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SOME INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR DIFFERENTIABLE (s, m)-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we present new inequalities connected with fractional integrals for twice differentiable functions derivatives which are $(s, m)-$ convex functions. To obtain this, integral inequalities were used classical inequalities as Hölder inequalitiy and power mean inequality.This results are related to the well-known integral inequality of the Hermite-Hadamard type. Also some applications to special means are provided.

Keywords: convex function,$(s, m)-$convex, Hermite–Hadamard inequalitiy, Riemann–Liouville fractional integral, power mean inequalitiy, Hölder inequality.

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1. Introduction

The property of convexity is fundamental in mathematics along monotony, continuity, differentiability,*etc.* This property widely used in the theory of extremal problems.

Definition 1.1. *The function* $f : [a, b] \to \mathbb{R}$*, is said to be convex, if we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a, b]$ *and* $\lambda \in [0, 1]$.

It is well known that in the nonlinear analysis the Hermite–Hadamard type double inequality plays a very important role. This inequality is stated as follows in literature (see [3])

Theorem 1.1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function and let* $a, b \in I$*, with* $a < b$*. The following double inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality.

J. Park asserted a new definition given in the following and gave some properties about this class of functions in [11].

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Definition 1.2. *([11]) For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be $(s,m)-$ convex in the second sense on I if*

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Remark 1.1. *In Definition 1.2 if we take $m = 1$, then obtain $s-$ convex second sense functions introduced by W. W. Breckner in [1] or if we choice $s = 1$ then obtain $m-$convex functions introduced by G.Toader in [17].*

The definition of a Riemann–Liouville fractional integral in the literature is given in the following way

Definition 1.3. *Let $f \in L_1[a,b]$. The Riemann Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt\,, \qquad x > a$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt\,, \qquad x < b$$

Here is $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$ and if $\alpha = 0$ then $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

For some recent results about Hermite–Hadamard type integral inequalities via Riemann – Liouville fractional integrals are reflected in [2], [4], [6]– [10], [13], [15], [16], [18] and references cited therein.

The purpose of this study is to establish new Hadmard type inequalities via fractional integrals for the classes of convex functions whose the second derivatives are $(s,m)-$ convex.

2. Some Results For Midpoint Inequalities

We formulate and prove lemma on which the obtained results are based.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° (I° is interior of I). If $f'' \in L[a,b]$, where $a, b \in I$ and $a < mb$, then $\forall \alpha > 1$ the following equality holds:*

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb) + J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right] - f\left(\frac{a+mb}{2}\right) \tag{2}$$
$$= \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}\left(I_1 + I_2\right)$$

where

$$I_1 = \int_0^{1/2} t^{\alpha}f''(at + m(1-t)b)dt \text{ and } I_2 = \int_{1/2}^1 (1-t)^{\alpha}f''(at + m(1-t)b)dt$$

Proof. Integrating both integrals by parts twice, we have

$$I_1=-\frac{1}{(mb-a)2^{\alpha}}f'\left(\frac{a+mb}{2}\right)-\frac{\alpha}{(bm-a)^2 2^{\alpha-1}}f\left(\frac{a+bm}{2}\right) \quad (3)$$
$$+\frac{\alpha(\alpha-1)}{(mb-a)^2}\int_0^{1/2}t^{\alpha-2}f(at+m(1-t)b)dt,$$
$$I_2=\frac{1}{(mb-a)2^{\alpha}}f'\left(\frac{a+mb}{2}\right)-\frac{\alpha}{(bm-a)^2 2^{\alpha-1}}f\left(\frac{a+bm}{2}\right) \quad (4)$$
$$+\frac{\alpha(\alpha-1)}{(mb-a)^2}\int_{1/2}^{1}(1-t)^{\alpha-2}f(at+m(1-t)b)dt$$

If we make $at+(1-t)b=z$ the transformation in both integrals in (3) and (4), and then summing these equalities, then we can write

$$I_1+I_2=-\frac{2\alpha}{(bm-a)^2 2^{\alpha-1}}f\left(\frac{a+bm}{2}\right)+\frac{\alpha(\alpha-1)\Gamma(\alpha-1)}{(mb-a)^{\alpha+1}} \quad (5)$$
$$\times\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb)+J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right]$$

