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CYCLIC ORTHOGONAL DOUBLE COVERS OF 6-REGULAR CIRCULANT GRAPHS BY DISCONNECTED FORESTS

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ABSTRACT. An orthogonal double cover (ODC) of a graph H is a collection $\mathcal{G} = \{G_v : v \in V(H)\}$ of $|V(H)|$ subgraphs of H such that every edge of H is contained in exactly two members of $\mathcal{G}$ and for any two members G_u and G_v in $\mathcal{G}$, $|E(G_u) \cap E(G_v)|$ is 1 if u and v are adjacent in H and it is 0 if u and v are nonadjacent in H. An ODC $\mathcal{G}$ of H is *cyclic* if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$; otherwise it is *noncyclic.* Recently, Sampathkumar and Srinivasan settled the problem of the existence of cyclic ODCs of 4-regular circulant graphs. An ODC $\mathcal{G}$ of H is *cyclic* (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$; otherwise it is *noncyclic.* In this paper, we have completely settled the existence problem of CODCs of 6-regular circulant graphs by four acyclic disconnected graphs.

Keywords: Orthogonal double covers of graphs, Labellings of graphs, Circulant graphs
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1. Introduction

Let H be any graph and let $\mathcal{G} = \{G_1, G_2, \ldots, G_{|V(H)|}\}$ be a collection of $|V(H)|$ subgraphs of H. $\mathcal{G}$ is a *double cover* (DC) of H if every edge of H is contained in exactly two members in $\mathcal{G}$. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is a *DC* of H by G. If $\mathcal{G}$ is a DC of H by G, then $|V(H)|\,|E(G)| = 2|E(H)|$.

A DC $\mathcal{G}$ of H is an *orthogonal double cover* (ODC) of H if there exists a bijective mapping $\phi : V(H) \to \mathcal{G}$ such that for every choice of distinct vertices u and v in $V(H)$, $|E(\phi(u)) \cap E(\phi(v))|$ is 1 if $uv \in E(H)$ and is 0 otherwise. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is an *ODC* of H by G.

An *automorphism* of an ODC $\mathcal{G} = \{G_1, G_2, \ldots, G_{|V(H)|}\}$ of H is a permutation $\pi : V(H) \to V(H)$ such that $\{\pi(G_1), \pi(G_2), \ldots, \pi(G_{|V(H)|})\} = \mathcal{G}$, where for $i \in \{1, 2, \ldots, |V(H)|\}$, $\pi(G_i)$ is a subgraph of H with $V(\pi(G_i)) = \{\pi(v) : v \in V(G_i)\}$ and $E(\pi(G_i)) = \{\pi(u)\pi(v) : uv \in E(G_i)\}$. An ODC $\mathcal{G}$ of H is *cyclic* (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$; otherwise it is *noncyclic.*

For results on ODCs of graphs, see [3], a survey by Gronau et al.

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Consider the complete graph $K_n = Circ(n; \{1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor\})$. Recall that: given a graph $G = (V, E)$ with $n-1$ edges, a 1-1 mapping $\psi : V \to \mathbb{Z}_n$ is an *orthogonal labelling* of G if:

(i) for every $\ell \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, G contains exactly *two* edges of length ℓ, and exactly *one* edge of length $\frac{n}{2}$ if n is even, and

(ii) $\{r(\ell) : \ell \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}\} = \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.

Following theorem of Gronau, Mullin and Rosa [2] relates CODCs of K_n and orthogonal labellings.

Theorem 1.1. *[2] A CODC of K_n by a graph G exists if and only if there exists an orthogonal labelling of G.*

Sampathkumar and Simaringa called an orthogonal labelling as an orthogonal $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$-labelling and generalized it to an orthogonal $\{d_1, d_2, \ldots, d_k\}$-labelling, where $\{d_1, d_2, \ldots, d_k\}$ is a sequence of positive integers with $1 \le d_1 < d_2 < \ldots < d_k \le \lfloor \frac{n}{2} \rfloor$.

I. *Either* n is odd *or* n is even and $d_k \neq \frac{n}{2}$:

Given a subgraph G of $Circ(n; \{d_1, d_2, \ldots, d_k\})$ with $2k$ edges, a labelling of G, in $\mathbb{Z}_n$, is an *orthogonal* $\{d_1, d_2, \ldots, d_k\}$*-labelling* of G if:

(i) for every $\ell \in \{d_1, d_2, \ldots, d_k\}$, G contains exactly *two* edges of length ℓ, and

(ii) $\{r(\ell) : \ell \in \{d_1, d_2, \ldots, d_k\}\} = \{d_1, d_2, \ldots, d_k\}$.

II. n is even and $d_k = \frac{n}{2}$:

Given a subgraph G of $Circ(n; \{d_1, d_2, \ldots, d_{k-1}, \frac{n}{2}\})$ with $2k-1$ edges, a labelling of G, in $\mathbb{Z}_n$, is an *orthogonal* $\{d_1, d_2, \ldots, d_{k-1}, \frac{n}{2}\}$*-labelling* of G if:

(i) for every $\ell \in \{d_1, d_2, \ldots, d_{k-1}\}$, G contains exactly *two* edges of length ℓ, and G contains exactly *one* edge of length $\frac{n}{2}$, and

(ii) $\{r(\ell) : \ell \in \{d_1, d_2, \ldots, d_{k-1}\}\} = \{d_1, d_2, \ldots, d_{k-1}\}$.

Following theorem, of Sampathkumar and Simaringa [4], is a generalization of Theorem 1.1. Proof of Theorem 1.2 is similar to that of Theorem 1.1.

Theorem 1.2. *A CODC of $Circ(n; \{d_1, d_2, \ldots, d_k\})$ by a graph G exists if and only if there exists an orthogonal $\{d_1, d_2, \ldots, d_k\}$-labelling of G.*

In [5] Sampathkumar and Srinivasan have completely settled the existence problem of CODCs 4-regular circulant graphs by any graph G with 4 edges,

In [6] Sampathkumar and Srinivasan have completely settled the existence problem of CODCs 5-regular circulant graphs by any graph G with 5 edges,
In [7], Sampathkumar and Srinivasan have completely settled the existence problem of CODCs of 6-regular circulant graphs by trees. In this paper we have completely settled the existence problem of CODCs of 6-regular circulant graphs by ten acyclic disconnected graphs. Recall that, for ODCs of 6-regular circulant graphs by a graph G, G has to have six edges.

Throughout the article we make use of the usual notations:

K_n for the complete graph on n vertices,

$K_{m,n}$ for the complete bipartite graph with independent sets of sizes m and n,

$K_{n_1,n_2,\ldots,n_k}$ for the complete k-partite graph in which partite sets are of sizes $n_1, n_2, \ldots, n_k$,

P_n for the path on n vertices,

C_n for the cycle on n vertices,

ℓG for ℓ disjoint copies of G and

$G + H$ for the disjoint union $G \cup H$ of G and H.

Let $n_1, n_2, \ldots, n_r$, $r \geq 1$, be integers, n_1, $n_r \geq 1$ and $n_i \geq 0$ for $i \in \{2, 3, \ldots, r-1\}$. The *caterpillar* $C_r(n_1, n_2, \ldots, n_r)$ is the tree obtained from the path $P_r := x_1\, x_2\, \ldots\, x_r$ by joining vertex x_i to n_i new vertices, $i \in \{1, 2, \ldots, r\}$.

Other terminology not defined here can be found in [1].

2. Section 2

Let $1 \leq d_1 < d_2 < d_3 \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and G be any simple acyclic disconnected graph with six edges. Then $G \in \{6K_2,\ P_3+4K_2,\ 2P_3+2K_2,\ K_{1,3}+3K_2,\ P_4+3K_2,\ 3P_3,\ P_4+P_3+K_2,\ K_{1,3}+P_3+K_2,\ P_5+2K_2,\ K_{1,4}+2K_2,\ C_2(1,2)+2K_2,\ 2P_4,\ K_{1,3}+P_4,\ 2K_{1,3},\ P_5+P_3,\ K_{1,4}+P_3,\ C_2(1,2)+P_3,\ P_6+K_2,\ C_3(1,0,2)+K_2,\ C_3(1,1,1)+K_2,\ C_2(1,3)+K_2,\ C_2(2,2)+K_2,\ K_{1,5}+K_2\}$. In this section, we find a CODC of the circulant graph $Circ(n; \{d_1, d_2, d_3\})$ by G, where $G \in \{K_{1,4}+2K_2, K_{1,5}+K_2, K_{1,3}+P_4, 2K_{1,3}\}$. By Theorem 1.2, we have to find a 1-1 mapping $\psi : V(G) \to \mathbb{Z}_n$ such that G contains two edges of length d_1, two edges of length d_2, two edges of length d_3, and $\{r(d_1), r(d_2), r(d_3)\} = \{d_1, d_2, d_3\}$.

