde{p}_0(x,z)$ parts, where $\hat{p}_0(x,z) = (p_0(x,z)+p_0(-x,z))/2$ and $\tilde{p}_0(x,z) = (p_0(x,z)-p_0(-x,z))/2$. Then the function $p(x,z,t)$ can be represented as the sum

$$p(x,z,t) = \hat{p}(x,z,t) + \tilde{p}(x,z,t),$$

where even $\hat{p}(x,z,t)$ and odd $\tilde{p}(x,z,t)$ parts satisfy the following problems

$$\begin{cases} \dfrac{\partial \hat{p}}{\partial t} = D\Delta \hat{p}, \\ \dfrac{\partial \hat{p}}{\partial z}\Big|_{z=\pm l_z} = 0, \ \dfrac{\partial \hat{p}}{\partial z}\Big|_{x=0} = 0, \ \hat{p}|_{x=l_x} = 0, \\ \hat{p}|_{t=0} = \hat{p}_0(x,z), \end{cases} \qquad \begin{cases} \dfrac{\partial \tilde{p}}{\partial t} = D\Delta \tilde{p}, \\ \dfrac{\partial \tilde{p}}{\partial z}\Big|_{z=\pm l_z} = 0, \ \tilde{p}|_{x=0} = 0, \ \tilde{p}|_{x=l_x} = 0, \\ \tilde{p}|_{t=0} = \tilde{p}_0(x,z). \end{cases}$$

The Fourier method gives the solution $p(x,z,t)$ for diffusion equation (2) in $[-l_x,l_x]\times[-l_z,l_z]$, which satisfies the initial and boundary conditions (5) and (6):

$$p(x,z,t) = \sum_{k=1}^{\infty} \hat{\rho}_k(z,t)\cos(\alpha_k x) + \sum_{k=1}^{\infty} \tilde{\rho}_k(z,t)\sin(\gamma_k x),$$

where

$$\begin{aligned} \hat{\rho}_k(z,t) &= \hat{p}_{0,k0}\,\mathrm{e}^{-D\alpha_k^2 t} + \sum_{n=1}^{\infty} \hat{p}_{0,kn}\,\mathrm{e}^{-D(\alpha_k^2+\beta_n^2)t}\,\cos(\beta_n z), \\ \tilde{\rho}_k(z,t) &= \tilde{p}_{0,k0}\,\mathrm{e}^{-D\gamma_k^2 t} + \sum_{n=1}^{\infty} \tilde{p}_{0,kn}\,\mathrm{e}^{-D(\gamma_k^2+\beta_n^2)t}\,\cos(\beta_n z), \end{aligned}$$

here $\hat{p}_{0,kn}$ and $\tilde{p}_{0,kn}$ are the Fourier coefficients corresponding to even and odd parts of the function $p_0(x,z)$, and $\alpha_k = \pi(2k-1)/2l_x$, $\gamma_k = \pi k/l_x$, and $\beta_n = \pi n/l_z$.

Appendix B. Basic functions

To solve the biharmonic equation are proposed many algorithms. First of all it is necessary to mention the decision in the forms of a polynomial, Filon, and Ribier. However, these solutions are not suitable for every type of boundary conditions. There is a so-called approach of an non-closed solution, where the solution is a sum of several rows, one row when the coefficient is expressed through all the coefficients of the second row, i.e., we obtain an endless system of linear equation. If you can prove it quite regularly, one can use the method of simple reduction. Also one can search for solutions with the help of the fundamental Krylov beam functions, but their records are present hyperbolic sines and cosines that the calculations can lead to large errors. To solve biharmonic equation, there are other approaches (see, for example, [1, 2, 19, 27]. Compare of solutions can be obtained by different methods, and can be found in [16]. The most promising is the approach suggested by S.A. Khalilov. He has proposed and investigated the functions $H_m(x)$ [10, 11], which in our case can be written as follows:

$$H_m(x) = \bar{P}^4_{m+4}(x), \quad m = 0,1,2....$$

where $\bar{P}^4_{m+4}(x)$ is the related normalized Legendre polynomials of degree m. The system of unctions $\{H_m(x)\}_{m=0}^{\infty}$ is complete and orthonormal on the interval $[-1,1]$. Function $s(x)$ with the boundary values $s(\pm1)=0$, $s'(\pm1)=0$ can be expanded in the Fourier series using the system of functions $\{H_m(x)\}_{n=0}^{\infty}$, the series converges absolutely and uniformly.