Finally, we multiply both parts of equality (5) by the expression $\frac{(bm-a)^2}{\alpha 2^{2-\alpha}}$ and taking into account the Gamma function property $(\alpha-1)\Gamma(\alpha-1)=\Gamma(\alpha)$ we complete the proof. □

Theorem 2.1. *Let $f:I=[0,b^*]\to R$ be a twice differentiable function on I° such that $f''\in L[a,b]$ where $a,b\in I^\circ$ with $a<b$ and $b^*>0$. If $|f''|$ is in the $(s,m)-$ convex function and $a<mb$, then for all $\alpha>1$ the following inequality holds:*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb)+J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right]-f\left(\frac{a+mb}{2}\right)\right| \quad (6)$$
$$\leq\frac{(mb-a)^2}{\alpha 2^{2-\alpha}}\left[\gamma+B_{\frac{1}{2}}(\alpha+1,s+1)\right]\left(\left|f''(a)\right|+m\left|f''(b)\right|\right)$$

where

$$\gamma=\left[(\alpha+s+1)2^{\alpha+s+1}\right]^{-1} \quad \textit{and } B \textit{ is incomplete Euler Beta function}:$$
$$B_x(p,q)=\int_0^x t^{p-1}(1-t)^{q-1}dt \quad p,q>0, x\in[0,1]$$

Proof. If we use the triangle inequality to the right-hand side of (2) from Lemma 2.1, we obtain:

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb)+J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right]-f\left(\frac{a+mb}{2}\right)\right| \quad (7)$$
$$=\frac{(mb-a)^2}{\alpha 2^{2-\alpha}}|I_1+I_2|\leq\frac{(mb-a)^2}{\alpha 2^{2-\alpha}}(|I_1|+|I_2|)$$

And since the $|f''|$ function is $(s,m)-$ convex with account of inequality (1), we can write

$$|I_1|\leq\left|f''(a)\right|\int_0^{1/2}t^{\alpha+s}dt+m\left|f''(b)\right|\int_0^{1/2}t^{\alpha}(1-t)^s dt$$

or

$$|I_1|\leq\gamma\left|f''(a)\right|+m\left|f''(b)\right|B_{\frac{1}{2}}(\alpha+1,s+1) \quad (8)$$

And likewise

$$|I_2|\leq B_{\frac{1}{2}}(\alpha+1,s+1)\left|f''(a)\right|+\gamma m\left|f''(b)\right| \quad (9)$$

Adding (8) and (9) we get

$$|I_1|+|I_2| \leq \left[\gamma + B_{\frac{1}{2}}(\alpha+1, s+1)\right]\left(\left|f''(a)\right| + m\left|f''(b)\right|\right) \tag{10}$$

And multiplying both sides of the inequality (10) by the expression $\frac{(mb-a)^2}{\alpha 2^{2-\alpha}}$ taking (7) into account, we obtain (6). The proof is completed. □

Corollary 2.1. *If we choise $m=1$, $s=1$ and $\alpha=2$ in Teorem 2.1, then from (6) we get the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)df - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{48}\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \tag{11}$$

This inequality for convex functions obtained M. Sarikaya and Aktan (see [14]*, Prposition* 1*) and Y. Erdem et al. (see* [5]*, Corollary* 2*, for* $c=0$*).*

Theorem 2.2. *Let $f: I=[0,b^*] \to R$ be a twice differentiable function on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a<b$ and $b^*>0$. If $|f''|^q$ is a $(s,m)-$ convex function and $a<mb$, then for all $\alpha>1$, $q \geq 1$ and $t \in (0,1)$ the following inequality holds*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb) + J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right] - f\left(\frac{a+mb}{2}\right)\right| \tag{12}$$
$$\leq \frac{(mb-a)^2}{\alpha 2^{2-\alpha}} \times \xi \times F$$

where

$$F = \left[\gamma\left|f''(a)\right|^q + B_{\frac{1}{2}}(\alpha+1, s+1)\, m\left|f''(b)\right|^q\right]^{1/q}$$
$$+\left[B_{\frac{1}{2}}(\alpha+1, s+1)\left|f''(a)\right|^q + \gamma m\left|f''(b)\right|^q\right]^{1/q}$$
$$\xi = \left[(\alpha+1)2^{\alpha+1}\right]^{\frac{1}{q}-1} \quad \text{and} \quad \gamma = \left[(\alpha+s+1)2^{\alpha+s+1}\right]^{-1}$$

and B is incomplete Euler Beta function.

