TWMS J. App. and Eng. Math. V.11, Special Issue, 2021, pp. 88-95

TRIPLE CONNECTED ETERNAL DOMINATION IN GRAPHS

G. MAHADEVAN[1], T. PONNUCHAMY[2], SELVAM AVADAYAPPAN[3], §

Abstract. The concept of Triple connected domination number was introduced by G. Mahadevan et. al., in [10]. The concept of eternal domination in graphs was introduced by W. Goddard., et. al., in [3]. The dominating set $S_0(\subseteq V(G))$ of the graph G is said to be an eternal dominating set, if for any sequence $v_1, v_2, v_3, \ldots v_k$ of vertices, there exists a sequence of vertices $u_1, u_2, u_3, \ldots u_k$ with $u_i \in S_{i-1}$ and u_i equal to or adjacent to v_i, such that each set $S_i = S_{i-1} - \{u_i\} \cup \{v_i\}$ is dominating set in G. The minimum cardinality taken over the eternal dominating sets in G is called the eternal domination number of G and it is denoted by $\gamma_\infty(G)$. In this paper we introduce another new concept Triple connected eternal domination in graph. The eternal dominating set $S_0(\subseteq V(G))$ of the graph G is said to be a triple connected eternal dominating set, if each dominating set S_i is triple connected. The minimum cardinality taken over the triple connected eternal dominating sets in G is called the triple connected eternal domination number of G and it is denoted by $\gamma_{tc,\infty}(G)$. We investigate this number for some standard graphs and obtain many results with other graph theoretical parameters.

Keywords: Triple connected domination number, Eternal domination in graphs, Triple connected eternal domination number of graphs.

AMS Subject Classification: 05C69

1. Introduction

By a graph we mean a finite, simple, connected and undirected graph $G(V, E)$, where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G has p vertices and q edges. We denote a cycle on m vertices by C_m, a path on m vertices by P_m, a complete graph on m vertices by K_m and a complete bipartite graph on m, n vertices by $K_{m,n}$. We denote a prism graph on n vertices by Y_n, $n \geq 3$ is defined by Cartesian product of a cycle with a single edge. The ladder graph can be obtained as the Cartesian Product of two paths, one of which has only one edge, denoted by L_n, $n \geq 1$. In [9], J. Paulraj

[1] Department of Mathematics, Gandhigram Rural Institute Deemed to be University, Gandhigram, Dindigul.
e-mail: drgmaha2014@gmail.com; ORCID: https://orcid.org/0000-0003-2438-1576.
[2] Full Time Research Scholar, Department of Mathematics, Gandhigram Rural Institute Deemed to be University, Gandhigram, Dindigul.
e-mail: tponnuchamy@gmail.com; ORCID: https://orcid.org/0000-0002-2537-4733.
[3] Department of Mathematics, VHNSN College, Virudhunagar.
e-mail: selvam_avadayappan@yahoo.co.in; ORCID: https://orcid.org/0000-0002-2498-6762.
The research work was supported by DSA (Departmental special assistance) Gandhigram Rural Institute-Deemed to be university, Gandhigram under University Grants Commission- New Delhi.
§ Manuscript received: October 02, 2019; Accepted: May 3, 2020.

Joseph et. al., introduced the concept of triple connected in graphs. A graph G is said to be triple connected if any three points lie on a path in G. In [10], G. Mahadevan et. al., introduced the concept of triple connected domination number of a graph. A subset S of V of a non-trivial graph G is said to be triple connected dominating set, if S is a dominating set in G and the sub graph $< S >$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called triple connected domination number of a graph G and it is denoted by $\gamma_{tc}(G)$. In [3], W. Goddard et. al., introduced by the concept of eternal domination in graphs. The dominating set $S_0(\subseteq V(G))$ of the graph G is said to be an eternal dominating set, if for any sequence $v_1, v_2, v_3, \ldots v_k$ of vertices, there exists a sequence of vertices $u_1, u_2, u_3, \ldots u_k$ with $u_i \in S_{i-1}$ and u_i equal to or adjacent to v_i, such that each set $S_i = S_{i-1} - \{u_i\} \cup \{v_i\}$ is dominating set in G. The minimum cardinality taken over the eternal dominating sets in G is called the eternal domination number of G and it is denoted by $\gamma_\infty(G)$. In this paper we introduce another new concept Triple connected eternal domination number of a graph also investigate this number for some standard graphs and obtain many results with other graph theoretical parameters.

