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EXISTENCE AND MULTIPLICITY OF WEAK SOLUTIONS FOR A CLASS OF THREE POINT BOUNDARY VALUE PROBLEMS OF KIRCHHOFF TYPE

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Abstract. In this paper we shall discuss the existence and multiplicity results of solutions for a three point boundary value problem of kirchhoff-type equations. We investigate the existence of one, two or three solutions for our problem under algebraic conditions by applying a different three critical point theorem.
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1. Introduction.

In the literature many results focus on the existence of multiple solutions to boundary value problems. There seems to be increasing interest in multiple solutions to boundary value problems, because of their applications in physical processes described by differential equations can exhibit more than one solution, and other fields. For example, certain chemical reactions in tubular reactors can be mathematically described by a nonlinear two-point boundary value problem and one is interested if multiple steady-states to the problem exist. For a instance treatment of chemical reactor theory and multiple solutions see [4, section 7], and the references therein. For additional approaches to the existence of multiple solutions to boundary value problems, see [25, 26] and references therein. Moreover, in [32], Ricceri obtained a three critical points theorem and in [31] gave a general version of the theorem to extend the results for a class of more extensive equations. By these results, many authors studied the existence of at least three solutions for BVPs (for instance, see [2, 9, 14]. The purpose of this paper is to establish the existence and

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multiplicity solutions for the three point boundary value problem of kirchhoff type

$$\begin{aligned} &-K(\int_a^b |u'(t)|^2 dt)u''(t) = \lambda f(t, u(t)) + h(u(t)),\ t \in (a, b) \\ &u(a) = 0, u(b) = \alpha u(\eta), \end{aligned} \tag{1}$$

where $K : [0, +\infty] \to \mathbb{R}$ is a continuous function such that there exist positive numbers m and M with $m < K(x) < M$ for all $x \geq 0$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, i.e. there is a constant $L > 0$ such that

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$ and $h(0) = 0, \alpha \in \mathbb{R}$ and $\eta \in (a, b)$. Multi-point boundary-value problems of ordinary differential equations play an important role in applied mathematics, physics and the vibration of cables with nonuniform weights [29], and as a consequence, have attracted a great deal of interest over the years. The study of these problems for linear second-order ordinary differential equations was initiated by Ii'in and Moiseev [27]. Motivated by the study of Ii'in and Moiseev [27], Gupta [17] studied certain three-point boundary value problems for nonlinear ordinary differential equations. In the past few years, there has been much attention focused on questions of solutions of three-point boundary-value problems for nonlinear ordinary differential equations. For background and recent results, we refer the reader to [6, 15, 18, 33, 34] and the references therein for details. For example, Xu in [34] by employing the fixed point index method, obtained some multiplicity results for positive solutions of some singular semi-position three-point boundary-value problem. Sun in [33] by using a fixed point theorem of cone expansion-compression type due to Krasnosel'skii, established various results on the existence of single and multiple positive solutions for the nonlinear singular third-order three-point boundary-value problem

$$\begin{aligned} &u''(t) - \lambda a(t)F(t, u(t)) = 0,\ t \in (0, 1) \\ &u(0) = 0, u'(\eta) = u''(1) = 0, \end{aligned}$$

with $\lambda > 1, \eta \in [1/2, 1)$ where $a(t)$ is a non-negative continuous function defined on $(0, 1)$ and $F : [0, 1] \times \mathbb{R} \to [0, \infty)$ is continuous. Du et al. in [15] based upon Leray-Schauder degree theory, ensured the existence of at least three solutions for the problem

$$\begin{aligned} u''(t) + f(t, u(t), u'(t)) = 0,\ t \in (0, 1) \\ u(0) = 0, u(1) = \xi u(\eta), \end{aligned}$$

where $\xi > 0, 0 < \eta < 1$ such that $\xi\eta < 1$ and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Lin [28] by using variational method and three-critical-point theorem, studied the existence of at least three solutions for a three-point boundary-value problem

$$\begin{aligned} u''(t) + \lambda f(t, u(t)) = 0,\ t \in [0, 1] \\ u(0) = 0, u(1) = \alpha u(\eta). \end{aligned}$$