We have the representation [10, 11]

$$H_m(x) = (1-x^2)^2 \sum_{k=0}^{[m/2]} W_{mk}\, x^{m-2k}, \quad W_{mk} = \frac{(-1)^k}{2^{m+3}} \sqrt{\frac{m!(2m+9)}{2(m+8)!}}\, \frac{(2m-2k+7)!}{(m-k+3)!k!(m-2k)!},$$

($[\,\cdot\,]$ is an integer part), the recurrence formula

$$H_m(x) = \xi_m x H_{m-1}(x) - \zeta_m H_{m-2}(x), \quad m=1,2,..., \quad H_{-1}(x)=0, \quad H_0(x)=W_{00}(1-x^2)^2,$$

$$\xi_m = \sqrt{\frac{(2m+9)(2m+7)}{m(m+8)}}, \quad \zeta_m = \sqrt{\frac{(m-1)(m+7)(2m+9)}{m(m+8)(2m+5)}},$$

and the following relations [13, 14]:

$$\begin{aligned}
\|H'_n\|^2_{[-1,1]} &= \frac{1}{15}(2n+9)(n^2+9n+5), \\
\|H''_n\|^2_{[-1,1]} &= \frac{1}{4}(2n+9)\Bigg((n+2)(n+7)\left[1+\frac{1}{60}n(n+2)(n+7)(n+9)\right] \\
&\qquad - n(n+9)\left[3+\frac{1}{84}(n-1)(n+4)(n+5)(n+10)\right]\Bigg).
\end{aligned}$$

Proved and shown in numerical examples [10, 15, 16] that the functions $H'_m(x)$ and $H''_m(x)$ are quasi-orthogonal in the sense of fulfillment of the conditions:

$$\frac{\langle H^{(k)}_n(x), H^{(k)}_m(x)\rangle}{\|H^{(k)}_n(x)\|\cdot\|H^{(k)}_m(x)\|} = \theta, \quad |\theta|\approx 0, \quad m\neq n, \quad k=1,2.$$

This remarkable property of these functions are enabled to use the Bubnov-Galerkin procedure to find a solution to the biharmonic equation, and it is much easier. This approach has been tested in the papers [10, 11, 12, 13, 14, 15, 17, 18, 22, 30] end others. It has been shown that the numerical solution of the biharmonic equation through the use of functions $H_m(x)$ is calculated fairly accurately. The maximum deviation is localized near the corner points of the field (see, for example, [10, 12, 15, 16] and othes) and is a small amount (about 1.2%), the largest error is achieved for a square area, more than rectangular area: the differs from the square, the less is the computing error.

We use the function $X_m(x;L) = \frac{1}{\sqrt{L}}H_m(x/L)$, which are orthonormal on the interval $[-L,L]$ and their derivatives true

$$\|X'_m\|^2_{[-L,L]} = \|H'_m\|^2_{[-1,1]}/L^2, \qquad \|X''_m\|^2_{[-L,L]} = \|H''_m\|^2_{[-1,1]}/L^4.$$

The expansion of functions in the derivatives of functions X_m can be found in [9]. Function $s(x)$ ($s(\pm L)=0$) can be represented as

$$s(x) = \frac{3}{4L}\int\limits_{-L}^{L} s(y)dy\cdot\left(1-\frac{x^2}{L^2}\right) + \sum_{n=0}^{\infty}\hat{c}_m X'_m(x;L),$$

where $\hat{c}_m$ are the coefficients of expansion of the function

$$\hat{S}(x) = S(x) + \frac{1}{4L^3}S(L)x^3 - \frac{3}{4L}S(L)x - \frac{1}{2}S(L), \qquad S(x) = \int\limits_{-L}^{x} s(y)dy$$

in a series of functions X_m.

Functions $s(x)$ ($|s(x)|<\infty$) can be represented as

$$s(x) = -\frac{3}{2L^3}\int\limits_{-L}^{L} y\,s(y)dy\cdot x + \frac{1}{2L}\int\limits_{-L}^{L} s(y)dy + \sum_{m=0}^{\infty}\tilde{c}_m X''_m(x;L).$$

where $\tilde{c}_m$ are the coefficients of expansion of the function

$$\begin{aligned}\tilde{S}(x) &= S(x) - \frac{1}{4L^2}\left(S'(L) + \frac{1}{L}S(L)\right)x^3 - \frac{1}{4L}S'(L)\,x^2 \\ &- \frac{1}{4}\left(\frac{3}{L}S(L) - S'(L)\right)x - \frac{L}{2}\left(\frac{1}{L}S(L) - \frac{1}{2}S'(L)\right), \quad S(x) = \int_{-L}^{x}\int_{-L}^{z} s(y)dydz.\end{aligned}$$

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Abdygali Redzhepovich Dzhandigulov finished the Novosibirsk State University, Mechanics and Mathematical Department at 1990, he received the Diploma of Candidate of Science in Mathematics at 1993. At 1990-1996 he works in the Institute of Applied Mathematics (Karaganda, Kazakhstan), at 1996-2014 in Kazakh-Russian University. Since 2014 he works in the L. Gumilev Eurasian National University as associative professor on the Algebra and Geometry Department. His research interests include coefficient inverse problems of mathematical physics, numerical methods for these solving, mathematical simulations and numerical calculations.

Andrey Leonidovich Karchevsky finished the Novosibirsk State University, Mechanics and Mathematical Department in 1990, he received the Diploma of Candidate of Science in Mathematics at 1995 and the Diploma of Doctor of Science in Mathematics and Geophysics in 2005. Since 1990, he has been working in the Sobolev Institute of Mathematics, and now, he is a leading research scientist of the Laboratory of Wave Phenomena. In 2016, he received the Professor of Russian Academy of Science. His research interests include coefficient inverse problems of mathematical physics, numerical methods for these solving, mathematical simulations and numerical calculations.