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ASYMPTOTIC RESULTS FOR AN INVENTORY MODEL OF TYPE (s, S) WITH A GENERALIZED BETA INTERFERENCE OF CHANCE

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ABSTRACT. In this study, asymptotic expansion for ergodic distribution of an inventory control model of type (s, S) with generalized beta interference of chance is obtained, when $S - s \to \infty$. Moreover, weak convergence theorem is proved for ergodic distribution. Finally, the accuracy of the asymptotic expansion is examined with Monte Carlo simulation method.

Keywords: Inventory model of type (s, S), renewal-reward process, generalized beta distribution, asymptotic expansion, weak convergence, Monte Carlo simulation

AMS Subject Classification: 60K15

1. INTRODUCTION

Random walks, renewal-reward process and their modifications are important mathematical tools which have a wide range of real-life application area (for examples, see Alsmeyer [2], Aras and Woodroofe [3], Borovkov [4], Brown and Solomon [5], Gihman and Skorohod [8], Khaniyev and Atalay [9], Khaniyev et. al. [10], Khaniyev and Mammadova [11], Korolyuk and Borovskikh [12], Prabhu [14]). Despite their importance, calculation of the proposed formulas in the literature for their ergodic distributions is very hard. In this paper, we study the ergodic distribution of an inventory model of type (s, S).

Let us consider an inventory model where the initial stock level of a depot is equal to $X_0 \equiv z \in (s, S)$. In addition, assume that there are demands for random amounts of material at random times. Until the amounts of stock in this depot falls below a certain control level s, these demands are met. If these demands cannot be met, that is the amount of material in the depot is lower than the control level s, we re-fill the stock immediately with a random amount of material. Let us denote these epochs with τ_n, $n = 1, 2, \ldots$. After the refillment, the process starts with a new initial level $\zeta_n \in (s, S)$, $n = 1, 2, \ldots$ and continuous with a similar way. This type of models are known as "inventory control model of type (s, S)", and in this study under some assumptions we prove a weak convergence theorem for the ergodic distribution of this process.

This paper is organized as follows: in the next section, we give a brief mathematical construction of the above-mentioned process. In Section 3, the ergodicity of the process

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is obtained and in Section 4 the exact and asymptotic forms are presented and a weak convergence theorem is proved. The accuracy of the proposed asymptotic expansion is examined with a Monte Carlo simulation in Section 5. In Section 6, a real-life application of this model is given as a case study which was previously studied by Aliyev et.al. [1]. In the last section some discussions are given.

2. Mathematical Construction of the Process X_t

Let $\{(\xi_n, \eta_n, \zeta_n)\}$, $n \geq 1$ be a vector of independent and identically distributed random variables. Here ξ_n is the inter-arrival time between consecutive demands and η_n is the amount of nth demand. Refillment level ζ_n takes values in the interval $[s, S]$, and represents the initial level of the stock immediately after the nth refillment. Moreover, assume that ξ_n, η_n and ζ_n are independent from each other and denote their distributions by $\Phi(t)$, $F(x)$ and $\pi(z)$, respectively; that is,

$$\Phi(t) = P\{\xi_n \leq t\}, \quad F(x) = P\{\eta_n \leq x\}, \quad \pi(z) = P\{\zeta_n \leq z\}, \quad n = 1, 2, \ldots$$

Everywhere in the sequel we assume that ζ_n has a generalized beta distribution with parameters (s, S, α, β), $\alpha, \beta > 0$, $0 \leq s < S < \infty$. In other words, let

$$\begin{aligned} \pi(z) &= \int_s^z f(x; s, S, \alpha, \beta)\, dx \\ &= C_{2\gamma} \int_s^z (x-s)^{\alpha-1} (S-x)^{\beta-1}\, dx, \quad 0 \leq s \leq z \leq S, \alpha, \beta > 0. \end{aligned}$$

Here

$$C_{2\gamma} = \frac{1}{(2\gamma)^{\alpha+\beta-1} B(\alpha, \beta)}$$

is the normalizing constant, $\gamma \equiv (S-s)/2$, and $B(\alpha, \beta)$ is beta function (for the properties of generalized beta distribution, see Pham-Gia and Turkhan [13]).