2. Triple Connected Eternal Domination in Graphs:

Definition 2.1. *The eternal dominating set $S_0(\subseteq V(G))$ is said to be a Triple Connected Eternal Dominating set in G if each dominating set S_i is triple connected. The minimum cardinality taken over the triple connected eternal dominating sets is called the Triple Connected eternal Domination Number of G and it is denoted by $\gamma_{tc,\infty}(G)$.*

Example 2.1. *Consider the graph G,*

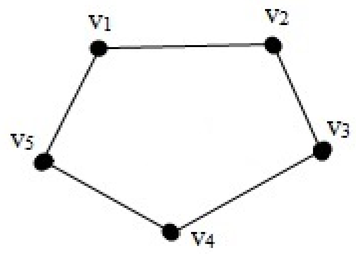

Figure 1

Here $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is a vertex set in G. Consider the set $S_0 = \{v_1, v_2, v_3, v_4\}$. $< S_0 >= P_4$ is triple connected. Also for every vertex in $V - S_0$ is adjacent to some vertex in S_0. This gives that S_0 is a triple connected dominating set in G. Now $S_1 = S_0 - \{v_4\} \cup \{v_5\}$ is a triple connected dominating set in G. Also $S_2 = S_1 - \{v_5\} \cup \{v_4\} = S_0$ is triple connected dominating set in G. Therefore S_0 is a triple connected eternal dominating set in G, which is minimum. This gives that $\gamma_{tc,\infty}(G) = 4$.

Theorem 2.1. *For any cycle C_n, $\gamma_{tc,\infty}(C_n) = n - 1$, $n > 3$.*

Proof. Consider the cycle on n vertices denoted by C_n. Here $V(C_n) = \{v_i, 1 \leq i \leq n\}$. Consider the set $S = \{v_i, 1 \leq i \leq n-1\}$.
Claim: S is a triple connected eternal dominating set in C_n

Here every vertex in $V - S$ is adjacent to some vertices in S. This gives that S is a dominating set in C_n. Also $< S >= P_{n-1}$ which is triple connected. Now $S_1 = S - \{v_1\} \cup \{v_n\}$ is also a triple connected dominating set in C_n. In this way we find $S_i = S_{i-1} - \{v_i\} \cup \{v_{i-1}\}$, $2 \leq i \leq n-1$ & $S_n = S$. All the sets S_i, $1 \leq i \leq n$ are triple connected dominating sets in C_n. This implies S is a triple connected eternal dominating set in C_n. Therefore

$$\gamma_{tc,\infty}(C_n) \leq n-1 \tag{1}$$

Consider the set S' with $|S'| < n-1$. That is $|S'| \leq n-2$. Now suppose $|S'| = n-2$. That is $S' = \{v_j, 1 \leq j \leq n-2\}$. Since every vertex in $V - S'$ is adjacent to some vertices in S' which implies S' is a dominating set in C_n. Also $< S' >= P_{n-2}$ which is triple connected. Now $S'_1 = S' - \{v_{n-2}\} \cup \{v_{n-1}\}$ is a dominating set but $< S'_1 >= P_{n-3} \cup K_1$ is disconnected. This gives that S'_1 is not a triple connected set in C_n. Therefore S' is not a triple connected eternal dominating set in C_n. Therefore $\gamma_{tc,\infty}(C_n) \neq n-2$. Also any set with cardinality less then $n-2$ is not a triple connected eternal dominating set in C_n. This gives that $\gamma_{tc,\infty}(C_n) \neq n-2$ which implies

$$\gamma_{tc,\infty}(C_n) > n-2 \Rightarrow \gamma_{tc,\infty}(C_n) \geq n-1 \tag{2}$$