Theorem 2.1. *Let $n \geq 9$. A CODC of $Circ(n; \{d_1, d_2, d_3\})$ by $K_{1,4}+2K_2$ exists if and only if $(d_1, d_2, d_3) \notin \{(\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}), (\frac{n}{8}, \frac{n}{4}, \frac{3n}{8})\}$.*

Proof: First assume that $(d_1, d_2, d_3) \notin \{(\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}), (\frac{n}{8}, \frac{n}{4}, \frac{3n}{8})\}$.
Case 1. $d_2 \neq 2d_1$.

Edges of length d_1 are $\{d_3-d_1, d_3\}$ and $\{d_3-d_1+d_2, d_3+d_2\}$; ones of d_2 are $\{d_3-d_2, d_3\}$ and $\{d_3-d_2+d_1, d_3+d_1\}$; and ones of d_3 are $\{0, d_3\}$ and $\{d_3, 2d_3\}$. $r(d_1) = d_2$, $r(d_2) = d_1$, and $r(d_3) = d_3$.
Case 2. $d_2 = 2d_1$.
Subcase 2.1. $d_3 \neq 3d_1$.

Edges of length d_1 are $\{n-d_3, n-d_3+d_1\}$ and $\{0, d_1\}$; ones of $2d_1$ are $\{n-2d_1, 0\}$ and $\{0, 2d_1\}$; and ones of d_3 are $\{n-d_1, d_3-d_1\}$ and $\{0, d_3\}$. $r(d_1) = d_3$, $r(2d_1) = 2d_1$, and $r(d_3) = d_1$.
Subcase 2.2. $d_3 = 3d_1$.

For $(d_1, d_2, d_3) \neq (\frac{n}{10}, \frac{n}{5}, \frac{3n}{10})$, edges of length d_1 are $\{n-d_1, 0\}$ and $\{0, d_1\}$; ones of $2d_1$ are $\{0, 2d_1\}$ and $\{3d_1, 5d_1\}$; and ones of $3d_1$ are $\{n-5d_1, n-2d_1\}$ and $\{n-3d_1, 0\}$. $r(d_1) = d_1$, $r(2d_1) = 3d_1$, and $r(3d_1) = 2d_1$. (If either $n-d_1 = 5d_1$ or $n-5d_1 = d_1$, then $d_3 = 3d_1 = \frac{n}{2}$, a contradiction.) (If either $n-2d_1 = 5d_1$ or $n-5d_1 = 2d_1$, then $(d_1, d_2, d_3) = (d_1, 2d_1, 3d_1) = (\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7})$, a contradiction.) (If either $n-3d_1 = 5d_1$ or $n-5d_1 = 3d_1$, then $(d_1, d_2, d_3) = (d_1, 2d_1, 3d_1) = (\frac{n}{8}, \frac{n}{4}, \frac{3n}{8})$, a contradiction.) (If $n-5d_1 = 5d_1$, then $(d_1, d_2, d_3) = (d_1, 2d_1, 3d_1) = (\frac{n}{10}, \frac{n}{5}, \frac{3n}{10})$, a contradiction.)

For $(d_1, d_2, d_3) = (\frac{n}{10}, \frac{n}{5}, \frac{3n}{10})$, edges of length $\frac{n}{10}$ are $\{0, \frac{n}{10}\}$ and $\{\frac{3n}{10}, \frac{4n}{10}\}$; ones of $\frac{n}{5}$ are $\{\frac{4n}{5}, 0\}$ and $\{0, \frac{n}{5}\}$; and ones of $\frac{3n}{10}$ are $\{\frac{6n}{10}, \frac{9n}{10}\}$ and $\{\frac{7n}{10}, 0\}$. $r(\frac{n}{10}) = \frac{3n}{10}$, $r(\frac{n}{5}) = \frac{n}{5}$, and $r(\frac{3n}{10}) = \frac{n}{10}$.

Conversely, assume that $(d_1, d_2, d_3) \in \{(\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}), (\frac{n}{8}, \frac{n}{4}, \frac{3n}{8})\}$. Suppose there exists a CODC of $Circ(n; \{d_1, d_2, d_3\})$ by $K_{1,4}+2K_2$. As the edge set of $K_{1,4}+2K_2$ cannot be partitioned into subsets inducing subgraphs isomorphic to P_3, $r(d_i) = d_i$ for every $i \in \{1, 2, 3\}$ is impossible; again as the edge set of $K_{1,4}+2K_2$ cannot be partitioned into subsets inducing subgraphs isomorphic to $2K_2$, $r(d_i) \neq d_i$ for every $i \in \{1, 2, 3\}$ is impossible. Hence, $r(d_i) = d_i$, $r(d_j) = d_k$ and $r(d_k) = d_j$ for $\{i, j, k\} = \{1, 2, 3\}$. We consider two cases and three subcases in each.
Case 1. $(d_1, d_2, d_3) = (\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7})$.
Subcase 1.1. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$.

Without loss of generality assume that the edges of length $\frac{n}{7}$ are $\{\frac{6n}{7}, 0\}$ and $\{0, \frac{n}{7}\}$. The edge of length $\frac{2n}{7}$ incident at 0 is either $\{0, \frac{2n}{7}\}$ or $\{0, \frac{5n}{7}\}$. By symmetry, assume that it is $\{0, \frac{2n}{7}\}$. As $r(\frac{2n}{7}) = \frac{3n}{7}$, the edge of length $\frac{2n}{7}$ not incident at 0 is $\{\frac{3n}{7}, \frac{5n}{7}\}$. This forces the edge of length $\frac{3n}{7}$ incident at 0 is $\{0, \frac{4n}{7}\}$. As $r(\frac{3n}{7}) = \frac{2n}{7}$, there is no edge of length $\frac{3n}{7}$ not incident at 0, a contradiction.

Subcase 1.2. $r(d_1) = d_3$, $r(d_2) = d_2$ and $r(d_3) = d_1$.

Without loss of generality assume that the edges of length $\frac{2n}{7}$ are $\{\frac{5n}{7}, 0\}$ and $\{0, \frac{2n}{7}\}$. The edge of length $\frac{n}{7}$ incident at 0 is either $\{0, \frac{n}{7}\}$ or $\{0, \frac{6n}{7}\}$. By symmetry, assume that it is $\{0, \frac{n}{7}\}$. As $r(\frac{n}{7}) = \frac{3n}{7}$, the edge of length $\frac{n}{7}$ not incident at 0 is $\{\frac{3n}{7}, \frac{4n}{7}\}$. This forces that there is no edge of length $\frac{3n}{7}$ incident at 0, a contradiction.

Subcase 1.3. $r(d_1) = d_2$, $r(d_2) = d_1$ and $r(d_3) = d_3$.

Without loss of generality assume that the edges of length $\frac{3n}{7}$ are $\{\frac{4n}{7}, 0\}$ and $\{0, \frac{3n}{7}\}$. The edge of length $\frac{n}{7}$ incident at 0 is either $\{0, \frac{n}{7}\}$ or $\{0, \frac{6n}{7}\}$. By symmetry, assume that it is $\{0, \frac{n}{7}\}$. As $r(\frac{n}{7}) = \frac{2n}{7}$, the edge of length $\frac{n}{7}$ not incident at 0 is $\{\frac{5n}{7}, \frac{6n}{7}\}$. This forces the edge of length $\frac{2n}{7}$ incident at 0 is $\{0, \frac{2n}{7}\}$. As $r(\frac{2n}{7}) = \frac{n}{7}$, there is no edge of length $\frac{2n}{7}$ not incident at 0, a contradiction.

Case 2. $(d_1, d_2, d_3) = (\frac{n}{8}, \frac{n}{4}, \frac{3n}{8})$.

Subcase 2.1. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$.

Without loss of generality assume that the edges of length $\frac{n}{8}$ are $\{\frac{7n}{8}, 0\}$ and $\{0, \frac{n}{8}\}$. The edge of length $\frac{n}{4}$ incident at 0 is either $\{0, \frac{n}{4}\}$ or $\{0, \frac{3n}{4}\}$. By symmetry, assume that it is $\{0, \frac{n}{4}\}$. As $r(\frac{n}{4}) = \frac{3n}{8}$, the edge of length $\frac{n}{4}$ not incident at 0 is $\{\frac{3n}{8}, \frac{5n}{8}\}$. This forces that there is no edge of length $\frac{3n}{8}$ incident at 0, a contradiction.