Moreover, define renewal sequences T_n and S_n as follows:

$$\begin{aligned} &T_0 = S_0 = 0, \\ &T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad n \geq 1. \end{aligned}$$

T_n is the time of the nth demand, and S_n is the sum of the amounts of the first n demands. Put

$$\begin{aligned} &N_0 = 0; N_1 = \inf\{k \geq 1 : z - S_k < s\}, \quad z \in [s, S], \\ &N_{n+1} = \inf\{k \geq N_n + 1 : \zeta_n - S_k + S_{N_n} < s\}, \quad n \geq 1, \\ &\tau_0 = 0; \tau_n = T_{N_n} = \sum_{i=1}^{N_n} \xi_i, \quad n \geq 1, \\ &\nu(t) = \max\{n \geq 0 : T_n \leq t\}, \quad t \geq 0. \end{aligned}$$

Note that N_n, $n \geq 1$ is a sequence of integer-valued random variables and τ_1 represents the first time when the stock level drops below the control level s. Using these sequences of random variables, we can now define the desired process as follows:

$$\begin{aligned} X_t &= \zeta_n - \left(\eta_{N_n+1} + \eta_{N_n+2} + \cdots + \eta_{\nu(t)}\right) \\ &= \zeta_n - \left(S_{\nu(t)} - S_{N_n}\right), \quad t \in [\tau_n, \tau_{n+1}), n \geq 0. \end{aligned}$$

The process X_t represents the amount of material in the depot at time $t > 0$. A realization of this process is given as in Figure 1.

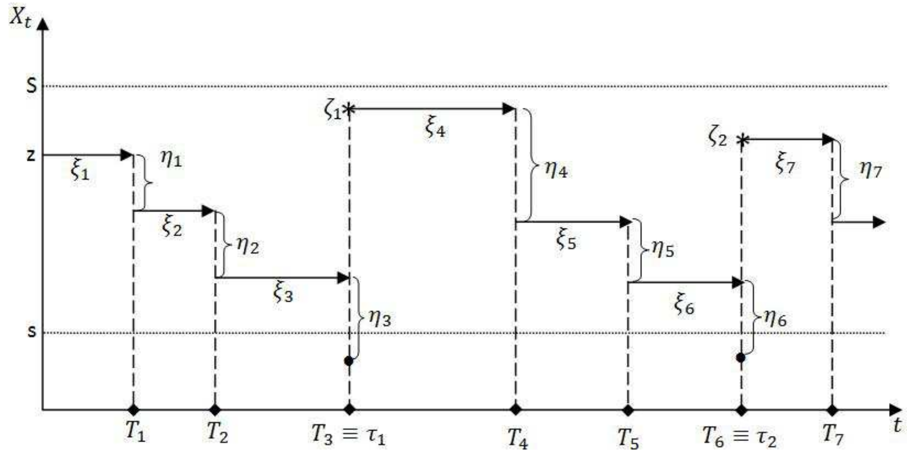

FIGURE 1. A realization of the process X_t

Similarly to Khaniyev and Atalay [9], in this study the process X_t is called "a renewal-reward process with a generalized beta interference of chance". The purpose of this study is to obtain an asymptotic expansion and to prove a weak convergence theorem for the ergodic distribution of the process X_t as $S-s \to \infty$. To obtain this asymptotic expansion, it is necessary to show that the process X_t is ergodic under some assumptions.

3. ERGODICITY OF THE PROCESS X_t

We will use the following proposition from Khaniyev and Atalay [9] to state the ergodicity of the process X_t.

Proposition 3.1. *(Khaniyev and Atalay* [9, Proposition 3.1]*) Let the sequence of the random variables* $\{(\xi_n, \eta_n, \zeta_n)\}$ *satisfy the following supplementary conditions:*

(1) $0 < E(\xi_1) < \infty$,
(2) $0 < E(\eta_1) < \infty$,
(3) η_1 *is a non-arithmetic random variable.*

Then, the process X_t *is ergodic and the following expression is correct for each measurable bounded function* $f : (s, S) \to \mathbb{R}$, *with probability 1:*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_u)\,du = \frac{\int_s^S \int_s^S f(x)\,[U_\eta(z-s) - U_\eta(z-x)]\,d\pi(z)\,dx}{\int_s^S U_\eta(z-s)\,d\pi(z)}$$

where

$$U_\eta(x) = \sum_{n=0}^{\infty} F^{*n}(x) \tag{1}$$

is the renewal function generated by the sequence $\{\eta_n\}$.

A direct result of this proposition can be given as follows:

Corollary 3.1. *Let the process X_t be satisfied the conditions of Proposition 3.1. Then, the ergodic distribution of the process X_t is given as follows:*

$$Q_X(v) \equiv \lim_{t\to\infty} P\{X_t \le v\} = 1 - \frac{\int_v^S U_\eta(z-v)\, d\pi(z)}{\int_s^S U_\eta(z-s)\, d\pi(z)}, \quad v \in [s, S].$$

Proof. Proof follows from Proposition 3.1 by choosing the f to be an indicator function. □

Now, let define a new process Y_t as a standardized version of the process X_t as follows:

$$Y_t = \frac{X_t - s}{\gamma}, \quad \gamma = \frac{S-s}{2}.$$

Moreover, denote the ergodic distribution of Y_t with

$$Q_Y(v) \equiv \lim_{t\to\infty} P\{Y_t \le v\}, \quad v \in [0,2].$$

4. Exact and asymptotic results for process Y_t

In this section, exact and asymptotic results for the process Y_t is presented.