From (1) and (2) $\gamma_{tc,\infty}(C_n) = n-1$. □

Theorem 2.2. *For any path P_n, $\gamma_{tc,\infty}(P_n) = n$, $n \geq 3$.*

Proof. Consider the path on n vertices denoted by P_n. Here $V(P_n) = \{v_i, 1 \leq i \leq n\}$. Consider the set $S = \{v_i, 1 \leq i \leq n\}$.
Claim: S is a triple connected eternal dominating set in P_n

Here every vertex in $V - S$ is adjacent to some vertices in S. This gives that S is a dominating set in P_n. Also $< S >= P_n$ which is triple connected. Now $S_1 = S$ is also a triple connected dominating set in P_n. In this way we find $S_i = S$, $2 \leq i \leq n$. All the sets S_i, $1 \leq i \leq n$ are triple connected dominating sets in P_n. This implies S is a triple connected eternal dominating set in P_n. Therefore

$$\gamma_{tc,\infty}(P_n) \leq n \tag{3}$$

Consider the set S' with $|S'| < n$. That is $|S'| \leq n-1$. Now suppose $|S'| = n-1$. That is $S' = \{v_j, 1 \leq j \leq n-1\}$. Since every vertex in $V - S'$ is adjacent to some vertices in S' which implies S' is a dominating set in P_n. Also $< S' >= P_{n-1}$ which is triple connected. Now $S'_1 = S' - \{v_{n-1}\} \cup \{v_n\}$ is a dominating set but $< S'_1 >= P_{n-2} \cup K_1$ is disconnected. This gives that S'_1 is not a triple connected set in P_n. Therefore S' is not a triple connected eternal dominating set in P_n. Therefore $\gamma_{tc,\infty}(P_n) \neq n-1$. Also any set with cardinality less then $n-1$ is not a triple connected eternal dominating set in P_n. This gives that $\gamma_{tc,\infty}(P_n) \nleq n-1$ which implies

$$\gamma_{tc,\infty}(P_n) > n-1 \Rightarrow \gamma_{tc,\infty}(P_n) \geq n \tag{4}$$

From (3) and (4) $\gamma_{tc,\infty}(P_n) = n$. □

Theorem 2.3. *For any complete graph K_n, $\gamma_{tc,\infty}(K_n) = 3$, $n \geq 3$.*

Proof. Consider the complete graph on n vertices denoted by K_n. Here $V(K_n) = \{v_i, 1 \leq i \leq n\}$. Consider the set $S = \{v_i, 1 \leq i \leq 3\}$.
Claim: S is a triple connected eternal dominating set in K_n

Here every vertex in $V - S$ is adjacent to some vertices in S. This gives that S is a dominating set in K_n. Also $< S >= K_3$ which is triple connected. Now $S_1 = S - \{v_1\} \cup \{v_n\}$ is also a triple connected dominating set in C_n. In this way we find $S_i = S_{i-1} - \{v_i\} \cup \{v_{i-1}\}$, $2 \leq i \leq n-1$ & $S_n = S$. All the sets S_i, $1 \leq i \leq n$ are triple connected dominating sets in K_n. This implies S is a triple connected eternal dominating set in K_n. Therefore

$$\gamma_{tc,\infty}(K_n) \leq 3 \tag{5}$$

We know that for any triple connected dominating set has at least 3 vertices. This implies that

$$\gamma_{tc,\infty}(K_n) \geq 3 \tag{6}$$

From (5) and (6) $\gamma_{tc,\infty}(K_n) = 3$. □

Theorem 2.4. *For any complete bipartite graph $K_{m,n}$, $\gamma_{tc,\infty}(K_{m,n}) = 3$, $m \geq 2$, $n \geq 2$.*

Proof. Consider the complete bipartite graph on m, n vertices denoted by $K_{m,n}$. Here $V(K_{m,n}) = \{X \cup Y\}$, where $X = \{u_i, 1 \leq i \leq m\}$ and $Y = \{v_j, 1 \leq j \leq n\}$. Consider the set $S = \{u_1, v_1, u_2\}$.
Claim: S is a triple connected eternal dominating set in $K_{m,n}$