Proof. If we use the triangle inequality to the right–hand side of (2) from Lemma 2.1, we obtain:

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb) + J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right] - f\left(\frac{a+mb}{2}\right)\right|$$
$$= \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}|I_1+I_2| \leq \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}(|I_1|+|I_2|)$$

Using the well–known power–mean integral inequality and since $|f''|^q$ is a $(s,m)-$ convex function with account of inequality (1) we obtained

$$|I_1| \leq \left(\int_0^{1/2} t^\alpha dt\right)^{1-\frac{1}{q}}\left[\left|f''(a)\right|^q \int_0^{1/2} t^{\alpha+s}dt + m\left|f''(b)\right|^q \int_0^{1/2} t^\alpha (1-t)^s dt\right]^{1/q}$$

Or

$$|I_1| \leq \xi \times \left[\gamma\left|f''(a)\right|^q + m\left|f''(b)\right|^q B_{\frac{1}{2}}(\alpha+1, s+1)\right]^{1/q} \tag{13}$$

Since

$$|I_2| = \left|\int_{1/2}^1 (1-t)^\alpha f''(at+m(1-t)b)dt\right| = \left|\int_0^{1/2} t^\alpha f''(a(1-t)+mtb)dt\right|$$

Similarly for I_2 we can write:

$$|I_2| \leq \xi \times \left[B_{\frac{1}{2}}\left(\alpha+1,s+1\right)\left|f''(a)\right|^q + \gamma m\left|f''(b)\right|^q\right]^{1/q} \tag{14}$$

Adding inequalties (13) and (14), we get:

$$|I_1| + |I_2| \leq \xi \times F$$

and multiplying both sides last inequality by the expression $\frac{(mb-a)^2}{\alpha 2^{2-\alpha}}$we obtain (12). The proof is completed. □

Corollary 2.2. *If we choise $m=1$, $s=1$ and $\alpha=2$ in Theorem 2.2, then from (12) we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{48}\times E \tag{15}$$

where

$$E = \left[\frac{3\left|f''(a)\right|^q + 5\left|f''(b)\right|^q}{8}\right]^{1/q} + \left[\frac{5\left|f''(a)\right|^q + 3\left|f''(b)\right|^q}{8}\right]^{1/q}$$

This inequality (15) for convex functions obtained by M. Sarikaya and Aktan (see [14], *Proposition* 5.*).*

Theorem 2.3. *Let $f : I = [0,b^*] \to R$ be a twice differentiable function on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a<b$ and $b^*>0$. If $|f''|^q$ is a (s,m)- convex function and $a<mb$, then for all α, q and $p>1$, such that $\frac{1}{q}+\frac{1}{p}=1$ the following inequality holds*

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb) + J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right] - f\left(\frac{a+mb}{2}\right)\right| \\ &\leq \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}\times 2^{-3/p}\times W\end{aligned} \tag{16}$$

where

$$\begin{aligned}W &= \left[\mu\left|f''(a)\right|^q + B_{\frac{1}{2}}\left(\left(\alpha-1\right)q+2,s+1\right)m\left|f''(b)\right|^q\right]^{1/q} \\ &+ \left[B_{\frac{1}{2}}\left(q\alpha-q+2,s+1\right)\left|f''(a)\right|^q + \mu m\left|f''(b)\right|^q\right]^{1/q} \\ \mu &= \left[(q\alpha-q+s+2)2^{q\alpha-q+s+2}\right]^{-1}\end{aligned}$$

and B is incomplete Euler Beta function.