Proposition 4.1. *Under the conditions of Proposition 3.1, the ergodic distribution function $Q_Y(v)$ of the process Y_t is given as follows:*

$$Q_Y(v) = 1 - \frac{\int_{\gamma v}^{2\gamma} U_\eta(x-\gamma v) f(x;0,2\gamma,\alpha,\beta)\, dx}{\int_0^{2\gamma} U_\eta(x) f(x;0,2\gamma,\alpha,\beta)\, dx}, \quad v \in (0,2). \tag{2}$$

Proof. From the definition of the process Y_t, for all $v \in (0,2)$ we have

$$Q_Y(v) = Q_X(\gamma v + s) = 1 - \frac{\int_{\gamma v+s}^{2\gamma+s} U_\eta(z-\gamma v - s)\, d\pi(z)}{\int_s^{2\gamma+s} U_\eta(z-s)\, d\pi(z)}.$$

On the other, since ζ_n has a generalized beta distribution with parameters (s, S, α, β), the random variable $\tilde{\zeta}_n \equiv \zeta_n - s$ will have the same distribution but with parameters $(0, 2\gamma, \alpha, \beta)$. Therefore, we have

$$Q_Y(v) = 1 - \frac{\int_{\gamma v}^{2\gamma} U_\eta(x-\gamma v) f(x;0,2\gamma,\alpha,\beta)\, dx}{\int_0^{2\gamma} U_\eta(x) f(x;0,2\gamma,\alpha,\beta)\, dx}.$$

This is the desired result. □

In general case, computation of the renewal function $U_\eta(x)$ is very hard. For this reason, under some additional weak assumptions, Feller [7] suggested to employ the expression in the following lemma for this renewal function.

Lemma 4.1. *(Feller* [7, page 366]*) Assume that η_1 is a non-arithmetic random variable, and the condition $E(\eta_1^2) < \infty$ is satisfied. Then, the renewal function $U_\eta(x)$ in* (1) *can be written as follows:*

$$U_\eta(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} + g(x). \tag{3}$$

Here $g(x)$ is a bounded function with $\lim_{x\to\infty} g(x) = 0$, *and* $m_k = E(\eta_1^k)$, $k = 1, 2, \ldots$

Following lemma will be used to prove the Theorem 4.1 where an asymptotic expansion is suggested, and states that for some g functions, the limit of $E(g(\zeta_1))$ tends to zero as the parameter γ tends to infinity.

Lemma 4.2. *For all measurable bounded $g:\mathbb{R}\to\mathbb{R}$ functions with $g(x)\to 0$ as $x\to\infty$, the following limit holds for all $u\in[s,S]$:*

$$\lim_{\gamma\to\infty}\int_u^S g(z)\, f(z;s,S,\alpha,\beta)\, dz = 0.$$

Here $\gamma\equiv(S-s)/2$.

Proof. Let $t=(z-s)/(2\gamma)$. Then,

$$\left|\int_u^S g(z)\, f(z;s,S,\alpha,\beta)\, dz\right| \le \frac{1}{B(\alpha,\beta)}\int_{(u-s)/(2\gamma)}^1 |g(s+2\gamma t)|\, t^{\alpha-1}(1-t)^{\beta-1}\, dt$$
$$\le \frac{1}{B(\alpha,\beta)}\left[I_1(\varepsilon)+I_2(\varepsilon)\right]. \tag{4}$$

Here, ε is an arbitrary fixed positive number and

$$I_1(\varepsilon)=\int_0^{\delta(\varepsilon)} |g(s+2\gamma t)|\, t^{\alpha-1}(1-t)^{\beta-1}\, dt,$$
$$I_2(\varepsilon)=\int_{\delta(\varepsilon)}^1 |g(s+2\gamma t)|\, t^{\alpha-1}(1-t)^{\beta-1}\, dt.$$

Moreover $\delta(\varepsilon)$ defined as

$$\delta(\varepsilon)=\sup\left\{\delta>0:\int_0^\delta t^{\alpha-1}(1-t)^{\beta-1}\, dt\le\frac{\varepsilon}{K}\right\}>0$$

where K is a fixed integer number which will be defined later. Since $g(x)\to 0$ as $x\to\infty$ and $\delta(\varepsilon)>0$, we can choose the parameter γ so large such that $s+2\delta(\varepsilon)\gamma\ge z(\varepsilon)$ holds where

$$z(\varepsilon)=\inf\left\{z>0:\sup_{u\ge z}|g(u)|\le\frac{\varepsilon}{K}\right\}.$$