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process that depends on the average of itself, for example, the population density. Problems of Kirchhoff-type have been widely investigated. We refer the reader to the papers [5, 7, 19, 20] and the references therein. For example in [20] based on a three critical point theorem, the existence of an interval of positive real parameters

λ for which the boundary-value problem of Kirchhoff-type

$$-K(\int_a^b |u'(t)|^2 dt)u''(t) = \lambda f(t, u(t)),\ t \in [a, b]$$
$$u(a) = u(b) = 0$$

where $K : [0, +\infty[\to \mathbb{R}$ is a continuous function, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\lambda > 0$, applying the mountain pass theorem as usual results studied the existence of solutions to nonlocal equations involving the p-Laplacian. In [1], authors establish the existence of at least three non-negative weak solutions for the following perturbed three-point boundary value problem of Kirchhoff type

$$\begin{aligned}&-K(\int_a^b |u'(t)|^2 dt)u''(t) = \lambda f(t, u(t)) + \mu g(t, u(t)) + h(u(t)),\ t \in (0, 1)\\ &u(0) = 0, u(1) = \alpha u(\gamma),\end{aligned}$$

where $K : [0, \infty] \to \mathbb{R}$ is a continuous function such that there exist positive numbers m and M with $m \leq K(x) \leq M$ for all $x \geq 0, \lambda > 0, \mu \geq 0, f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.

$$|h(\xi_1) - h(\xi_2)| \leq L|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}$ and $h(0) = 0, \gamma \in (0, 1)$ and $0 < \alpha < 1$. More precisely, they proved the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. We refer to [3] in which using variational methods and critical point theory the existence of at least one weak solution for a three-point boundary-value problem of Kirchhoff-type was discussed. See [22] which concerned with existence and multiplicity results for the following Kirchhoff-type three-point boundary value problem

$$\begin{aligned}&-K(\int_a^b |u'(t)|^2 dt)u''(t) = \lambda f(t, u(t)) + h(u(t)),\ t \in (a, b)\\ &u(a) = 0, u(b) = \alpha u(\eta),\end{aligned}$$

where $K : [0, +\infty] \to \mathbb{R}$ is a continuous function such that there exist positive numbers m and M with $m < K(x) < M$ for all $x \geq 0$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, i.e. there is a constant $L > 0$ such that

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$ and $h(0) = 0, \alpha \in \mathbb{R}$ and $\eta \in (a, b)$. The main aim of this paper is to continue the further study on the problem (1). When the nonlinearity f has a sub-critical growth, by using three different variational methods which are the main tools established in [10, 12] (see Theorems 2.1, 2.2 and 2.3 below), we discuss the existence of at least one, two or three weak solutions for the parameter λ belonging to precise positive intervals.

The paper is arranged as follows. In section 2, we recall some basic definitions and three critical points theorem and in section 3 the existence of one weak solution for the problem (1) is obtained. In section 4, we apply one of the main tools to establish the existence of two distinct weak solutions for the problem (1). Finally, in section 5, the existence of three weak solutions for the problem (1) is achieved.

2. PRELIMINARIES

Theorem 2.1. *([10, Theorem 2.3]) Let X be a real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that:

(1) $\frac{\sup_{\Phi(x)\leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(2) *for each* $\lambda \in \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\right[$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *satisfies* $(P.S)^{[r]}$ *condition.*

Then for each

$$\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\right[,$$

there is $x_{0,\lambda} \in \Phi^{-1}(]0, r[)$ such that

$$I'_\lambda(x_{0,\lambda}) = \vartheta_{X^*},$$

and

$$I_\lambda(x_{0,\lambda}) \leq I_\lambda(x) \text{ for all } x \in \Phi^{-1}(]0, r[).$$

Other tool is the following abstract result.

Theorem 2.2. *([10, Theorem 3.6]) Let X be a real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$.*

Fix $r > 0$ and assume that, for each

$$\lambda \in \left[0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}\right],$$

the functional I_λ admits two distinct critical points.