Subcase 2.2. $r(d_1) = d_3$, $r(d_2) = d_2$ and $r(d_3) = d_1$.

Without loss of generality assume that the edges of length $\frac{n}{4}$ are $\{\frac{3n}{4}, 0\}$ and $\{0, \frac{n}{4}\}$. The edge of length $\frac{n}{8}$ incident at 0 is either $\{0, \frac{n}{8}\}$ or $\{0, \frac{7n}{8}\}$. By symmetry, assume that it is $\{0, \frac{n}{8}\}$. As $r(\frac{n}{8}) = \frac{3n}{8}$, the edge of length $\frac{n}{8}$ not incident at 0 is $\{\frac{3n}{8}, \frac{n}{2}\}$. This forces the edge of length $\frac{3n}{8}$ incident at 0 is $\{0, \frac{5n}{8}\}$. As $r(\frac{3n}{8}) = \frac{n}{8}$, there is no edge of length $\frac{3n}{8}$ not incident at 0, a contradiction.

Subcase 2.3. $r(d_1) = d_2$, $r(d_2) = d_1$ and $r(d_3) = d_3$.

Without loss of generality assume that the edges of length $\frac{3n}{8}$ are $\{\frac{5n}{8}, 0\}$ and $\{0, \frac{3n}{8}\}$. The edge of length $\frac{n}{8}$ incident at 0 is either $\{0, \frac{n}{8}\}$ or $\{0, \frac{7n}{8}\}$. By symmetry, assume that it is $\{0, \frac{n}{8}\}$. As $r(\frac{n}{8}) = \frac{n}{4}$, the edge of length $\frac{n}{8}$ not incident at 0 is $\{\frac{3n}{4}, \frac{7n}{8}\}$. This forces the edge of length $\frac{n}{4}$ incident at 0 is $\{0, \frac{n}{4}\}$. As $r(\frac{n}{4}) = \frac{n}{8}$, there is no edge of length $\frac{n}{8}$ not incident at 0, a contradiction.

This completes the proof.

Theorem 2.2. *Let $n \geq 8$. There is no CODC of $Circ(n; \{d_1, d_2, d_3\})$ by $K_{1,5} + K_2$.*

Proof: Suppose a CODC of $Circ(n; \{d_1, d_2, d_3\})$ by $K_{1,5} + K_2$ exists. If $r(d_i) = d_i$ for every $i \in \{1, 2, 3\}$, then the edge set of $K_{1,5} + K_2$ can be partitioned into subsets each inducing a subgraph isomorphic to P_3, which is impossible. If $r(d_i) \neq d_i$ for every $i \in \{1, 2, 3\}$, then the edge set of $K_{1,5} + K_2$ can be partitioned into subsets each inducing a subgraph isomorphic to $2K_2$, which is again impossible. Hence, $r(d_i) = d_i$, $r(d_j) = d_k$ and $r(d_k) = d_j$, where $\{i, j, k\} = \{1, 2, 3\}$. Consequently, the edge set of $K_{1,5} + K_2$ can be partitioned into three subsets one inducing a subgraph isomorphic to P_3 and the remaining two each inducing a subgraph isomorphic to $2K_2$. As this partition is also impossible, we have the required contradiction.

This completes the proof.

Theorem 2.3. *Let $n \geq 8$. A CODC of $Circ(n; \{d_1, d_2, d_3\})$ by $K_{1,3} + P_4$ exists if and only if $(d_1, d_2, d_3) \notin \{(\frac{n}{8}, \frac{2n}{8}, \frac{3n}{8}), (\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7})\}$.*

Proof: First assume that $(d_1, d_2, d_3) \notin \{(\frac{n}{8}, \frac{2n}{8}, \frac{3n}{8}), (\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7})\}$.
Case 1. $n \neq d_1 + d_2 + d_3$, $n \neq d_1 + 2d_3$, $n \neq d_2 + 2d_3$, and $n \neq d_1 + d_2 + 2d_3$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_3, d_1 + d_3\}$; ones of d_2 are $\{n - d_2, 0\}$ and $\{0, d_2\}$; and ones of d_3 are $\{d_3, 2d_3\}$ and $\{d_3 + d_1, 2d_3 + d_1\}$. $r(d_1) = d_3$, $r(d_2) = d_2$ and $r(d_3) = d_1$.
Case 2. $d_2 \neq 2d_1$, $d_3 \neq 2d_1$, $n \neq 2d_1 + d_2$, and $n \neq 2d_1 + d_3$.

Edges of length d_1 are $\{n - 2d_1, n - d_1\}$ and $\{d_2 - 2d_1, d_2 - d_1\}$; ones of d_2 are $\{n - d_1, d_2 - d_1\}$ and $\{0, d_2\}$; and ones of d_3 are $\{n - d_3, 0\}$ and $\{0, d_3\}$. $r(d_1) = d_2$, $r(d_2) = d_1$ and $r(d_3) = d_3$.

By Cases 1 and 2, we have to consider 16 possible cases.
Case 3. $n = d_1 + d_2 + d_3$ and $d_2 = 2d_1$.

Edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{n - d_3, n - d_3 + d_2\}$ and $\{0, d_2\}$; and ones of d_3 are $\{n - 2d_3, n - d_3\}$ and $\{n - 2d_3 + d_2, n - d_3 + d_2\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $n - d_1 = n - d_3 + d_2$, then $d_3 = d_1 + d_2$, and hence $d_3 = \frac{n}{2}$, a contradiction.) (If $n - 2d_3 = d_2$, then $n = d_2 + 2d_3$, a contradiction to $n = d_1 + d_2 + d_3$.) (If $n - 2d_3 = d_1$, then $n = d_1 + 2d_3$, a contradiction to $n = d_1 + d_2 + d_3$.)
Case 4. $n = d_1 + 2d_3$ and $d_2 = 2d_1$.

For $(d_1, d_2, d_3) \neq (\frac{n}{9}, \frac{2n}{9}, \frac{4n}{9})$, edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{d_2, 2d_2\}$ and $\{d_2 + d_3, 2d_2 + d_3\}$; and ones of d_3 are $\{0, d_3\}$ and $\{d_2, d_2 + d_3\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $d_3 = 2d_2$, then $(d_1, d_2, d_3) = (\frac{n}{9}, \frac{2n}{9}, \frac{4n}{9})$.) (If $n - d_1 = 2d_2$, then $n = d_1 + 2d_2$, a contradiction to $n = d_1 + 2d_3$.) (If $n - d_1 = d_2 + d_3$, then $n = d_1 + d_2 + d_3$, a contradiction to $n = d_1 + 2d_3$.) (If $n - d_1 = 2d_2 + d_3$, then $n = d_1 + 2d_2 + d_3$. As $n = d_1 + 2d_3$, $d_3 = 2d_2$, and hence $(d_1, d_2, d_3) = (\frac{n}{9}, \frac{2n}{9}, \frac{4n}{9})$.) (If $n = 2d_2 + d_3$, then as $n = d_1 + 2d_3$, $d_1 + d_3 = 2d_2$, and hence $(d_1, d_2, d_3) = (\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7})$, a contradiction.) (If $n + d_1 = 2d_2 + d_3$, then $n = -d_1 + 2d_2 + d_3$. As $n = d_1 + 2d_3$, $2d_2 = 2d_1 + d_3$, and hence $d_3 = 2d_1$, a contradiction to $d_2 = 2d_1$.)

For $(d_1, d_2, d_3) = (\frac{n}{9}, \frac{2n}{9}, \frac{4n}{9})$, edges of length $\frac{n}{9}$ are $\{\frac{6n}{9}, \frac{7n}{9}\}$ and $\{\frac{8n}{9}, 0\}$; ones of $\frac{2n}{9}$ are $\{\frac{5n}{9}, \frac{7n}{9}\}$ and $\{0, \frac{2n}{9}\}$; and ones of $\frac{4n}{9}$ are $\{0, \frac{4n}{9}\}$ and $\{\frac{n}{9}, \frac{5n}{9}\}$. $r(\frac{n}{9}) = \frac{2n}{9}$, $r(\frac{2n}{9}) = \frac{4n}{9}$ and $r(\frac{4n}{9}) = \frac{n}{9}$.
Case 5. $n = d_2 + 2d_3$ and $d_2 = 2d_1$.