On the other hand, since g is given as a bounded function, there exists an $M>0$ such that $\sup_{x\ge 0}|g(x)|\equiv M<\infty$ holds. Thus we have,

$$I_1(\varepsilon)\le M\frac{\varepsilon}{K} \tag{5}$$

and

$$I_2(\varepsilon)\le\frac{\varepsilon}{K}\int_{\delta(\varepsilon)}^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt$$
$$\le\frac{\varepsilon}{K}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt=\frac{\varepsilon}{K}B(\alpha,\beta). \tag{6}$$

Substituting inequalities (5) and (6) in (4) yields to

$$\left|\int_u^S g(z)\, f(z;s,S,\alpha,\beta)\, dz\right|\le\frac{\varepsilon}{K}\left(\frac{M}{B(\alpha,\beta)}+1\right)$$

By choosing $K\equiv[M/B(\alpha,\beta)]+2$ we obtain

$$\frac{M+B(\alpha,\beta)}{KB(\alpha,\beta)}\le 1.$$

Hence for all $\varepsilon>0$, when $\gamma\to\infty$ we have

$$\left|\int_u^S g(z)\, f(z;s,S,\alpha,\beta)\, dz\right|\le\varepsilon.$$

Therefore,

$$\lim_{\gamma\to\infty}\int_u^S g(z)\, f(z;s,S,\alpha,\beta)\,dz=0.$$

This completes the proof. □

For $\alpha,\beta>0$ and $x\in[0,1]$ let

$$B_x(\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt,$$
$$I_x(\alpha,\beta)=\frac{B_x(\alpha,\beta)}{B(\alpha,\beta)}$$

be the incomplete beta and regularized incomplete beta functions, respectively.

Lemma 4.3. *For all $v\in(0,2)$, the following equation holds, as $\gamma\to\infty$,*

$$\begin{aligned}J(v)&\equiv\int_{\gamma v}^{2\gamma}U_\eta(x-\gamma v)\, f(x;0,2\gamma,\alpha,\beta)\,dx\\&=\frac{2\gamma}{m_1}\left\{\frac{B(\alpha+1,\beta)-B_{v/2}(\alpha+1,\beta)}{B(\alpha,\beta)}-\frac{v}{2}\left[1-I_{v/2}(\alpha,\beta)\right]\right\}\\&\quad+\frac{m_2}{2m_1^2}\left[1-I_{v/2}(\alpha,\beta)\right]+o(1),\end{aligned}$$

where $m_k=E\left(\eta_1^k\right)$, $k=1,2,\ldots$.

Proof. Let $g:\mathbb{R}\to\mathbb{R}$ be a measurable bounded function with $\lim_{x\to\infty}g(x)=0$. Then, we have

$$\begin{aligned}J(v)&=\int_{\gamma v}^{2\gamma}U_\eta(x-\gamma v)\, f(x;0,2\gamma,\alpha,\beta)\,dx\\&=\int_{\gamma v}^{2\gamma}\left(\frac{x-\gamma v}{m_1}+\frac{m_2}{2m_1^2}+g(x)\right)C_{2\gamma}x^{\alpha-1}(2\gamma-x)^{\beta-1}\,dx\\&=\frac{1}{m_1}\int_{\gamma v}^{2\gamma}C_{2\gamma}x^{\alpha}(2\gamma-x)^{\beta-1}\,dx\\&\quad+\left(-\frac{\gamma v}{m_1}+\frac{m_2}{2m_1^2}\right)\int_{\gamma v}^{2\gamma}C_{2\gamma}x^{\alpha-1}(2\gamma-x)^{\beta-1}\,dx\\&\quad+\int_{\gamma v}^{2\gamma}g(x)\,C_{2\gamma}x^{\alpha-1}(2\gamma-x)^{\beta-1}\,dx\\&=\frac{2\gamma}{m_1}\left\{\frac{B(\alpha+1,\beta)-B_{v/2}(\alpha+1,\beta)}{B(\alpha,\beta)}-\frac{v}{2}\left[1-I_{v/2}(\alpha,\beta)\right]\right\}\\&\quad+\frac{m_2}{2m_1^2}\left[1-I_{v/2}(\alpha,\beta)\right]+o(1).\end{aligned}$$

This completes the proof. □

By passing to the limit as $v\to0$ in Lemma 4.3, following result can be obtained.