Proof. If we use the triangle inequality to the right-hand side of (2) from Lemma 2.1, we obtain

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb) + J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right] - f\left(\frac{a+mb}{2}\right)\right| \\ &= \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}\left|I_1+I_2\right| \leq \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}\left(|I_1|+|I_2|\right)\end{aligned}$$

Using the well-known Hölder integral inequality and since $|f''|^q$ is a $(s,m)-$ convex function with account of inequality (1) we get

$$|I_1| = \left|\int_0^{1/2} t^{1/p}t^{1/q}t^{\alpha-1}f''(at+m(1-t)b)dt\right| \leq \left(\int_0^{1/2}\left(t^{1/p}\right)^p dt\right)^{1/p}$$
$$\times\left[\left|f''(a)\right|^q\int_0^{1/2}t^{q(\alpha-1)+s+1}dt + m\left|f''(b)\right|^q\int_0^{1/2}t^{q\alpha-q+1}(1-t)^s dt\right]^{1/q}$$

and so

$$|I_1| \leq 2^{-3/p}\left[\mu\left|f''(a)\right|^q + m\left|f''(b)\right|^q B_{\frac{1}{2}}(q\alpha-q+2,s+1)\right]^{1/q} \tag{17}$$

Since

$$|I_2| = \left|\int_{1/2}^1(1-t)^\alpha f''(at+m(1-t)b)dt\right| = \left|\int_0^{1/2}t^\alpha f''((1-t)a+mbt)dt\right|$$

Similarly for I_2, we can write

$$|I_2| \leq 2^{-\frac{3}{p}}\left[B_{\frac{1}{2}}(q\alpha-q+2,s+1)\left|f''(a)\right|^q + \mu m\left|f''(b)\right|^q\right]^{\frac{1}{q}} \tag{18}$$

Adding inequalties (17) and (18) we get

$$|I_1| + |I_2| \leq 2^{-\frac{3}{p}} \times W$$

and multiplying both sides last inequality by the expression $\frac{(mb-a)^2}{\alpha 2^{2-\alpha}}$we obtain (16). The proof is completed. □

Corollary 2.3. *Since $\frac{1}{p} = 1 - \frac{1}{q}$ if we choise $m = 1$, $s = 1$, $\alpha = 2$, in Theorem 2.3 then from (16) we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)df - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{32}\times\psi(q)\times D \tag{19}$$

where

$$\psi(q) = [(q+3)(q+2)]^{-1/q},$$
$$D = \left[(q+2)\left|f''(a)\right|^q + (q+4)\left|f''(b)\right|^q\right]^{1/q} + \left[(q+4)\left|f''(a)\right|^q + (q+2)\left|f''(b)\right|^q\right]^{1/q}$$

Here, since the $\lim\limits_{q\to 1^+}\psi(q) = \frac{1}{12}$ and $\lim\limits_{q\to\infty}\psi(q) = 1$ then $\frac{1}{12} \leq \left(\frac{1}{q+3}\right)^{1/q} \leq 1$ for all $q > 1$. For $q \to 1^+$ from (19) we get (11).

3. Some Results For Trapezoid Inequalities

We formulate and prove the following lemma

Lemma 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°. If $f'' \in L[a,b]$, where $a, b \in I$ and $a \neq b$, then for all $\alpha > 1$ the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U = \frac{(b-a)^2}{2}(I_1+I_2) \tag{20}$$

where

$$U = \frac{(\alpha+1)}{b-a}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] - \left[J_{a^+}^{\alpha-1} f(b) + J_{b^-}^{\alpha-1} f(a)\right],$$

$$I_1 = \int_0^1 t(1-t)^{\alpha} f''(at + (1-t)b)dt \text{ and } I_2 = \int_0^1 t(1-t)^{\alpha} f''((1-t)a + tb)dt$$

Proof. To calculate the integrals we first make a transformation of variables $1-t = z$, and then twice integrating by parts we obtain:

$$I_1 = \int_0^1 z^{\alpha}(1-z)f''((1-z)a+zb)dz = \frac{f(b)}{(b-a)^2} + \frac{\alpha(\alpha-1)}{(b-a)^2}$$
$$\times \int_0^1 z^{\alpha-2} f((1-z)a+zb)dz - \frac{\alpha(\alpha+1)}{(b-a)^2}\int_0^1 z^{\alpha-1} f((1-z)a+zb)dz$$