Theorem 2.3. *([12, Theorem 5.1]) Let X be a reflective real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:

(a_1) $\frac{\sup_{\Phi(x)\leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a_2) *for each* $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

We refer [11, 21] in which Theorems 2.1-2.3 have been successfully to the existence of one, two and three weak solutions for boundary value problems.

Here and in the sequel, we take

$$X := W_1^{1,2}(a, b) = \left\{u \in W^{1,2}(a, b) : u(a) = 0, u(b) = \alpha u(\eta)\right\}.$$

The space X, equipped with the norm

$$\| u \| := \left(\int_a^b |u'(t)|^2 dt \right)^{1/2}.$$

Corresponding to the functions f, K and h, we introduce the functions $F : [a,b] \times \mathbb{R} \to \mathbb{R}, \bar{K} : [a, +\infty] \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$, defined as follows

$$F(t,x) := \int_0^x f(t,\xi) d\xi \qquad \text{for every } (t,x) \in [a,b] \times \mathbb{R},$$

$$\bar{K}(x) := \int_0^x K(\xi) d\xi \qquad \text{for every } x \geq 0,$$

$$H(x) := \int_0^x h(\xi) d\xi \qquad \text{for every } x \in \mathbb{R}.$$

We say that a function $u \in X$ is a weak solution of (1) if

$$K\left(\int_a^b |u^{'}(t)|^2 dt\right) \int_a^b u^{'}(t) v^{'}(t) dt - \int_a^b h(u(t)) v(t) dt - \int_a^b f(t, u(t)) v(t) dt = 0$$

holds for all $v \in X$.

Proposition 2.1. *Assume that $u \in X$ is a weak solution of the problem* (1)*. Then u is also a generalized solution of* (1)*.*

Lemma 2.1. *(*[23, Lemma 2.3]*). For all $u \in X$, we have*

$$\max_{t \in [a,b]} |u(t)| \leq \frac{(1+|\alpha|)\sqrt{b-a}}{2} \| u \| .$$

We assume throughout and without further mention, that the following condition holds:
(H_1) $\qquad m > \frac{L(1+|\alpha|)^2 (b-a)^2}{4}$.

3. Existence of one weak solution

Before introducing our result we observe that, putting

$$\delta(x) = \sup\{\delta > 0 : \; B(x,\delta) \subseteq (a,b)\}$$

for all $x \in (a,b)$, one can prove that there exists $x_0 \in (a,b)$ such that $B(x_0, D) \subseteq (a,b)$, where

$$D := \sup_{x \in (a,b)} \delta(x).$$

With the above notations, we deal with the existence of one weak solution for the problem (1).

Theorem 3.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfies in the following condition:*

(A) *there exist a_1, $a_2 \in [0, +\infty]$ and a suitable positive constant $q \geq 1$ such that,*

$$|f(x,t)| \leq a_1 + a_2 |t|^{q-1},$$

for every $(x,t) \in (a,b) \times \mathbb{R}$.

Moreover, assume that

$$\limsup_{t\to 0^+} \frac{F(x,t)}{t^2} = +\infty. \tag{2}$$

Then, put $\lambda^* := a_1 k_1 \sqrt{\frac{8}{m(1+|\alpha|)^2(b-a)^2}} + \frac{a_2}{q} k_q (\frac{8}{m(1+|\alpha|)^2(b-a)^2})^{\frac{q}{2}}$, *where* k_1 *and* k_q *denote respectively the constants of the embeddings* $W^{1,2}(a,b) \hookrightarrow L^1(a,b)$ *and* $W^{1,p}(a,b) \hookrightarrow L^q(a,b)$. *Then, for each* $\lambda \in]0, \lambda^*[$ *the problem* (1) *admits at least one nontrivial weak solution.*

Proof. Our aim is to apply Theorem 2.1 in the case $r = 1$ to the space $X := W^{1,p}(a,b)$ and to the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined as:

$$\Phi(u) = \frac{1}{2}\tilde{K}(\| u \|^2) - \int_a^b H(u(t))dt \tag{3}$$

$$\Psi(u) = \int_a^b F(t,u(t))dt.$$

for all $u \in X$. The functional Φ is $C^1(X,\mathbb{R})$, as it was said in the previous section. Moreover, thanks to condition (A) and to the compact embedding $W^{1,2}(a,b) \hookrightarrow L^q(a,b)$, Ψ is in $C^1(X,\mathbb{R})$ and has compact derivative. This ensures that function $I_\lambda = \Phi - \lambda\Psi$ verifies $(P.S)^{[}r]$ condition for each $r > 0$ (see [8, proposition 2.1]) and so condition (a_2) of Theorem 2.1 is verified.