For $(d_1, d_2, d_3) \neq (\frac{n}{10}, \frac{2n}{10}, \frac{4n}{10})$, edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{d_2, 2d_2\}$ and $\{d_2 + d_3, 2d_2 + d_3\}$; and ones of d_3 are $\{0, d_3\}$ and $\{d_2, d_2 + d_3\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $d_3 = 2d_2$, then $(d_1, d_2, d_3) = (\frac{n}{10}, \frac{2n}{10}, \frac{4n}{10})$.) (If $n - d_1 = 2d_2$, then $n = d_1 + 2d_2 < d_2 + 2d_3 = n$, a contradiction.) (If $n - d_1 = d_2 + d_3$, then $n = d_1 + d_2 + d_3$, a contradiction to $n = d_2 + 2d_3$.) (If $n - d_1 = 2d_2 + d_3$, then $n = d_1 + 2d_2 + d_3$. As $n = d_2 + 2d_3$, $d_3 = d_1 + d_2$, and hence $(d_1, d_2, d_3) = (\frac{n}{8}, \frac{2n}{8}, \frac{3n}{8})$, a contradiction.) (If $n = 2d_2 + d_3$, then as $n = d_2 + 2d_3$, $d_2 = d_3$, a contradiction.) (If $n + d_1 = 2d_2 + d_3$, then $n = -d_1 + 2d_2 + d_3$. As $n = d_2 + 2d_3$, $d_3 = d_2 - d_1$, a contradiction.)

For $(d_1, d_2, d_3) = (\frac{n}{10}, \frac{2n}{10}, \frac{4n}{10})$, edges of length $\frac{n}{10}$ are $\{\frac{9n}{10}, 0\}$ and $\{\frac{3n}{10}, \frac{4n}{10}\}$; ones of $\frac{2n}{10}$ are $\{0, \frac{2n}{10}\}$ and $\{\frac{n}{10}, \frac{3n}{10}\}$; and ones of $\frac{4n}{10}$ are $\{\frac{4n}{10}, \frac{8n}{10}\}$ and $\{\frac{6n}{10}, 0\}$. $r(\frac{n}{10}) = \frac{4n}{10}$, $r(\frac{2n}{10}) = \frac{n}{10}$ and $r(\frac{4n}{10}) = \frac{2n}{10}$.
Case 6. $n = d_1 + d_2 + 2d_3$ and $d_2 = 2d_1$.

For $(d_1, d_2, d_3) \neq (\frac{n}{9}, \frac{2n}{9}, \frac{3n}{9})$, edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{n - d_3, n - d_3 + d_2\}$ and $\{0, d_2\}$; and ones of d_3 are $\{n - 2d_3, n - d_3\}$ and $\{n - 2d_3 + d_2, n - d_3 + d_2\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $n - d_1 = n - d_3 + d_2$, then $d_3 = d_1 + d_2$, and hence $(d_1, d_2, d_3) = (\frac{n}{9}, \frac{2n}{9}, \frac{3n}{9})$.) (If $n - 2d_3 = d_2$, then $n = d_2 + 2d_3$, a contradiction to $n = d_1 + d_2 + 2d_3$.) (If $n - 2d_3 = d_1$, then $n = d_1 + 2d_3$, a contradiction to $n = d_1 + d_2 + 2d_3$.)

For $(d_1, d_2, d_3) = (\frac{n}{9}, \frac{2n}{9}, \frac{3n}{9})$, edges of length $\frac{n}{9}$ are $\{\frac{7n}{9}, \frac{8n}{9}\}$ and $\{0, \frac{n}{9}\}$; ones of $\frac{2n}{9}$ are $\{0, \frac{2n}{9}\}$ and $\{\frac{3n}{9}, \frac{5n}{9}\}$; and ones of $\frac{3n}{9}$ are $\{\frac{5n}{9}, \frac{8n}{9}\}$ and $\{\frac{6n}{9}, 0\}$. $r(\frac{n}{9}) = \frac{2n}{9}$, $r(\frac{2n}{9}) = \frac{3n}{9}$ and $r(\frac{3n}{9}) = \frac{n}{9}$.

Case 7. $n = d_1 + d_2 + d_3$ and $d_3 = 2d_1$.

Edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{n - d_3, n - d_3 + d_2\}$ and $\{0, d_2\}$; and ones of d_3 are $\{n - 2d_3, n - d_3\}$ and $\{n - 2d_3 + d_2, n - d_3 + d_2\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $n - d_1 = n - d_3 + d_2$, then $d_3 = d_1 + d_2$, a contradiction to $d_3 = 2d_1$.) (If $n - 2d_3 = d_2$, then $n = d_2 + 2d_3$, a contradiction to $n = d_1 + d_2 + d_3$.) (If $n - 2d_3 = d_1$, then $n = d_1 + 2d_3$, a contradiction to $n = d_1 + d_2 + d_3$.)

Case 8. $n = d_1 + 2d_3$ and $d_3 = 2d_1$.

For $(d_1, d_2, d_3) \neq (\frac{2n}{10}, \frac{3n}{10}, \frac{4n}{10})$, edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{d_2, 2d_2\}$ and $\{d_2+d_3, 2d_2+d_3\}$; and ones of d_3 are $\{0, d_3\}$ and $\{d_2, d_2+d_3\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $d_3 = 2d_2$, then we have a contradiction to $d_3 = 2d_1$.) (If $n - d_1 = 2d_2$, then $n = d_1 + 2d_2 < d_1 + 2d_3 = n$, a contradiction.) (If $n - d_1 = d_2 + d_3$, then $n = d_1 + d_2 + d_3$, a contradiction to $n = d_1 + 2d_3$.) (If $n - d_1 = 2d_2 + d_3$, then $n = d_1 + 2d_2 + d_3$. As $n = d_1 + 2d_3$, $d_3 = 2d_2$, a contradiction to $d_3 = 2d_1$.) (If $n = 2d_2 + d_3$, then as $n = d_1 + 2d_3$, $2d_2 = d_1 + d_3$, and hence $(d_1, d_2, d_3) = (\frac{2n}{10}, \frac{3n}{10}, \frac{4n}{10})$.) (If $n + d_1 = 2d_2 + d_3$, then $n = -d_1 + 2d_2 + d_3$. As $n = d_1 + 2d_3$, $d_3 = 2(d_2 - d_1)$. This together with $d_3 = 2d_1$ implies that $d_2 = 2d_1$, and hence $d_3 = d_2$, a contradiction.)

For $(d_1, d_2, d_3) = (\frac{2n}{10}, \frac{3n}{10}, \frac{4n}{10})$, edges of length $\frac{2n}{10}$ are $\{\frac{8n}{10}, 0\}$ and $\{\frac{2n}{10}, \frac{4n}{10}\}$; ones of $\frac{3n}{10}$ are $\{\frac{7n}{10}, 0\}$ and $\{\frac{9n}{10}, \frac{2n}{10}\}$; and ones of $\frac{4n}{10}$ are $\{\frac{6n}{10}, 0\}$ and $\{\frac{9n}{10}, \frac{3n}{10}\}$. $r(\frac{2n}{10}) = \frac{4n}{10}$, $r(\frac{3n}{10}) = \frac{2n}{10}$ and $r(\frac{4n}{10}) = \frac{3n}{10}$.

Case 9. $n = d_2 + 2d_3$ and $d_3 = 2d_1$.

Edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{d_2, 2d_2\}$ and $\{d_2 + d_3, 2d_2 + d_3\}$; and ones of d_3 are $\{0, d_3\}$ and $\{d_2, d_2 + d_3\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $d_3 = 2d_2$, then we have a contradiction to $d_3 = 2d_1$.) (If $n-d_1 = 2d_2$, then $n = d_1+2d_2 < d_2+2d_3 = n$, a contradiction.) (If $n-d_1 = d_2+d_3$, then $n = d_1+d_2+d_3$, a contradiction to $n = d_2 + 2d_3$.) (If $n - d_1 = 2d_2 + d_3$, then $n = d_1 + 2d_2 + d_3$. As $n = d_2 + 2d_3$, $d_3 = d_1 + d_2$, a contradiction to $d_3 = 2d_1$.) (If $n = 2d_2 + d_3$, then as $n = d_2 + 2d_3$, $d_2 = d_3$, a contradiction.) (If $n + d_1 = 2d_2 + d_3$, then $n = -d_1 + 2d_2 + d_3$. As $n = d_2 + 2d_3$, $d_3 = d_2 - d_1$, a contradiction.)