Corollary 4.1. *The following expansion holds as $\gamma\to\infty$,*

$$J(0)=\frac{2\alpha}{\alpha+\beta}\frac{\gamma}{m_1}+\frac{m_2}{2m_1^2}+o(1).$$

Theorem 4.1. *In addition to the conditions in Proposition 4.1, assume that* $E\left(\eta_1^2\right) < \infty$. *Then, for all* $v \in (0,2)$ *the following asymptotic expansion can be written for the ergodic distribution function* $Q_Y(v)$ *of the process* Y_t*, as* $\gamma \equiv (S-s)/2 \to \infty$,

$$Q_Y(v) = G(v) - \frac{m_{21}}{\gamma} R(v) + o\left(\frac{1}{\gamma}\right)$$

where

$$G(v) = I_{v/2}(\alpha+1,\beta) + \frac{(\alpha+\beta)\,v}{2\alpha}\left[1 - I_{v/2}(\alpha,\beta)\right], \tag{7}$$

$$R(v) = \frac{\alpha+\beta}{2\alpha^2}\left\{\frac{v(\alpha+\beta)}{2}\left[1 - I_{v/2}(\alpha,\beta)\right] - \frac{v^\alpha (2-v)^\beta}{2^{\alpha+\beta} B(\alpha,\beta)}\right\}.$$

and $m_{21} = m_2/(2m_1)$.

Proof. Using the Lemma 4.3 and Corollary 4.1, as $\gamma \to \infty$ we have

$$\begin{aligned}
Q_Y(v) &= 1 - \frac{J(v)}{J(0)} \\
&= 1 - \left[\frac{2\gamma}{m_1}\frac{\alpha}{\alpha+\beta} + \frac{m_2}{2m_1^2} + o(1)\right]^{-1} \\
&\quad \times \left\{\frac{2\gamma}{m_1}\frac{B(\alpha+1,\beta) - B_{v/2}(\alpha+1,\beta)}{B(\alpha,\beta)}\right. \\
&\quad \left. + \left(-\frac{\gamma v}{m_1} + \frac{m_2}{2m_1^2}\right)\left[1 - I_{v/2}(\alpha,\beta)\right] + o(1)\right\} \\
&= 1 - \frac{4m_1\gamma\alpha}{4m_1\gamma\alpha + (\alpha+\beta)m_2}\left\{1 - G(v) + \frac{\alpha+\beta}{\alpha}\frac{m_2}{4m_1\gamma}\left[1 - I_{v/2}(\alpha,\beta)\right]\right\} + o\left(\frac{1}{\gamma}\right) \\
&= 1 - \left[1 - \frac{\alpha+\beta}{\alpha}\frac{m_2}{4m_1}\frac{1}{\gamma} + o\left(\frac{1}{\gamma}\right)\right] \\
&\quad \times \left\{1 - G(v) + \frac{\alpha+\beta}{\alpha}\frac{m_2}{4m_1}\left[1 - I_{v/2}(\alpha,\beta)\right]\frac{1}{\gamma}\right\} + o\left(\frac{1}{\gamma}\right) \\
&= G(v) - \frac{m_{21}}{\gamma} R(v) + o\left(\frac{1}{\gamma}\right)
\end{aligned}$$

□

Now, the following weak convergence theorem can be given using the Theorem 4.1.

Theorem 4.2. *Assume that the conditions of Theorem 4.1 are satisfied. Then, the ergodic distribution* $(Q_Y(v))$ *of* Y_t *converges to* $G(v)$ *as* $\gamma \to \infty$*; that is,*

$$Q_Y(v) \to G(v)$$

where $G(v)$ *is defined in* (7).

Proof. Since for all $v \in (0,2)$, and $\alpha, \beta > 0$, the regularized incomplete beta function $I_{v/2}(\alpha,\beta)$ takes values in the interval $[0,1]$, we have

$$\begin{aligned} |R(v)| &= \left| \frac{\alpha+\beta}{2\alpha^2} \left\{ \frac{v(\alpha+\beta)}{2} \left[1 - I_{v/2}(\alpha,\beta)\right] - \frac{v^\alpha (2-v)^\beta}{2^{\alpha+\beta} B(\alpha,\beta)} \right\} \right| \\ &\le \frac{\alpha+\beta}{2\alpha^2} \left\{ \left| \frac{v(\alpha+\beta)}{2} \left[1 - I_{v/2}(\alpha,\beta)\right] \right| + \left| \frac{v^\beta (2-v)^\beta}{2^{\alpha+\beta} B(\alpha,\beta)} \right| \right\} \\ &\le \frac{\alpha+\beta}{2\alpha^2} \left(\alpha+\beta + \frac{1}{B(\alpha,\beta)} \right) < \infty. \end{aligned}$$

Moreover, according to the conditions of Teorem 4.1 and Proposition 4.1 we have $m_2 \equiv E\left(\eta_1^2\right) < \infty$ and $m_1 \equiv E(\eta_1) > 0$, respectively. Therefore, as $\gamma \to \infty$ we have $m_{21} R(v)/\gamma \to 0$. Hence, from Theorem 4.1 as $\gamma \to \infty$, $Q_Y(v) \to G(v)$ holds. This completes the proof of Theorem 4.2. □