Fixed $\lambda \in]0, \lambda^*[$, by (2), there exists

$$0 < \delta < \min\{1, K_0\} \tag{4}$$

such that

$$\frac{K_0 \inf F(x,\delta_\lambda)(\frac{D}{2})^N}{\delta_\lambda^2} > \frac{1}{\lambda}, \tag{5}$$

where

$$K_0 = D\sqrt{\frac{2}{4M + L(1+|\alpha|)^2(b-a)^2}(D^N - (\frac{D}{2})^N)}.$$

We denote by v_λ the function of X defined by

$$v_\lambda(x) = \begin{cases} 0, & x \in (a,b)\backslash B(x_0,D) \\ \frac{2\delta_\lambda}{D}(D - |x - x_0|), & x \in B(x_0,D)\backslash B(x_0,\frac{D}{2}) \\ \delta_\lambda, & x \in B(x_0,\frac{D}{2}), \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. We have

$$\begin{aligned} \Phi(v_\lambda) &= \frac{1}{2}\tilde{K}(\| v_\lambda \|^2) - \int_a^b H(v_\lambda(t))\,dt \\ &\le \frac{4M + L(1+|\alpha|)^2(b-a)^2}{8}\left((\frac{2\delta_\lambda}{D})^2[D^N - (\frac{D}{2})^N]\right) \\ &= \frac{{\delta_\lambda}^2}{K_0^2}. \end{aligned} \tag{6}$$

Moreover, thanks to (5), we observe that

$$\Psi(v_\lambda) \ge \int_{B(x_0,\frac{D}{2})} F(x, v_\lambda(x))\,dx \ge \inf F(x,\delta_\lambda)(\frac{D}{2})^N$$

and so we obtain that

$$\frac{\Psi(v_\lambda)}{\Phi(v_\lambda)} \geq \frac{K_0(\frac{D}{2})^N \inf F(x,\delta_\lambda)}{\delta_\lambda^2} > \frac{1}{\lambda}. \tag{7}$$

From (4) it results $\frac{\delta_\lambda^2}{K_0^2} < 1$ and so, from (6), $\Phi(v_\lambda) < 1$. For each $u \in \Phi^{-1}(]-\infty,1])$, one has

$$\begin{aligned}\Phi^{-1}(]-\infty,1[) &= \{u \in X : \Phi(u) < 1\}\\ &= \{u \in X : \frac{1}{2}\tilde{K}(\| u \|^2) - \int_a^b H(u(t))\,dt < 1\} \qquad (8)\\ &\subseteq \{u \in X : \frac{1}{2}\tilde{K}(\| u \|^2) \leq 1\}\\ &\subseteq \{u \in X :\| u \|^2 < \frac{8}{m(1+|\alpha|)^2(b-a)^2}\}.\end{aligned}$$

Moreover, (8), condition (A) .imply that, for each $u \in \Phi^{-1}(]-\infty,1])$ we have

$$\begin{aligned}\Psi(u) &= \int_a^b F(x,u(x))\,dx \leq a_1 \int_a^b |u(x)|\,dx + \frac{a_2}{q}\int_a^b |u(x)|^q\,dx\\ &\leq a_1 k_1\|u\| + \frac{a_2}{q}k_q^q\|u\|^q\\ &\leq a_1 k_1\sqrt{\frac{8}{m(1+|\alpha|)^2(b-a)^2}} + \frac{a_2}{q}(\frac{8}{m(1+|\alpha|)^2(b-a)^2})^{\frac{q}{2}}\end{aligned}$$

$$\sup_{u\in\Phi^{-1}(]-\infty,1])} \Psi(u) \leq \frac{8}{m(1+|\alpha|)^2(b-a)^2} = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{9}$$