Case 10. $n = d_1 + d_2 + 2d_3$ and $d_3 = 2d_1$.

Edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of d_2 are $\{n - d_3, n - d_3 + d_2\}$ and $\{0, d_2\}$; and ones of d_3 are $\{n - 2d_3, n - d_3\}$ and $\{n - 2d_3 + d_2, n - d_3 + d_2\}$. $r(d_1) = d_1$, $r(d_2) = d_3$ and $r(d_3) = d_2$. (If $n - d_1 = n - d_3 + d_2$, then $d_3 = d_1 + d_2$, a contradiction to $d_3 = 2d_1$.) (If $n - 2d_3 = d_2$, then $n = d_2 + 2d_3$, a contradiction to $n = d_1 + d_2 + 2d_3$.) (If $n - 2d_3 = d_1$, then $n = d_1 + 2d_3$, a contradiction to $n = d_1 + d_2 + 2d_3$.)

Case 11. $n = d_1 + d_2 + d_3$ and $n = 2d_1 + d_2$.

Then $d_3 = d_1$, a contradiction.

Case 12. $n = d_1 + 2d_3$ and $n = 2d_1 + d_2$.

Then $n = d_1 + 2d_3 > 2d_1 + d_2 = n$, a contradiction.

Case 13. $n = d_2 + 2d_3$ and $n = 2d_1 + d_2$.

Then $d_3 = d_1$, a contradiction.

Case 14. $n = d_1 + d_2 + 2d_3$ and $n = 2d_1 + d_2$.

Then $n = d_1 + d_2 + 2d_3 > 2d_1 + d_2 = n$, a contradiction.

Case 15. $n = d_1 + d_2 + d_3$ and $n = 2d_1 + d_3$.

Then $d_2 = d_1$, a contradiction.

Case 16. $n = d_1 + 2d_3$ and $n = 2d_1 + d_3$.

Then $d_3 = d_1$, a contradiction.

Case 17. $n = d_2 + 2d_3$ and $n = 2d_1 + d_3$.

Then $n = d_2 + 2d_3 > 2d_1 + d_3 = n$, a contradiction.

Case 18. $n = d_1 + d_2 + 2d_3$ and $n = 2d_1 + d_3$.

Then $n = d_1 + d_2 + 2d_3 > 2d_1 + d_3 = n$, a contradiction.

Conversely, assume that $(d_1, d_2, d_3) \in \{(\frac{n}{8}, \frac{2n}{8}, \frac{3n}{8}), (\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7})\}$. Suppose there exists a CODC of $Circ(n; \{d_1, d_2, d_3\})$ by $K_{1,3} + P_4$. We consider two cases.

Case 1. $(d_1, d_2, d_3) = (\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7})$.

Observe that: $Circ(n; \{\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}\}) \cong \frac{n}{7}K_7$; for each $i, j \in \{1, 2, 3\}$, any two edges of length $\frac{in}{7}$ with rotation-distance $\frac{jn}{7}$ are in the same component of $\frac{n}{7}K_7$; and in the CODC there exists $i \in \{1, 2, 3\}$, an edge e' of $K_{1,3}$, and an edge e'' of P_4 such that the edges e' and e'' are of same length d_i. Consequently, in the CODC, all the edges of $K_{1,3} + P_4$ are in the same component of $\frac{n}{7}K_7$. Hence, we have a component of $\frac{n}{7}K_7$ with at least 8 vertices, a contradiction.

Case 2. $(d_1, d_2, d_3) = (\frac{n}{8}, \frac{2n}{8}, \frac{3n}{8})$.

Observe that: $Circ(n; \{\frac{n}{8}, \frac{2n}{8}, \frac{3n}{8}\}) \cong \frac{n}{8}Circ(8; \{1, 2, 3\})$; for each $i, j \in \{1, 2, 3\}$, any two edges of length $\frac{in}{8}$ with rotation-distance $\frac{jn}{8}$ are in the same component of $\frac{n}{8}Circ(8; \{1, 2, 3\})$; and in the CODC there exists $i \in \{1, 2, 3\}$, an edge e' of $K_{1,3}$, and an edge e'' of P_4 such that the edges e' and e'' are of same length d_i. Consequently, in the CODC, all the edges of $K_{1,3} + P_4$ are in the same component of $\frac{n}{8}Circ(8; \{1, 2, 3\})$. Hence, the CODC of $\frac{n}{8}Circ(8; \{1, 2, 3\})$ by $K_{1,3} + P_4$ yields a CODC of $Circ(8; \{1, 2, 3\})$ by $K_{1,3} + P_4$.

Now consider a CODC of $Circ(8; \{1, 2, 3\})$ by $K_{1,3} + P_4$. If $r(1) = 1$, $r(2) = 2$ and $r(3) = 3$, then the edge set of $K_{1,3}+P_4$ can be partitioned into subsets inducing subgraphs isomorphic to P_3, which is impossible. Thus we consider five subcases.

Subcase 2.1. $r(1) = 1$, $r(2) = 3$ and $r(3) = 2$.

Let $A = \{\{7, 0\}, \{0, 1\}\}$. Without loss of generality assume that the edges of length 1 are the edges of A. Let $B_0 = \{\{0, 2\}, \{3, 5\}\}$, $B_1 = \{\{1, 3\}, \{4, 6\}\}$, $B_2 = \{\{2, 4\}, \{5, 7\}\}$ and $B_3 = \{\{3, 5\}, \{6, 0\}\}$. The edges of length 2 are the edges of one of the sets B_0, B_1, B_2, B_3. Let $C_0 = \{\{0, 3\}, \{2, 5\}\}$, $C_1 = \{\{1, 4\}, \{3, 6\}\}$, $C_2 = \{\{2, 5\}, \{4, 7\}\}$ and $C_3 = \{\{3, 6\}, \{5, 0\}\}$. The edges of length 3 are the edges of one of the sets C_0, C_1, C_2, C_3. Observe that, for any $i, j \in \{0, 1, 2, 3\}$, the subgraph induced by the edge set $A \cup B_i \cup C_j$ is not isomorphic to $K_{1,3} + P_4$, a contradiction.

Subcase 2.2. $r(1) = 2$, $r(2) = 1$ and $r(3) = 3$.

Let $A = \{\{0, 1\}, \{2, 3\}\}$. Without loss of generality assume that the edges of length 1 are the edges of A. Let $B_2 = \{\{2, 4\}, \{3, 5\}\}$, $B_3 = \{\{3, 5\}, \{4, 6\}\}$, $B_5 = \{\{5, 7\}, \{6, 0\}\}$ and $B_6 = \{\{6, 0\}, \{7, 1\}\}$. The edges of length 2 are the edges of one of the sets B_2, B_3, B_5, B_6. Let $C_1 = \{\{1, 4\}, \{4, 7\}\}$, $C_4 = \{\{4, 7\}, \{7, 2\}\}$, $C_6 = \{\{6, 1\}, \{1, 4\}\}$ and $C_7 = \{\{7, 2\}, \{2, 5\}\}$. The edges of length 3 are the edges of one of the sets C_1, C_4, C_6, C_7. Observe that, for any $i \in \{2, 3, 5, 6\}$ and for any $j \in \{1, 4, 6, 7\}$, the subgraph induced by the edge set $A \cup B_i \cup C_j$ is not isomorphic to $K_{1,3} + P_4$, a contradiction.

Subcase 2.3. $r(1) = 2$, $r(2) = 3$ and $r(3) = 1$.

Let $A = \{\{0, 1\}, \{2, 3\}\}$. Without loss of generality assume that the edges of length 1 are the edges of A. Let $B_1 = \{\{1, 3\}, \{4, 6\}\}$, $B_2 = \{\{2, 4\}, \{5, 7\}\}$, $B_3 = \{\{3, 5\}, \{6, 0\}\}$, $B_4 = \{\{4, 6\}, \{7, 1\}\}$, $B_5 = \{\{5, 7\}, \{0, 2\}\}$ and $B_7 = \{\{7, 1\}, \{2, 4\}\}$. The edges of length 2 are the edges of one of the sets B_1, B_2, B_3, B_4, B_5, B_7. Let $C_1 = \{\{1, 4\}, \{2, 5\}\}$, $C_2 = \{\{2, 5\}, \{3, 6\}\}$, $C_3 = \{\{3, 6\}, \{4, 7\}\}$, $C_4 = \{\{4, 7\}, \{5, 0\}\}$, $C_5 = \{\{5, 0\}, \{6, 1\}\}$

and $C_6 = \{\{6,1\},\{7,2\}\}$. The edges of length 3 are the edges of one of the sets C_1, C_2, C_3, C_4, C_5, C_6. Observe that, for any $i \in \{1,2,3,4,5,7\}$ and for any $j \in \{1,2,3,4,5,6\}$, the subgraph induced by the edge set $A \cup B_i \cup C_j$ is not isomorphic to $K_{1,3} + P_4$, a contradiction.