If we make $(1-z)a + zb = x$ transformation in both integrals obtained and taking into account the property of the Gamma function, we obtain:

$$I_1 = \frac{f(b)}{(b-a)^2} + \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{b^-}^{\alpha-1} f(a) - \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}} J_{b^-}^{\alpha} f(a)$$

Similarly for the other integral

$$I_2 = \frac{f(a)}{(b-a)^2} + \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{a^+}^{\alpha-1} f(b) - \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}} J_{a^+}^{\alpha} f(b)$$

Summing these equalites and then grouping the summands we get

$$I_1 + I_2 = \frac{1}{(b-a)^2}[f(a) + f(b)] + -\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \times U \tag{21}$$

And multiplying both sides of the equality (21) by the expression $\frac{(b-a)^2}{2}$ we obtain (20). The proof is completed. □

Theorem 3.1. *Let $f : I = [0, b^*] \to R$ be a twice differentiable function on I° such that $f'' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$ and $b^* > 0$. If $|f''|$ is in the $(s, m)-$ convex function and $\frac{b}{m} \in I^\circ$, then for all $\alpha > 1$ the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \tag{22}$$
$$\leq \frac{(b-a)^2}{2}\left[B(s+2, \alpha+1) + \zeta\right]\left(\left|f''(a)\right| + m\left|f''\left(\frac{b}{m}\right)\right|\right)$$

where

$$U = \frac{(\alpha+1)}{b-a}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] - \left[J_{a^+}^{\alpha-1} f(b) + J_{b^-}^{\alpha-1} f(a)\right],$$
$$\zeta = [(s+\alpha+1)(s+\alpha+2)]^{-1}$$

and B is Euler Beta function: $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \forall x, y > 0$

Proof. From Lemma 3.1 and from the triangle inequality we obtain:

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \leq \frac{(b-a)^2}{2}\left(|I_1| + |I_2|\right) \tag{23}$$

Since $|f''|$ is $(s,m)-$ convex with account of inequality (1), we can write

$$|I_1| \leq \left|f''(a)\right| \int_0^1 t^{s+1}(1-t)^{\alpha}dt \; + m\left|f''\left(\frac{b}{m}\right)\right| \int_0^1 t(1-t)^{\alpha+s}dt$$
$$\leq \left|f''(a)\right| B(s+2,\alpha+1) + \zeta m\left|f''\left(\frac{b}{m}\right)\right| \tag{24}$$

Since

$$|I_2| = \left|\int_0^1 t(1-t)^{\alpha} f''((1-t)a+tb)dt\right| = \left|\int_0^1 (1-t)t^{\alpha} f''(ta+(1-t)b)dt\right|$$

Similarly for the second integral $|I_2|$ we can write:

$$|I_2| \leq \zeta\left|f''(a)\right| + m\left|f''\left(\frac{b}{m}\right)\right| B(\alpha+1,s+2) \tag{25}$$

Summing these inequalities (24) and (25) and the since Beta function is simmetric $(B(x,y)=B(y,x))$ then can write:

$$|I_1|+|I_2| \leq [B(\alpha+1,s+2)+\zeta]\left[\left|f''(a)\right| + m\left|f''\left(\frac{b}{m}\right)\right|\right] \tag{26}$$

And we multiply both sides of inequality (26) by the expression $\frac{(b-a)^2}{2}$ and taking into account inequality (23) we obtain (22). The proof is completed. □

Corollary 3.1. *In Theorem 3.1 if we choise $m=1$, $\alpha=2$ and $s=1$ from (22) ve get Trapezoid inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{24}\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \tag{27}$$

This inequaliti for convex functions obtained by M. Sarıkaya and Aktan (see [14]*, Proposition* 2.*).*