From (6) and (8) one has

$$\sup_{\Phi(u)\leq 1} \Psi(u) < \frac{\Psi(v_\lambda)}{\Phi(v_\lambda)},$$

and so condition (1) of Theorem 2.1 is verified. Since $\lambda \in \left]\frac{\Phi(v_\lambda)}{\Psi(v_\lambda)}, \frac{1}{\sup_{\Phi(u)\leq 1}\Psi(u)}\right[$, Theorem 2.1 guarantees the existence of a local minimum point u_λ for the functional I_λ such that

$$0 < \Phi(u_\lambda) < 1,$$

and so u_λ is a nontrivial weak solution of the problem (1). □

Remark 3.1. Here, we can present a special case of one of our results where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

(h_1') there exist two nonnegative constants a_1, a_2 such that

$$|f(t)| \leq a_1 + a_2|t|^{q-1}, \quad \text{for all } t \in \mathbb{R},$$

for some convenient positive constant $q \geq 1$;

(h_2') $\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$.

Then, there exists $\lambda^* > 0$ such that for each $\lambda \in]0,\lambda^*[$, the problem

$$\begin{aligned}&-K\left(\int_a^b |u'(t)|^2 dt\right) u''(t) = \lambda f(u(t)) + h(u(t)),\ t \in (a,b) \qquad (10)\\ &u(a) = 0, u(b) = \alpha u(\eta),\end{aligned}$$

admits at least one nontrivial weak solution.

Now we present the following example in which the nonlinearity f verifies the hypotheses of Theorem 3.1. Recall that the construction of the nonlinear function f is partly motivated by [13, Example 5.1].

Example 3.1. *Suppose that r be a positive constant. Moreover, the function $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ is given as*

$$f(x,t) := \begin{cases} 1 + x\,|t|^{q-1}, & \text{if } \ x \in (a,b), \ |t| \leq r, \\ \dfrac{(1+r^2)(1+r^{q-1})}{1+t^2}, & \text{if } \ x \in (a,b), \ |t| > r, \end{cases}$$

where r is a fixed constant. It is easy to see that $|f(t)| \leq (1 + \max\{1, |b|\}\,|t|^{q-1})$ for every $t \in \mathbb{R}$, and thus condition (A) holds with $a_1 = 1$ and $a_2 = \max\{1, |b|\}$. On the other hand since $q > 2$, we have

$$\limsup_{t \to 0^+} \frac{F(x,t)}{t^2} = +\infty.$$

By using Theorem 3.1, for every $\lambda \in]0, \lambda^*[$, where

$$\lambda^* := k_1 \sqrt{\frac{8}{m(1+|\alpha|)^2(b-a)^2}} + \frac{k_q \max\{1,|b|\}}{q} \left(\frac{8}{m(1+|\alpha|)^2(b-a)^2}\right)^{\frac{q}{2}},$$

the problem (1) has at least one nontrivial weak solution.

Remark 3.2. We can assume that $f(t) = \frac{1}{1+t^2}$ for every $t \in \mathbb{R}$. Clearly, we have

$$|f(t)| \leq 1 \leq 1 + |t|^{q-1}, \quad \text{for all} \ \ q \geq 1.$$

On the other hand, we obtain that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = \lim_{t \to 0^+} \frac{1}{t(1+t^2)} = +\infty.$$

Thus, Remark 3.1 is applicable for the problem (10) by above notations.

4. Existence of two weak solutions

In this section, our goal is to obtain the existence of two distinct weak solutions for the problem (1). The following result is obtained by applying Theorem 2.2.