Example 4.1. *Particullarly, let choose* $\alpha = \beta = \frac{1}{2}$*. Since, for each* $v \in (0,2)$

$$\begin{aligned} B\left(\frac{1}{2},\frac{1}{2}\right) &= \pi; \\ B_{v/2}\left(\frac{1}{2},\frac{1}{2}\right) &= 2\arcsin\left(\sqrt{v/2}\right); \\ I_{v/2}(\alpha+1,\beta) &= I_{v/2}(\alpha,\beta) - \frac{(v/2)^\alpha (1-v/2)^\beta}{\alpha B(\alpha,\beta)} \end{aligned}$$

we have

$$\begin{aligned} I_{v/2}\left(\frac{1}{2},\frac{1}{2}\right) &= \frac{2}{\pi}\arcsin\left(\sqrt{\frac{v}{2}}\right), \\ I_{v/2}\left(\frac{3}{2},\frac{1}{2}\right) &= \frac{2}{\pi}\arcsin\left(\sqrt{\frac{v}{2}}\right) - \frac{\sqrt{v(2-v)}}{\pi}. \end{aligned}$$

Therefore, for the ergodic distribution $Q_Y(v)$*,* $v \in (0,2)$ *of* $Y_t = (X_t - s)/\gamma$*,* $\gamma = (S-s)/2$*, we have*

$$Q_Y(v) = G(v) - \frac{m_{21}}{\gamma} R(v) + o\left(\frac{1}{\gamma}\right)$$

where the evidence forms of $G(v)$ *and* $R(v)$ *are as follows:*

$$\begin{aligned} G(v) &= \frac{v}{2} - \frac{\sqrt{v(2-v)}}{\pi} + \frac{2}{\pi}\left(1 - \frac{v}{2}\right)\arcsin\left(\sqrt{\frac{v}{2}}\right), \\ R(v) &= v - \frac{\sqrt{v(2-v)}}{\pi} - \frac{2}{\pi} v \arcsin\left(\sqrt{\frac{v}{2}}\right). \end{aligned}$$

Remark 4.1. *Since, the exact values of incomplete beta function is hard to calculate for all values of* α, β*, and* v*, we examined the accuracy of the proposed asymptotic expansion with using functions in GNU Octave* [6].

5. Simulation Results

In this section, Monte Carlo simulation results are given for the examination of the accuracy of the proposed asymptotic expansion in Theorem 4.1. For this purpose, we choose η_1 from an exponential distribution with a parameter $\lambda = 1$, and ζ_1 from a generalized beta distribution with parameters $(0, 2\gamma, 3, 1)$, $\gamma = 3, 4, 5, 10$. Let denote the ergodic distribution of Y_t obtained by Monte Carlo simulation with $\widehat{Q}_Y(v)$ and the asymptotic expansion in Theorem 4.1 with $\tilde{Q}_Y(v)$ using the reminder term; that is,

$$\tilde{Q}_Y(v) \equiv G(v) - \frac{m_{21}}{\gamma} R(v).$$

In this study, we will use absolute difference $\Delta = \left|\widehat{Q}_Y(v) - \tilde{Q}_Y(v)\right|$, relative error $\delta = \Delta / \widehat{Q}_Y(v) \times 100\%$, and accuracy percentage $AP = 100 - \delta$ for the measure of the accuracy of proposed asymptotic expansion. We simulated 10^6 trajectories to calculate the ergodic distribution of the process X_t. Table 1 - Table 4 are presented the values of $\tilde{Q}_Y(v)$ and simulated values of $\widehat{Q}_Y(v)$ with their comparisons for $\gamma = 3, 4, 5, 10$.

TABLE 1. Comparison of the asymptotic and the simulation values of the ergodic distribution for the case $\gamma = 3$, $(\gamma \equiv (S - s)/2)$