Theorem 3.2. *Let $f: I=[0,b^*] \to R$ be a twice differentiable function on I° such that $f'' \in L[a,b]$ where $a,b \in I^{\circ}$ with $a<b$ and $b^*>0$. If $|\,f''|^q$ is a (s,m)- convex function and $\frac{b}{m} \in I^{\circ}$, then for all $\alpha>1$, $q \geq 1$ and $t \in (0,1)$ the following inequality holds:*

$$\left|\frac{f\,(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \leq \frac{(b-a)^2}{2} \times \nu \times V \tag{28}$$

where

$$U = \frac{(\alpha+1)}{b-a}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right] - \left[J^{\alpha-1}_{a^+}f(b) + J^{\alpha-1}_{b^-}f(a)\right],$$
$$V = \left[B\left(s+2,\alpha+1\right)\left|f''(a)\right|^q + \mu m\left|f''(b)\right|^q\right]^{1/q}$$
$$+ \left[\mu\left|f''(a)\right|^q + m\left|f''(b)\right|^q B\left(s+2,\alpha+1\right)\right]^{1/q},$$
$$\nu = \left[(\alpha+1)\left(\alpha+2\right)\right]^{\frac{1}{q}-1}, \mu = \left[(\alpha+s+1)(\alpha+s+2)\right]^{-1}$$

and B is Euler Beta function.

Proof. From Lemma 3.1 and from the triangle inequality we obtain

$$\left|\frac{f\,(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \leq \frac{(b-a)^2}{2}\left(|I_1|+|I_2|\right) \tag{29}$$

Using the well–known power–mean integral inequality and since $|f''|^q$ is a $(s,m)-$ convex function, we have

$$|I_1| \le \int_0^1 t(1-t)^\alpha \left|f''(at+(1-t)b)\right| dt \le \left(\int_0^1 t(1-t)^\alpha dt\right)^{1-\frac{1}{q}}$$
$$\times \left[\int_0^1 (1-t)^\alpha t \left[t^s \left|f''(a)\right|^q + m(1-t)^s \left|f''\left(\frac{b}{m}\right)\right|^q\right] dt\right]^{1/q}$$

Or

$$|I_1| \le \nu \times \left[\left|f''(a)\right|^q B(s+2,\alpha+1) + \mu m \left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}$$

Since

$$|I_2| = \left|\int_0^1 t(1-t)^\alpha f''((1-t)a+tb)dt\right| = \left|\int_0^1 t^\alpha (1-t) f''(ta+(1-t)b)dt\right|$$

Similarly to the first, for the second integral, we can write:

$$|I_2| \le \nu \times \left[\mu \left|f''(a)\right|^q + m \left|f''\left(\frac{b}{m}\right)\right|^q B(s+2,\alpha+1)\right]^{1/q}$$

And adding the last inequalites we get

$$|I_1| + |I_2| \le \nu \times V \tag{30}$$

Multiplying both sides of the last inequality by the expression $\frac{(b-a)^2}{2}$ and taking into account inequality (29) we obtain (28). The proof is completed. □

Corollary 3.2. *In Theorem 3.2 if we choise $m = 1$, $\alpha = 2$ and $s = 1$ from (28) ve get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{24} \times E \tag{31}$$

where

$$E = \left[\frac{2\left|f''(a)\right|^q + 3\left|f''(b)\right|^q}{5}\right]^{1/q} + \left[\frac{3\left|f''(a)\right|^q + 2\left|f''(b)\right|^q}{5}\right]^{1/q}$$

This inequality is of the same order as the Trapezoid inequality for convex functions obtained by M. Sarıkaya and Aktan (see [14]*, Prposition 6.)*

Theorem 3.3. *Let $f : I = [0, b^*] \to R$ be a twice differentiable function on I° such that $f'' \in L[a,b]$ where $a, b \in I^\circ$ with $a < b$ and $b^* > 0$. If $|f''|^q$ is a (s,m)- convex function and $\frac{b}{m} \in I^\circ$, then for all $\alpha, q > 1$ and $t \in (0,1)$ the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}} \times U\right| \le \frac{(b-a)^2}{2} \times 2^{-1/p} \times D \tag{32}$$

where

$$U = \frac{(\alpha+1)}{b-a}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - \left[J_{a^+}^{\alpha-1} f(b) + J_{b^-}^{\alpha-1} f(a)\right],$$
$$D = \left[\left|f''(a)\right|^q B(s+2,\alpha q+1) + \xi m \left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}$$
$$+ \left[\left|\xi f''(a)\right|^q + m \left|f''\left(\frac{b}{m}\right)\right|^q B(s+2,\alpha q+1)\right]^{1/q}$$
$$\xi = \left[(\alpha q+s+1)(\alpha q+s+2)\right]^{-1}$$

and B *is Euler Beta function.*

Proof. From Lemma 3.1 and from the triangle inequality we obtain:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha-1}}\times U\right|\le\frac{(b-a)^2}{2}\left(|I_1|+|I_2|\right)\tag{33}$$