Subcase 2.4. $r(1) = 3$, $r(2) = 1$ and $r(3) = 2$.

Let $A = \{\{0,1\},\{3,4\}\}$. Without loss of generality assume that the edges of length 1 are the edges of A. Let $B_2 = \{\{2,4\},\{3,5\}\}$, $B_3 = \{\{3,5\},\{4,6\}\}$, $B_4 = \{\{4,6\},\{5,7\}\}$, $B_5 = \{\{5,7\},\{6,0\}\}$, $B_6 = \{\{6,0\},\{7,1\}\}$ and $B_7 = \{\{7,1\},\{0,2\}\}$. The edges of length 2 are the edges of one of the sets B_2, B_3, B_4, B_5, B_6, B_7. Let $C_0 = \{\{0,3\},\{2,5\}\}$, $C_2 = \{\{2,5\},\{4,7\}\}$, $C_3 = \{\{3,6\},\{5,0\}\}$, $C_4 = \{\{4,7\},\{6,1\}\}$, $C_5 = \{\{5,0\},\{7,2\}\}$ and $C_7 = \{\{7,2\},\{1,4\}\}$. The edges of length 3 are the edges of one of the sets C_0, C_2, C_3, C_4, C_5, C_7. Observe that, for any $i \in \{2,3,4,5,6,7\}$ and for any $j \in \{0,2,3,4,5,7\}$, the subgraph induced by the edge set $A \cup B_i \cup C_j$ is not isomorphic to $K_{1,3} + P_4$, a contradiction.

Subcase 2.5. $r(1) = 3$, $r(2) = 2$ and $r(3) = 1$.

Let $A = \{\{0,1\},\{3,4\}\}$. Without loss of generality assume that the edges of length 1 are the edges of A. Let $B_2 = \{\{2,4\},\{4,6\}\}$, $B_3 = \{\{3,5\},\{5,7\}\}$, $B_5 = \{\{5,7\},\{7,1\}\}$ and $B_6 = \{\{6,0\},\{0,2\}\}$. The edges of length 2 are the edges of one of the sets B_2, B_3, B_5, B_6. Let $C_1 = \{\{1,4\},\{2,5\}\}$, $C_2 = \{\{2,5\},\{3,6\}\}$, $C_3 = \{\{3,6\},\{4,7\}\}$, $C_4 = \{\{4,7\},\{5,0\}\}$, $C_5 = \{\{5,0\},\{6,1\}\}$ $C_6 = \{\{6,1\},\{7,2\}\}$ and $C_7 = \{\{7,2\},\{0,3\}\}$. The edges of length 3 are the edges of one of the sets C_1, C_2, C_3, C_4, C_5, C_6, C_7. Observe that, for any $i \in \{2,3,5,6\}$ and for any $j \in \{1,2,3,4,5,6,7\}$, the subgraph induced by the edge set $A \cup B_i \cup C_j$ is not isomorphic to $K_{1,3} + P_4$, a contradiction.

This completes the proof.

Theorem 2.4. *Let $n \geq 8$. A CODC of $Circ(n;\{d_1,d_2,d_3\})$ by $2K_{1,3}$ exists if and only if $(n,d_3) = (3d_1+3d_2, 2d_1+d_2)$..*

Proof: Suppose a CODC of $Circ(n;\{d_1,d_2,d_3\})$ by $2K_{1,3}$ exists. If $r(d_i) = d_i$ for every $i \in \{1,2,3\}$, then the edge set of $2K_{1,3}$ can be partitioned into subsets inducing subgraphs isomorphic to P_3, which is impossible. If $r(d_i) = d_i$, $r(d_j) = d_k$ and $r(d_k) = d_j$ with $\{i,j,k\} = \{1,2,3\}$, then the edge set of $2K_{1,3}$ can be partitioned into three subsets one inducing a subgraph isomorphic to P_3 and the remaining two each inducing a subgraph isomorphic to $2K_2$, which is again impossible. Hence, $r(d_i) \neq d_i$ for every $i \in \{1,2,3\}$. Consequently, the edge set of $2K_{1,3}$ can be partitioned into subsets inducing subgraphs isomorphic to $2K_2$. Also, each vertex of degree 3 in $2K_{1,3}$ is incident with one edge of length d_1, one edge of length d_2, and one edge of length d_3. Without loss of generality, assume that one vertex of degree 3 in $2K_{1,3}$ is 0 and the edge of length d_1 incident at 0 is $\{0,d_1\}$. The edge of length d_2 incident at 0 is either $\{0,d_2\}$ or $\{0,n-d_2\}$ and the edge of length d_3 incident at 0 is either $\{0,d_3\}$ or $\{0,n-d_3\}$. We consider four cases and in each of the four cases we consider two subcases.

Case 1. Edges incident at 0 are $\{0,d_1\}$, $\{0,d_2\}$ and $\{0,d_3\}$.

Subcase 1.1. $r(d_1) = d_2$, $r(d_2) = d_3$ and $r(d_3) = d_1$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0,d_1,d_2,d_3\}$ are, respectively, $\{n-d_2, n+d_1-d_2\}$, $\{n-d_3, n+d_2-d_3\}$ and $\{n-d_1, d_3-d_1\}$.

First, consider the two adjacent edges $\{n-d_2, n+d_1-d_2\}$ and $\{n-d_1, d_3-d_1\}$. As $n-d_2 \neq n-d_1$, $n-d_2 \neq d_3-d_1$ and $n+d_1-d_2 \neq d_3-d_1$, we have $n+d_1-d_2 = n-d_1$, and hence $d_2 = 2d_1$.

Next, consider the two adjacent edges $\{n-d_3, n+d_2-d_3\}$ and $\{n-d_1, d_3-d_1\}$. As $n-d_3 \neq n-d_1$, $n-d_3 \neq d_3-d_1$ and $n+d_2-d_3 \neq d_3-d_1$, we have $n+d_2-d_3 = n-d_1$, and hence $d_3 = d_1 + d_2$, i.e., $d_3 = 3d_1$.

Now the edges incident at 0 are $\{0, d_1\}$, $\{0, 2d_1\}$ and $\{0, 3d_1\}$; and the edges not incident at 0 are $\{n-2d_1, n-d_1\}$, $\{n-3d_1, n-d_1\}$ and $\{n-d_1, 2d_1\}$; a contradiction to the fact that $2d_1$ belongs to both $K_{1,3}$'s.

Subcase 1.2. $r(d_1) = d_3$, $r(d_2) = d_1$ and $r(d_3) = d_2$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0, d_1, d_2, d_3\}$ are, respectively, $\{n-d_3, n+d_1-d_3\}$, $\{n-d_1, d_2-d_1\}$ and $\{n-d_2, d_3-d_2\}$.

First, consider the two adjacent edges $\{n-d_1, d_2-d_1\}$ and $\{n-d_3, n+d_1-d_3\}$. As $n-d_1 \neq n-d_3$, $d_2-d_1 \neq n-d_3$ and $d_2-d_1 \neq n+d_1-d_3$, we have $n-d_1 = n+d_1-d_3$.

Next, consider the two adjacent edges $\{n-d_2, d_3-d_2\}$ and $\{n-d_3, n+d_1-d_3\}$. As $n-d_2 \neq n-d_3$, $d_3-d_2 \neq n-d_3$ and $d_3-d_2 \neq n+d_1-d_3$, we have $n-d_2 = n+d_1-d_3$. Consequently, $n-d_1 = n-d_2$, i.e., $d_1 = d_2$, a contradiction.

Case 2. Edges incident at 0 are $\{0, d_1\}$, $\{0, d_2\}$ and $\{0, n-d_3\}$.

Subcase 2.1. $r(d_1) = d_2$, $r(d_2) = d_3$ and $r(d_3) = d_1$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0, d_1, d_2, n-d_3\}$ are, respectively, $\{n-d_2, n+d_1-d_2\}$, $\{d_3, d_3+d_2\}$ and $\{n-d_3-d_1, n-d_1\}$.