Theorem 4.1. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be carathéodory function satisfying conditions (A) and*

(AR) *there exist $\mu > 2$ and $R > 0$ such that $0 < \mu F(x,t) \leq t f(x,t)$, for $|t| \geq R$.*

Then, for each $\lambda \in]0, \lambda^*[$, the problem (1) admits at least two distinct weak solutions, where λ^* is the constant introduced in the statement of Theorem 3.1

Proof. We apply Theorem 2.2 in the case $r = 1$ to the space $X = W_1^{1,2}(]a, b[)$ with the usual norm and $\Phi, \Psi : X \to \mathbb{R}$ are defined in Theorem 3.1. Integrating condition (A) there exist $a_3, a_4 > 0$ such that

$$F(x,t) \geq a_3 |t|^{\mu} - a_4. \tag{11}$$

In fact, setting $\eta(x) := \min_{|\xi|=R} F(x, \xi)$ for every $x \in]a, b[$, and

$$\varphi_t(x,s) := F(x, st), \quad \forall s > 0. \tag{12}$$

By (AR), one has for every $(x, t) \in]a, b[\times \mathbb{R}$,

$$0 \leq \mu\varphi_t(x, s) = \mu F(x, st) \leq stf(x, st) = s\varphi_t'(x, s), \quad \forall\ s > \frac{R}{|t|}.$$

By direct computations, we have

$$\varphi_t(x, 1) \geq \varphi_t\Big(x, \frac{R}{|t|}\Big)\ \frac{|t|^\mu}{R^\mu}.$$

Taking into account (12), we obtain

$$F(x, t) \geq F\Big(x, \frac{R}{|t|}t\Big)\ \frac{|t|^\mu}{R^\mu} \geq \eta(x)\ \frac{|t|^\mu}{R^\mu} \geq a_1|t|^\mu - a_2.$$

Thus (11) is proved.

Now fixed $\bar{u} \in X\backslash\{0_X\}$, for each $|\eta| > R$, from the relation (11), one has

$$\begin{aligned} I_\lambda(\eta u) &= \Phi(\eta u) - \lambda\Psi(\eta u) \\ &= \frac{1}{2}\tilde{K}(\| \eta u \|^2) - \int_a^b H(\eta u(t))dt - \lambda\int_a^b F(t, \eta u(t))dt \\ &\leq \frac{\eta^2}{2}\left[\frac{4M + L(1 + |\alpha|)^2(b - a)^2}{8} \| u \|^2\right] - \lambda\int_a^b [a_3\eta|u|^\mu - a_4]\, dt. \end{aligned}$$

Since $\mu > 2$, this condition guarantees that I_λ is unbounded from below. By standard computation the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(P.S.)$ condition (see for instance [21, 30]) and so all hypotheses of Theorem 2.2 are verified. Then, for each $\lambda \in]0, \lambda^*[$, the functional I_λ admits two distinct critical points that are weak solutions of the problem (1). □

Remark 4.1. We observe that, if $f(x, 0) \neq 0$, then Theorem 4.1 ensures the existence of two nontrivial weak solutions for the problem (1). Moreover, we point out that in the most of the papers concerning the existence of solutions for the problem (1) the following condition is requested:

$$\limsup_{t\to 0^+} \frac{F(t)}{t^2} = 0.$$

It is easily proved that the previous condition is in conflict with condition $f(x, 0) \neq 0$.

Example 4.1. *We consider the function f defined by*

$$f(x, t) = 1 + |t|^{\gamma-2}t$$

and $\mu \leq \gamma$, for any fixed $x \in R^+$. We prove that f verifies the assumptions requested in Theorem 4.1. condition (A) of Theorem 4.1 is verifies. We observe that

$$F(x, t) = t + \frac{1}{\gamma}|t|^\gamma.$$

Now, we observe that

$$\begin{aligned} tf(x, t) - \mu F(x, t) &= t + |t|^\gamma - \mu(t + \frac{|t|^\gamma}{\gamma}) \\ &= t(1 - \mu) + |t|^\gamma(1 - \frac{\mu}{\gamma}) \\ &= |t|(\mu - 1) + |t|^\gamma(1 - \frac{\mu}{\gamma}). \end{aligned}$$

Since $\mu > 2$ and $\mu \leq \gamma$ then

$$tf(x, t) - \mu F(x, t) > 0.$$

This implies that (AR) is verified.

5. Existence of three weak solutions

In this section we deal the existence of at least three weak solutions for the problem (1). Our main result is the following theorem.