Using the well-known Hölder integral inequality and since $|f''|^q$ is a $(s,m)-$ convex function, we have

$$\begin{aligned}|I_1|&=\left|\int_0^1 t(1-t)^{\alpha}f''(at+(1-t)b)dt\right|\le\int_0^1 t^{1/p}t^{1/q}(1-t)^{\alpha}\left|f''(at+(1-t)b)\right|dt\\&\le\left(\int_0^1\left(t^{1/p}\right)^p dt\right)^{1/p}\left\{\int_0^1 t(1-t)^{\alpha q}\left[t^s\left|f''(a)\right|^q+m\left(1-t\right)^s\left|f''\left(\frac{b}{m}\right)\right|^q\right]dt\right\}^{1/q}\end{aligned}$$

and so

$$|I_1|\le 2^{-1/p}\times\left[|f''(a)|^q B\left(s+2,\alpha q+1\right)+\xi m\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\tag{34}$$

In the second integral, making the change of variables $z=1-t$, we can write

$$\begin{aligned}|I_2|&=\left|\int_0^1 z^{\alpha}(1-z)^{1/p}(1-z)^{1/q}f''(za+(1-z)\,b)dz\right|\\&\le\left(\int_0^1(1-z)dz\right)^{1/p}\left[\int_0^1 z^{\alpha q}(1-z)\left|f''(za+(1-z)\,b)\right|^q dz\right]^{1/q}\end{aligned}$$

and so

$$|I_2|\le 2^{-1/p}\times\left[\xi\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q B\left(\alpha q+1,s+2\right)\right]^{1/q}\tag{35}$$

Adding the last inequalites (34) and (35) and taking into account that the Beta symmetric function we get:

$$|I_1|+|I_2|\le 2^{-1/p}\times D\tag{36}$$

And we multiply both sides of inequality (36) by the expression $\frac{(b-a)^2}{2}$ and taking into account inequality (33) we obtain (32). The proof is completed. □

Corollary 3.3. *Since* $\frac{1}{p}=1-\frac{1}{q}$ *if we choise* $m=1, s=1$ *and* $\alpha=2$*, in Theorem 3.3 then from (32) we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{4}\times\varphi(q)\times F\tag{37}$$

where

$$\begin{aligned}\varphi(q)&=\left[(q+1)(2q+3)\right]^{-1/q},\\F&=\left[\frac{2|f''(a)|^q+(2q+1)|f''(b)|^q}{2q+1}\right]^{1/q}+\left[\frac{(2q+1)|f''(a)|^q+2|f''(b)|^q}{2q+1}\right]^{1/q}\end{aligned}$$

Here, since the $\lim_{q\to1^+}\varphi(q)=\frac{1}{10}$ *and* $\lim_{q\to+\infty}\varphi(q)=1$ *then* $\frac{1}{10}<\varphi(q)<1$ *for all* $q>1$*. For* $q\to1^+$ *from (37) we get (27).*

4. Applications To Special Means

We now consider the means (see Pearce, C.M.E. and Pečarič, J. in [12]) for arbitrary real numbers α and β.

(1) *Arithmetic mean* : $A(\alpha,\beta)=\frac{\alpha+\beta}{2}$;
(2) *Quadratic mean* : $Q(\alpha,\beta)=\sqrt{\alpha^2+\beta^2}$;
(3) *Geometric mean* : $G(\alpha,\beta)=\sqrt{\alpha\beta}$, $\alpha\beta\geq 0$;
(4) *Harmonic mean* : $H(\alpha,\beta)=\frac{2\alpha\beta}{\alpha+\beta}$, $\alpha+\beta\neq 0$;
(5) *Logarithmic mean* : $L(\alpha,\beta)=\frac{\beta-\alpha}{\ln\beta-\ln\alpha}$, $\alpha,\beta>0$ and $\alpha\neq\beta$.