First, consider the two adjacent edges $\{n-d_2, n+d_1-d_2\}$ and $\{n-d_3-d_1, n-d_1\}$. As $n-d_2 \neq n-d_1$, $n-d_2 \neq n-d_3-d_1$ and $n-d_3-d_1 \neq n+d_1-d_2$ (i.e., $d_2 \neq d_3+2d_1$), we have $n-d_1 = n+d_1-d_2$, and hence, $d_2 = 2d_1$.

Next, consider the two adjacent edges $\{n-d_2, n+d_1-d_2\}$ and $\{d_3, d_3+d_2\}$. As $n-d_2 \neq d_3$ and $n+d_1-d_2 \neq d_3$, we have either $n-d_2 = d_3+d_2$ or $n+d_1-d_2 = d_3+d_2$. Hence, we have either $n = 2d_2+d_3$ or $n = -d_1+2d_2+d_3$. As $d_2 = 2d_1$, we have either $n = 4d_1+d_3$ or $n = 3d_1+d_3$. As $d_2 = 2d_1$ and $n-d_3-d_1$ are vertices of disjoint $K_{1,3}$'s, they are not equal, and hence $n \neq 3d_1+d_3$. Thus, $n = 4d_1+d_3$.

Finally, consider the two adjacent edges $\{d_3, d_3+d_2\}$ and $\{n-d_3-d_1, n-d_1\}$. As $n-d_1 \neq d_3$, we have one of the following: $n-d_1 = d_3+d_2$, $n-d_3-d_1 = d_3$ and $n-d_3-d_1 = d_3+d_2$. As $d_2 = 2d_1$, we have one of the following: $n = 3d_1+d_3$, $n = d_1+2d_3$ and $n = 3d_1+2d_3$. As $d_2 = 2d_1$ and $n-d_3-d_1$ are vertices of disjoint $K_{1,3}$'s, they are not equal, and hence $n \neq 3d_1+d_3$. Thus, we have either $n = d_1+2d_3$ or $n = 3d_1+2d_3$. If $n = 3d_1+2d_3$, then as $n = 4d_1+d_3$, we have $d_1 = d_3$, a contradiction. Thus, $n = d_1+2d_3$.

$n = 4d_1+d_3$ and $n = d_1+2d_3$ implies that $d_3 = 3d_1$ and $n = 7d_1$. Now $n-d_3 = 4d_1$ and $n+d_1-d_2 = 4d_1$ are vertices of disjoint $K_{1,3}$'s, a contradiction.

Subcase 2.2. $r(d_1) = d_3$, $r(d_2) = d_1$ and $r(d_3) = d_2$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0, d_1, d_2, n-d_3\}$ are, respectively, $\{d_3, d_1+d_3\}$, $\{n-d_1, d_2-d_1\}$ and $\{n-d_2-d_3, n-d_2\}$.

First, consider the two adjacent edges $\{d_3, d_1+d_3\}$ and $\{n-d_1, d_2-d_1\}$. As $d_3 \neq n-d_1$, $d_3 \neq d_2-d_1$ and $d_1+d_3 \neq d_2-d_1$, we have $d_3+d_1 = n-d_1$, and hence $n = 2d_1+d_3$.

Next, consider the two adjacent edges $\{d_3, d_1+d_3\}$ and $\{n-d_2-d_3, n-d_2\}$. As $d_3 \neq n-d_2$, we have one of the following: $d_3 = n-d_2-d_3$, $d_1+d_3 = n-d_2-d_3$, and $d_1+d_3 = n-d_2$; i.e., we have one of the following: $n = d_2+2d_3$, $n = d_1+d_2+2d_3$, and $n = d_1+d_2+d_3$.

If $n = d_2+2d_3$, then as $n = 2d_1+d_3$, we have $n = 2d_1+d_3 < d_2+2d_3 = n$, a contradiction. If $n = d_1+d_2+2d_3$, then again as $n = 2d_1+d_3$, we have $n = 2d_1+d_3 < d_1+d_2+2d_3 = n$, a contradiction. If $n = d_1+d_2+d_3$, then once again as $n = 2d_1+d_3$, we have $d_1 = d_2$, a contradiction.

Case 3. Edges incident at 0 are $\{0, d_1\}$, $\{0, n-d_2\}$ and $\{0, d_3\}$.

Subcase 3.1. $r(d_1) = d_2$, $r(d_2) = d_3$ and $r(d_3) = d_1$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0, d_1, n-d_2, d_3\}$

are, respectively, $\{d_2, d_1 + d_2\}$, $\{n - d_3 - d_2, n - d_3\}$ and $\{n - d_1, d_3 - d_1\}$.

First, consider the two adjacent edges $\{n - d_3 - d_2, n - d_3\}$ and $\{n - d_1, d_3 - d_1\}$. As $n-d_3 \neq n-d_1$, $n-d_3 \neq d_3-d_1$ and $n-d_3-d_2 \neq n-d_1$, we have $n-d_3-d_2 = d_3-d_1$, and hence $n = d_2 + 2d_3 - d_1$.

Next, consider the two adjacent edges $\{d_2, d_1 + d_2\}$ and $\{n - d_3 - d_2, n - d_3\}$. As $d_2 \neq n - d_3$, we have one of the following: $d_2 = n - d_3 - d_2$, $d_1 + d_2 = n - d_3 - d_2$, and $d_1 + d_2 = n - d_3$; i.e., we have one of the following: $n = 2d_2 + d_3$, $n = d_1 + 2d_2 + d_3$, and $n = d_1 + d_2 + d_3$.

If $n = 2d_2 + d_3$, then as $n = d_2 + 2d_3 - d_1$, we have $d_3 = d_1 + d_2$, a contradiction to the fact that d_3 and $d_1 + d_2$ are vertices of disjoint $K_{1,3}$'s. Thus, we have either $n = d_1 + 2d_2 + d_3$ or $n = d_1 + d_2 + d_3$.

Finally, consider the two adjacent edges $\{d_2, d_1+d_2\}$ and $\{n-d_1, d_3-d_1\}$. As $d_2 \neq n-d_1$, we have one of the following: $d_2 = d_3 - d_1$, $d_1 + d_2 = n - d_1$, and $d_1 + d_2 = d_3 - d_1$; i.e., we have one of the following: $d_3 = d_1 + d_2$, $n = 2d_1 + d_2$, and $d_3 = 2d_1 + d_2$.

As d_3 and $d_1 + d_2$ are vertices of disjoint $K_{1,3}$'s, $d_3 \neq d_1 + d_2$. Thus, we have either $n = 2d_1 + d_2$ or $d_3 = 2d_1 + d_2$.

If $n = d_1 + 2d_2 + d_3$ and $n = 2d_1 + d_2$, then $n = 2d_1 + d_2 < d_1 + 2d_2 + d_3 = n$, a contradiction.

If $n = d_1 + d_2 + d_3$ and $n = 2d_1 + d_2$, then $n = 2d_1 + d_2 < d_1 + d_2 + d_3 = n$, a contradiction.

If $n = d_1 + d_2 + d_3$ and $d_3 = 2d_1 + d_2$, then as $n = d_2 + 2d_3 - d_1$, we have $d_3 = 2d_1$, a contradiction to $d_3 = 2d_1 + d_2$.

Hence, $n = d_1 + 2d_2 + d_3$ and $d_3 = 2d_1 + d_2$. Consequently, $n = 3d_1 + 3d_2$ and $d_3 = 2d_1 + d_2$.

Subcase 3.2. $r(d_1) = d_3$, $r(d_2) = d_1$ and $r(d_3) = d_2$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0, d_1, n - d_2, d_3\}$ are, respectively, $\{n - d_3, n - d_3 + d_1\}$, $\{n - d_1 - d_2, n - d_1\}$ and $\{d_2, d_2 + d_3\}$.

First, consider the two adjacent edges $\{n - d_3, n - d_3 + d_1\}$ and $\{d_2, d_2 + d_3\}$. As $n-d_3 \neq d_2$ and $n-d_3+d_1 \neq d_2$, we have either $n-d_3 = d_2+d_3$ or $n-d_3+d_1 = d_2+d_3$. Hence, we have either $n = d_2 + 2d_3$ or $n = d_2 + 2d_3 - d_1$.

Next, consider the two adjacent edges $\{n-d_3, n-d_3+d_1\}$ and $\{n-d_1-d_2, n-d_1\}$. As $n-d_3 \neq n-d_1$, we have one of the following: $n-d_3 = n-d_1-d_2$, $n-d_3+d_1 = n-d_1-d_2$, and $n-d_3+d_1 = n-d_1$. Hence, we have one of the following: $d_3 = d_1+d_2$, $d_3 = 2d_1+d_2$, and $d_3 = 2d_1$. If $d_3 = d_1 + d_2$, then $n - d_2$ and $n - d_3 + d_1 = n - d_2$ are vertices of disjoint $K_{1,3}$'s, a contradiction. Thus, we have either $d_3 = 2d_1 + d_2$ or $d_3 = 2d_1$.