Theorem 5.1. *Assume that there exist five positive constants c, d, σ, β and j such that $c < d < j$ and $\beta + \sigma < b - a$ and*

$$(\frac{\sigma+\beta}{\sigma\beta}c^2)^{\frac{1}{2}}\frac{\int_{a+\sigma}^{b-\beta}F(x,d^2(\frac{\sigma+\beta}{\sigma\beta}))dx}{Md^2(\frac{\sigma+\beta}{\sigma\beta})}-\int_a^b\sup_{\Phi(x)\le r}F(x,t)dx>0, \tag{13}$$

and $j = \frac{1}{2}\left[m - Ld^2\left(\frac{\sigma+\beta}{\sigma\beta}\right)(b-a)\right]$. Then, for each parameter λ belongs to

$$\lambda := \Big[\frac{\int_{a+\sigma}^{b-\beta}F(t,d^2(\frac{\sigma+\beta}{\sigma\beta})}{Md^2(\frac{\sigma+\beta}{\sigma\beta})}, a_1c_1(\frac{8}{4m-L(1+|\alpha|)^2(b-a)^2})^{\frac{1}{2}}(\frac{\sigma+\beta}{\sigma\beta}c^2)^{-\frac{1}{2}}$$
$$+\frac{a_2}{q}c_q^q(\frac{8}{4m-L(1+|\alpha|)^2(b-a)^2})^{\frac{q}{2}}(\frac{\sigma+\beta}{\sigma\beta}c^2)^{\frac{q-2}{2}}\Big],$$

the problem (1) at least three weak solutions in X.

Proof. Let us define in X two functional Φ and Ψ by setting, for each $u \in X$,

$$\Phi(u) = \frac{1}{2}\tilde{K}(\| u \|^2) - \int_a^b H(u(t))\,dt$$

and

$$\Psi(u) = \int_a^b F(t,u(t))\,dt,$$

and it is well known that these functionals are well-defined and satisfy the hypotheses of Theorem 2.3. In particular,

$$\Phi'(u)(v) = K(\int_a^b |u'(t)|^2\,dt)\int_a^b u'(t)v'(t)\,dt - \int_a^b h(u(t))v(t)\,dt, \quad \forall\, u,v \in X,$$

admits a continuous inverse in X^* (see Proof of [24, Theorem 2.1]), and

$$\Psi'(u)(v) = \lambda\int_a^b f(t,u(t))v(t)\,dt, \quad \forall\, u,v \in X,$$

is compact (see Proof of [24, Theorem 2.1]). Clearly, the weak solutions to the problem (1) are exactly the critical points of the functional $I_\lambda := \Phi - \lambda\Psi$ and, owing to proposition 2.1, they are also generalized solutions. So, our aim is apply Theorem 2.3 to the space X with the norm and to the functional Φ and Ψ. As seen before, the functional Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.3. Now, let $\bar{v} \in X$ be defined by

$$\bar{v}(x) = \begin{cases} \frac{d}{\sigma}(x-a) & a \le x < a+\sigma \\ d & a+\sigma \le x \le b-\beta \\ \frac{d}{\beta}(b-x) & b-\beta < x \le b \end{cases}$$

and $r = c^2(\frac{\sigma+\beta}{\sigma\beta})$ where constants c, σ, β are given in the statement of the theorem. One has $\| \bar{v} \|^2 = d^2(\frac{\sigma+\beta}{\sigma\beta})$, so

$$\begin{aligned}\Phi(\bar{v}) &= \frac{1}{2}\tilde{K}(\| \bar{v} \|^2) - \int_a^b H(\bar{v}(t))dt \\ &\geq \frac{1}{2}md^2(\frac{\sigma+\beta}{\sigma\beta}) - \frac{L}{2}d^4(\frac{\sigma+\beta}{\sigma\beta})^2(b-a) \\ &\geq \frac{1}{2}d^2(\frac{\sigma+\beta}{\sigma\beta})\,[m - Ld^2(\frac{\sigma+\beta}{\sigma\beta})(b-a)] \\ &\geq d^2(\frac{\sigma+\beta}{\sigma\beta})j \\ &\geq r\end{aligned} \tag{14}$$