Now, using results we give some applications to special means of positive real numbers.

Proposition 4.1. *Let $a=0$ and $b\in\mathbb{R}^+$, then, we have*

$$\left|A\left[Q\left(b,1\right),\frac{1}{b}\ln\left(2A\left(b,Q\left(b,1\right)\right)\right)\right]-Q\left(1,\frac{b}{2}\right)\right|\leq\frac{b^2}{96}H^{-1}\left(Q^3\left(b,1\right),1\right)$$

Proof. The assertion follows from Corollary 2.1 applied to the function $f(x)=\sqrt{1+x^2}$. ☐

Proposition 4.2. *Let $a,b\in\mathbb{R}^+$, $a<b$, then, we have*

$$\left|G^{-1}\left(a^2,b^2\right)-A^{-2}\left(a,b\right)\right|\leq\frac{\left(b-a\right)^2}{2^{\frac{3q+2}{q}}}\left\{A^{\frac{1}{q}}\left(3a^{-4q},5b^{-4q}\right)+A^{\frac{1}{q}}\left(5a^{-4q},3b^{-4q}\right)\right\}.$$

Proof. The assertion follows from Corollary 2.2 applied to the function $f(x)=\frac{1}{x^2}$, $x>0$. ☐

Proposition 4.3. *Let $a,b\in\mathbb{R}^+$, $a<b$. Then, we have*

$$\left|L^{-1}\left(a,b\right)-A^{-1}\left(a,b\right)\right|\leq 16^{-1}(b-a)^2\left(q^2+5q+6\right)^{-1/q}\times\left[A^{\frac{1}{q}}\left(\frac{q+2}{a^{3q}},\frac{q+4}{b^{3q}}\right)+A^{\frac{1}{q}}\left(\frac{q+4}{a^{3q}},\frac{q+2}{b^{3q}}\right)\right]$$

Proof. The assertion follows from Corollary 2.3 applied to the function $f(x)=\frac{1}{x}$, $x>0$. ☐

Proposition 4.4. *Let $a=0$ and $b>0$ then we have*

$$\left|A(1,Q(b,1))-A\left[Q(b,1),\frac{1}{b}ln(2A(b,Q(b,1)))\right]\right|\leq\frac{b^2}{48}H^{-1}\left(Q^3(b,1),1\right)$$

Proof. The assertion follows from Corollary 3.1 applied to the function $f(x)=\sqrt{1+x^2}$. ☐

Proposition 4.5. *Let $a,b>0$ and $a<b$ then we have*

$$\left|H^{-1}(a^2,b^2)-L^{-1}(a^2,b^2)\right|\leq\frac{(b-a)^2}{4}\left(\frac{2}{5}\right)^{1/q}\times\left[A^{\frac{1}{q}}\left(3a^{-4q},5b^{-4q}\right)+A^{\frac{1}{q}}\left(5a^{-4q},3b^{-4q}\right)\right]$$

Proof. The assertion follows from Corollary 3.2 applied to the function $f(x)=\frac{1}{x^2}$, $x>0$. ☐

Proposition 4.6. *Let $a,b>0$ and $a<b$ then we have*

$$\left|H^{-1}(a,b)-L^{-2}(a,b)\right|\leq 2^{(1-q)/q}(b-a)^2[(q+1)(2q+1)(2q+3)]^{-1/q}\times\left[A^{\frac{1}{q}}(2a^{-3q},(2q+1)b^{-3q})+A^{\frac{1}{q}}((2q+1)a^{-4q},2b^{-4q})\right]$$

Proof. The assertion follows from Corollary 3.3 applied to the function $f(x) = \frac{1}{x}, \quad x > 0$. □

5. Conclusion

Two lemmas are formulated. On the basis of these lemmas, through fractional integrals, we obtain new integral Hadamard-type inequalities for functions whose second-order derivatives are (s, m) – convex functions. As a consequence of these inequalities, upper bound estimates are obtained for Midpoint and Trapezoid inequalities. The obtained estimations correspond to the estimations in the literature.

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Bahtiyar BAYRAKTAR for the photograph and short biography, see TWMS J. Appl. and Eng. Math., V.6, No.2, 2016.