v	$\widehat{Q}_Y(v)$	$\tilde{Q}_Y(v)$	Δ	$\delta(\%)$	$AP(\%)$
0,1	0,05927	0,05437	0,00491	9,02273	90,97727
0,2	0,11860	0,10902	0,00958	8,78239	91,21761
0,3	0,17800	0,16385	0,01416	8,63963	91,36037
0,4	0,23745	0,21906	0,01839	8,39729	91,60271
0,5	0,29688	0,27450	0,02238	8,15118	91,84882
0,6	0,35616	0,32983	0,02633	7,98286	92,01714
0,7	0,41513	0,38523	0,02990	7,76222	92,23778
0,8	0,47360	0,44092	0,03268	7,41202	92,58798
0,9	0,53131	0,49669	0,03462	6,97002	93,02998
1,0	0,58796	0,55134	0,03663	6,64331	93,35669
1,1	0,64323	0,60564	0,03759	6,20618	93,79382
1,2	0,69671	0,65883	0,03788	5,74976	94,25024
1,3	0,74799	0,71054	0,03745	5,27054	94,72946
1,4	0,79660	0,76048	0,03612	4,74963	95,25037
1,5	0,84201	0,80852	0,03349	4,14210	95,85790
1,6	0,88367	0,85415	0,02952	3,45618	96,54382
1,7	0,92098	0,89616	0,02481	2,76869	97,23131
1,8	0,95327	0,93539	0,01788	1,91136	98,08864
1,9	0,97985	0,97004	0,00981	1,01157	98,98843
2,0	1,00000	1,00000	0,00000	0,00000	100,00000

Table 2. Comparison of the asymptotic and the simulation values of the ergodic distribution for the case $\gamma = 4$, $(\gamma \equiv (S-s)/2)$

v	$\widehat{Q}_Y(v)$	$\tilde{Q}_Y(v)$	Δ	$\delta\,(\%)$	$AP\,(\%)$
0,1	0,06112	0,05703	0,00409	7,16724	92,83276
0,2	0,12228	0,11388	0,00839	7,36708	92,63292
0,3	0,18346	0,17130	0,01216	7,09985	92,90015
0,4	0,24462	0,22836	0,01626	7,12131	92,87869
0,5	0,30566	0,28629	0,01937	6,76542	93,23458
0,6	0,36644	0,34373	0,02271	6,60679	93,39321
0,7	0,42677	0,40149	0,02528	6,29541	93,70459
0,8	0,48640	0,45833	0,02807	6,12371	93,87629
0,9	0,54506	0,51492	0,03015	5,85454	94,14546
1,0	0,60243	0,57057	0,03186	5,58325	94,41675
1,1	0,65813	0,62515	0,03298	5,27549	94,72451
1,2	0,71173	0,67903	0,03270	4,81618	95,18382
1,3	0,76279	0,73055	0,03224	4,41303	95,58697
1,4	0,81077	0,78004	0,03074	3,94018	96,05982
1,5	0,85514	0,82647	0,02867	3,46874	96,53126
1,6	0,89529	0,87021	0,02508	2,88241	97,11759
1,7	0,93056	0,90970	0,02086	2,29329	97,70671
1,8	0,96028	0,94512	0,01516	1,60350	98,39650
1,9	0,98368	0,97561	0,00807	0,82760	99,17240
2,0	1,00000	1,00000	0,00000	0,00000	100,00000

TABLE 3. Comparison of the asymptotic and the simulation values of the ergodic distribution for the case $\gamma = 5$, $(\gamma \equiv (S - s)/2)$

v	$\widehat{Q}_Y(v)$	$\tilde{Q}_Y(v)$	Δ	$\delta\,(\%)$	$AP\,(\%)$
0,1	0,06223	0,05877	0,00346	5,87959	94,12041
0,2	0,12448	0,11800	0,00648	5,49063	94,50937
0,3	0,18673	0,17648	0,01025	5,80979	94,19021
0,4	0,24892	0,23603	0,01290	5,46439	94,53561
0,5	0,31094	0,29534	0,01559	5,28013	94,71987
0,6	0,37261	0,35422	0,01839	5,19233	94,80767
0,7	0,43375	0,41273	0,02101	5,09050	94,90950
0,8	0,49408	0,47087	0,02321	4,92917	95,07083
0,9	0,55332	0,52831	0,02501	4,73408	95,26592
1,0	0,61111	0,58473	0,02638	4,51185	95,48815
1,1	0,66707	0,63978	0,02729	4,26498	95,73502
1,2	0,72075	0,69275	0,02799	4,04108	95,95892
1,3	0,77166	0,74404	0,02762	3,71187	96,28813
1,4	0,81928	0,79300	0,02628	3,31387	96,68613
1,5	0,86302	0,83885	0,02417	2,88142	97,11858
1,6	0,90226	0,88106	0,02120	2,40641	97,59359
1,7	0,93632	0,91873	0,01759	1,91444	98,08556
1,8	0,96448	0,95160	0,01288	1,35319	98,64681
1,9	0,98598	0,97888	0,00710	0,72528	99,27472
2,0	1,00000	1,00000	0,00000	0,00000	100,00000

As can be seen from these tables, the accuracy percentage is greater than 96% for $\gamma \geq 10$. This indicates that the proposed asymptotic expansion can be applied to different problems of inventory models even for not too large values of the parameter $\gamma \equiv (S - s)/2$.