Finally, consider the two adjacent edges $\{n - d_1 - d_2, n - d_1\}$ and $\{d_2, d_2 + d_3\}$. As $n - d_1 \neq d_2$, we have one of the following: $n - d_1 = d_2 + d_3$, $n - d_1 - d_2 = d_2$, and $n-d_1-d_2 = d_2+d_3$. Hence, we have one of the following: $n = d_1+d_2+d_3$, $n = d_1+2d_2$ and $n = d_1 + 2d_2 + d_3$. If $n = d_1 + d_2 + d_3$, then d_3 and $n - d_1 - d_2 = d_3$ are vertices of disjoint $K_{1,3}$'s, a contradiction. Thus, we have either $n = d_1 + 2d_2$ or $n = d_1 + 2d_2 + d_3$.

We consider all the eight possibilities.

If $n = d_2 + 2d_3$, either $d_3 = 2d_1 + d_2$ or $d_3 = 2d_1$, and $n = d_1 + 2d_2$, then $n = d_1 + 2d_2 < d_2 + 2d_3 = n$, a contradiction.

If $n = d_2 + 2d_3$, either $d_3 = 2d_1 + d_2$ or $d_3 = 2d_1$, and $n = d_1 + 2d_2 + d_3$, then $d_3 = d_1 + d_2$, a contradiction to either $d_3 = 2d_1 + d_2$ or $d_3 = 2d_1$.

If $n = d_2 + 2d_3 - d_1$, $d_3 = 2d_1 + d_2$, and $n = d_1 + 2d_2$, then $n > 2d_3 = 4d_1 + 2d_2 > d_1 + 2d_2 = n$, a contradiction.

If $n = d_2 + 2d_3 - d_1$, $d_3 = 2d_1$, and $n = d_1 + 2d_2$, then $d_3 = d_2$, a contradiction.

If $n = d_2 + 2d_3 - d_1$, $d_3 = 2d_1$, and $n = d_1 + 2d_2 + d_3$, then $d_3 = 2d_1 + d_2$, a contradiction to $d_3 = 2d_1$.

Hence, $n = d_2 + 2d_3 - d_1$, $d_3 = 2d_1 + d_2$, and $n = d_1 + 2d_2 + d_3$. Consequently, $n = 3d_1 + 3d_2$ and $d_3 = 2d_1 + d_2$.

Case 4. Edges incident at 0 are $\{0, d_1\}$, $\{0, n - d_2\}$ and $\{0, n - d_3\}$.

Subcase 4.1. $r(d_1) = d_2$, $r(d_2) = d_3$ and $r(d_3) = d_1$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0, d_1, n-d_2, n-d_3\}$ are, respectively, $\{d_2, d_1 + d_2\}$, $\{d_3 - d_2, d_3\}$ and $\{n - d_1 - d_3, n - d_1\}$.

First, consider the two adjacent edges $\{d_2, d_1 + d_2\}$ and $\{d_3 - d_2, d_3\}$. As $d_2 \neq d_3$, we have one of the following: $d_2 = d_3 - d_2$, $d_1 + d_2 = d_3 - d_2$, and $d_1 + d_2 = d_3$. Hence, we have one of the following: $d_3 = 2d_2$, $d_3 = d_1 + 2d_2$, and $d_3 = d_1 + d_2$. If $d_3 = d_1 + d_2$, then d_1 and $d_3 - d_2 = d_1$ are vertices of disjoint $K_{1,3}$'s, a contradiction. Thus, we have either $d_3 = 2d_2$ or $d_3 = d_1 + 2d_2$.

Next, consider the two adjacent edges $\{d_2, d_1 + d_2\}$ and $\{n - d_1 - d_3, n - d_1\}$. As $d_2 \neq n - d_1$, we have one of the following: $d_2 = n - d_1 - d_3$, $d_1 + d_2 = n - d_1 - d_3$, and $d_1 + d_2 = n - d_1$. Hence, we have one of the following: $n = d_1 + d_2 + d_3$, $n = 2d_1 + d_2 + d_3$, and $n = 2d_1 + d_2$. If $n = d_1 + d_2 + d_3$, then $n - d_3 = d_1 + d_2$ and $d_1 + d_2$ are vertices of disjoint $K_{1,3}$'s, a contradiction. Thus, we have either $n = 2d_1 + d_2 + d_3$ or $n = 2d_1 + d_2$.

Finally, consider the two adjacent edges $\{d_3 - d_2, d_3\}$ and $\{n - d_1 - d_3, n - d_1\}$. As $d_3 - d_2 \neq n - d_1$ and $d_3 \neq n - d_1$, we have either $d_3 - d_2 = n - d_1 - d_3$ or $d_3 = n - d_1 - d_3$. Hence, we have either $n = d_1 - d_2 + 2d_3$ or $n = d_1 + 2d_3$.

We consider all the eight possibilities.

If $d_3 = 2d_2$, $n = 2d_1 + d_2 + d_3$, and $n = d_1 - d_2 + 2d_3$, then $d_3 = d_1 + 2d_2$, a contradiction to $d_3 = 2d_2$.

If $d_3 = 2d_2$, $n = 2d_1 + d_2 + d_3$, and $n = d_1 + 2d_3$, then $d_3 = d_1 + d_2$, a contradiction to $d_3 = 2d_2$.

If $d_3 = 2d_2$, $n = 2d_1 + d_2$, and $n = d_1 - d_2 + 2d_3$, then $2d_3 = d_1 + 2d_2$, and this together with $d_3 = 2d_2$ implies that $d_3 = d_1$, a contradiction.

If $n = 2d_1 + d_2$, $n = d_1 + 2d_3$, and either $d_3 = 2d_2$ or $d_3 = d_1 + 2d_2$, then $n = 2d_1 + d_2 < d_1 + 2d_3 = n$, a contradition.

If $d_3 = d_1 + 2d_2$, $n = 2d_1 + d_2 + d_3$, and $n = d_1 + 2d_3$, then $d_3 = d_1 + d_2$, a contradiction to $d_3 = d_1 + 2d_2$.

If $d_3 = d_1 + 2d_2$, $n = 2d_1 + d_2$, and $n = d_1 - d_2 + 2d_3$, then $2d_3 = d_1 + 2d_2$, a contradiction to $d_3 = d_1 + 2d_2$.

If $d_3 = d_1 + 2d_2$, $n = 2d_1 + d_2 + d_3$, and $n = d_1 - d_2 + 2d_3$, then $d_3 = d_1 + 2d_2$ and $n = 3d_1 + 3d_2$. As $d_3 < \frac{n}{2}$, $2d_3 < n$, and hence $2(d_1 + 2d_2) < 3d_1 + 3d_2$.
This implies that $d_2 < d_1$, a contradiction.

Subcase 4.2. $r(d_1) = d_3$, $r(d_2) = d_1$ and $r(d_3) = d_2$.

Then the edges of lengths d_1, d_2 and d_3 not incident at vertices in $\{0, d_1, n-d_2, n-d_3\}$ are, respectively, $\{d_3, d_1 + d_3\}$, $\{n - d_1 - d_2, n - d_1\}$ and $\{n + d_2 - d_3, d_2\}$.

Consider the two adjacent edges $\{d_3, d_1 + d_3\}$ and $\{n + d_2 - d_3, d_2\}$. As $d_3 \neq n + d_2 - d_3$, $d_3 \neq d_2$, and $d_1 + d_3 \neq d_2$, we have $d_1 + d_3 = n + d_2 - d_3$, and hence $n = d_1 - d_2 + 2d_3$, a contradiction to $d_1 < d_2$ and $2d_3 < n$.

Conversely, assume that $(n, d_3) = (3d_1 + 3d_2, 2d_1 + d_2)$. Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, d_1 + d_2\}$; ones of d_2 are $\{d_1 + d_2, d_1 + 2d_2\}$ and $\{3d_1 + 2d_2, 0\}$; and ones of $d_3 = 2d_1 + d_2$ are $\{2d_1 + 3d_2, d_1 + d_2\}$ and $\{0, 2d_1 + d_2\}$. $r(d_1) = d_2$, $r(d_2) = 2d_1 + d_2$ and $r(2d_1 + d_2) = d_1$.

This completes the proof.

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