and to verify condition (a_1) of Theorem 2.3, from the compact embedding $X \hookrightarrow L^q(a,b)$, we have

$$\begin{aligned}\sup_{\Phi(x)\leq r} \Psi(u) &= \sup_{\Phi(x)\leq r} \int_a^b F(x,u)\,dx \\ &\leq \int_a^b a_1 c_1 \| u \| + \frac{a_2}{q} c_q^q \| u \|^q \\ &\leq a_1 c_1 (\frac{8r}{4m - L(1+|\alpha|)^2(b-a)^2})^{\frac{1}{2}} \\ &\quad + \frac{a_2}{q} c_q^q (\frac{8r}{4m - L(1+|\alpha|)^2(b-a)^2})^{\frac{q}{2}},\end{aligned}$$

hence

$$\begin{aligned}\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} &\leq a_1 c_1 (\frac{8}{4m - L(1+|\alpha|)^2(b-a)^2})^{\frac{1}{2}}(r)^{-\frac{1}{2}} \\ &\quad + \frac{a_2}{q} c_q^q (\frac{8}{4m - L(1+|\alpha|)^2(b-a)^2})^{\frac{q}{2}}(r)^{\frac{q-2}{2}}.\end{aligned}$$

Since $F(x,u) > 0$, then

$$\Psi(\bar{v}) = \int_a^b F(x,\bar{v})\,dx \geq \int_{a+\sigma}^{b-\beta} F(x,\bar{v})\,dx$$

and

$$\begin{aligned}\Phi(\bar{v}) &= \frac{1}{2}\tilde{K}(\| \bar{v} \|^2) - \int_a^b H(\bar{v}(t))\,dt \\ &\leq \frac{1}{2}\tilde{K}(\| \bar{v} \|^2) \leq \frac{1}{2}Md^2(\frac{\sigma+\beta}{\sigma\beta}).\end{aligned}$$

Taking into account (13), we get

$$\begin{aligned}\frac{\sup_{\Phi(x)\leq r}\Psi(u)}{r} &\leq a_1 c_1 (\frac{8}{4m - L(1+|\alpha|)^2(b-a)^2})^{\frac{1}{2}}(r)^{-\frac{1}{2}} \\ &\quad + \frac{a_2}{q} c_q^q (\frac{8}{4m - L(1+|\alpha|)^2(b-a)^2})^{\frac{q}{2}}(r)^{\frac{q-2}{2}} \\ &\leq \frac{\int_{a+\sigma}^{b-\beta} F(x, d^2(\frac{\sigma+\beta}{\sigma\beta}))\,dx}{Md^2(\frac{\sigma+\beta}{\sigma\beta})} \leq \frac{\Psi(\bar{v})}{\Phi(\bar{v})}.\end{aligned}$$

$\square$

Remark 5.1. If $f(x,0) \neq 0$, then, by Theorem 5.1, we obtain the existence of at least three non-zero weak solutions.

Example 5.1. *Consider the system*

$$3 + \frac{1}{\pi}\arctan\left(\int_0^1 |u^{'}(t)|^2\,dt\right) u^{''}(t) = \exp^{-t}(t)^{\gamma-1}(\gamma - t) + 1 + \sin(t),\ t \in (0,1)$$
$$u(0) = 0,\ u(1) = 1/2\,u(1/4).$$

By choosing $f(x,t) = f(t) = \exp^{-t}(t)^{\gamma-1}(\gamma - t) + 1$ *then* $F(t) = t^{\gamma}\exp^{-t} + t$ *for every* $t \geq 0$*. On the other hand* $2 < K(t) < 4$*, for every* $t \in \mathbb{R}$*.*

Assumptions (14) is satisfied by choosing, $\sigma, \beta = \frac{1}{4}, c = \frac{1}{4}, d = \frac{1}{3}, L = \frac{1}{3}$*. So there exist an open interval* $\Lambda \subseteq]0,+\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$*, the previous problem admits three non-trivial solutions belonging to* $C([0,1])$*.*

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