6. Case Study

A company operating in the energy sector in Turkey distributes liquefied petrole-um gas (LPG) to 30 dealers with pipelines and land transport. LPG is distributed with tankers from LPG production center to a dealer if there is no pipeline installation between them. Each tanker has a capacity of 22 m^3 (approx. 10-11 tons) and 35 m^3 (approx. 17-18 tons).

Each dealer has a storage capacity of $S = 30m^3$ (approx. 15 tons). Random amount of LPG (η_n) are sold from these storage tanks at random inter-arrival times (ξ_n). Since the amount and arrival times of these sales are random, the gas level falls below the control level $s = S/5$ (approx. 3 tons) at random moments τ_n, $n \geq 1$. Whenever this happens, an on-line signal automatically sent to the production center. The demands of the dealer are met by the nearest tanker around it. If there is no such tanker, a tanker with full storage is sent from the production center. After delivering the needed amount of gas to the dealer, if more than 10 % of the capacity of the tanker is left over, the tanker waits its position until the next order take place.

For security concerns, each dealer usually prefers to fill their storage up to 85 % of their capacity (that is approx. 13.2 tons). But in some rare situations, the dealers may choose to use their storage capacity more or less than 85 % of their capacity.

For a more detailed description of this model see Aliyev et. al. [1].

TABLE 4. Comparison of the asymptotic and the simulation values of the ergodic distribution for the case $\gamma = 10$, $(\gamma \equiv (S-s)/2)$

v	$\widehat{Q}_Y(v)$	$\tilde{Q}_Y(v)$	Δ	$\delta\,(\%)$	$AP\,(\%)$
0,1	0,06445	0,06200	0,00245	3,94948	96,05052
0,2	0,12889	0,12459	0,00430	3,44717	96,55283
0,3	0,19328	0,18709	0,00619	3,30725	96,69275
0,4	0,25753	0,24946	0,00807	3,23330	96,76670
0,5	0,32148	0,31195	0,00953	3,05638	96,94362
0,6	0,38496	0,37384	0,01112	2,97447	97,02553
0,7	0,44770	0,43520	0,01251	2,87408	97,12592
0,8	0,50944	0,49600	0,01344	2,71009	97,28991
0,9	0,56982	0,55521	0,01462	2,63278	97,36722
1,0	0,62847	0,61326	0,01521	2,48055	97,51945
1,1	0,68495	0,66945	0,01550	2,31524	97,68476
1,2	0,73877	0,72289	0,01589	2,19748	97,80252
1,3	0,78941	0,77375	0,01566	2,02433	97,97567
1,4	0,83629	0,82132	0,01497	1,82292	98,17708
1,5	0,87878	0,86502	0,01376	1,59026	98,40974
1,6	0,91620	0,90385	0,01235	1,36589	98,63411
1,7	0,94782	0,93763	0,01020	1,08757	98,91243
1,8	0,97289	0,96545	0,00744	0,77031	99,22969
1,9	0,99057	0,98670	0,00387	0,39238	99,60762
2,0	1,00000	1,00000	0,00000	0,00000	100,00000

Therefore, in our opinion, the process which explains the working of the storage explained above can be considered as a stochastic process with a generalized beta distributed interference of chance.

It's known that if $\zeta_1 \sim Beta\,(s, S, \alpha, \beta)$, then $E(\zeta_1) = \frac{s\beta + S\alpha}{\alpha+\beta}$ and its mode is $\frac{s(\beta-1)+S(\alpha-1)}{\alpha+\beta-2}$.

We will choose the parameters as $\alpha = 2$ and $\beta = 22/3$, so in this case we have

$$P\{\zeta_1 \le \kappa\} = 0.85,$$

where κ is the mode of generalized beta distribution with parameters $(3, 15, 2, 22/3)$

Using the Theorem 4.2, the ergodic distribution of the process Y_t weakly convergence to $G(v)$, where

$$G(v) = I_{v/2}(3, 22/3) + \frac{7}{3} v \left[1 - I_{v/2}(2, 22/3)\right].$$

TABLE 5. Table Values of $G(v)$

v	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
$G(v)$	0.43200	0.73004	0.89251	0.96506	0.99122	0.99845	0.99984	0.99999	1.00000

7. Conclusion

In this paper an inventory control model of type (s, S) is studied. Particularly, under the assumption of there exist interferences with a generalized beta distribution, we obtain the ergodic distribution of the underlying process. Since the exact forms are not useful for practical calculations, using the results of Feller [7] and Khaniyev and Atalay [9], we obtain an asymptotic expansion, when $\gamma \equiv (S - s)/2 \to \infty$. The accuracy of the proposed asymptotic expansion is examined with a Monte Carlo simulation. Results are indicate that the accuracy of the proposed asymptotic expansion is fairly good even not too large values of the parameter γ.

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