Here every vertex in $V - S$ is adjacent to some vertices in S. This gives that S is a dominating set in $K_{m,n}$. Also $< S >= P_3$ which is triple connected. Now $S_1 = S - \{u_1\} \cup \{v_2\}$ is also a triple connected dominating set in $K_{m,n}$. In this way we find $S_i = [S_{i-1} - \{v_j\} \cup \{u_{i+1}\}] \cup [S_{i-1} - \{u_1\} \cup \{v_{j+1}\}]$, $2 \leq j \leq n$ & $S_m = S$. All the sets S_i, $1 \leq i \leq n$ are triple connected dominating sets in $K_{m,n}$. This implies S is a triple connected eternal dominating set in $K_{m,n}$. Therefore

$$\gamma_{tc,\infty}(K_{m,n}) \leq 3 \tag{7}$$

Since any triple connected dominating set has at least 3 vertices, we have

$$\gamma_{tc,\infty}(K_{m,n}) \geq 3 \tag{8}$$

From (7) and (8) $\gamma_{tc,\infty}(K_{m,n}) = 3$. □

Corollary 2.1. *For any graph G, $3 \leq \gamma_{tc,\infty}(G) \leq n$ and the bounds are sharp*

Proof. Consider the graph G with $n \geq 3$ vertices. In this graph any triple connected eternal dominating set has at least 3 vertices. This gives that $\gamma_{tc,\infty}(G) \geq 3$. Also in theorem 2.2 $\gamma_{tc,\infty}(G) \leq n$. Therefore $3 \leq \gamma_{tc,\infty}(G) \leq n$. □

Theorem 2.5. *For any Prism graph Y_n, $n \geq 3$, $\gamma_{tc,\infty}(Y_n) = 2n - 2$.*

Proof. Consider the prism graph Y_n, $n \geq 3$. Here $V(Y_n) = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots v_n\}$ has $2n$ vertices. Consider the set $S = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots v_{n-2}\}$.
Claim: S is a triple connected eternal dominating set in Y_n.

Here every vertex in $V - S$ is adjacent to some vertices in S. This gives that S is a dominating set in Y_n. Also $< S >$ is triple connected. Now $S_1 = S - \{v_1\} \cup \{v_n\}$ is also a triple connected dominating set in Y_n. In this way we find $S_i = S_{i-1} - \{v_{i+1}\} \cup \{v_i\}$ & $S_n = S$. All the sets S_i, $1 \leq i \leq n$ are triple connected dominating sets in Y_n. This implies S is a triple connected eternal dominating set in Y_n. Therefore

$$\gamma_{tc,\infty}(Y_n) \leq 2n - 2 \tag{9}$$

Consider the set S' with $|S'| < 2n-2$. That is $|S'| \leq 2n-3$. Now suppose $|S'| = 2n-3$. That is $S' = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots v_{n-3}\}$. Since every vertex in $V - S'$ is adjacent to some vertices in S' which implies S' is a dominating set in Y_n. Also $< S' >$ is triple connected. Now $S'_1 = S' - \{u_{n-1}\} \cup \{v_{n-1}\}$ is a dominating set but $< S'_1 >$ is disconnected. This gives that S'_1 is not a triple connected set in Y_n. Therefore S' is not a triple connected eternal dominating set in Y_n. Therefore $\gamma_{tc,\infty}(Y_n) \neq 2n - 3$. Also any set with cardinality less then $2n - 3$ is not a triple connected eternal dominating set in Y_n. This gives that $\gamma_{tc,\infty}(Y_n) \not\leq 2n - 3$ which implies

$$\gamma_{tc,\infty}(Y_n) > 2n - 3 \Rightarrow \gamma_{tc,\infty}(Y_n) \geq 2n - 2 \tag{10}$$

From (9) and (10) $\gamma_{tc,\infty}(Y_n) = 2n - 2$. □

Example 2.2. *Consider the Prism graph* Y_3

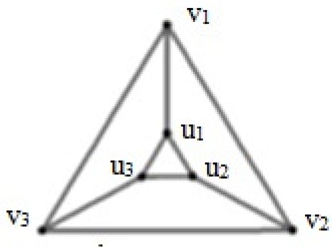

FIGURE 2

Here $V(Y_3) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ *is a vertex set in* Y_3. *Consider the set* $S_0 = \{u_1, u_2, u_3, v_1\}$. $< S_0 >$ *is triple connected. Also for every vertex in* $V - S_0$ *is adjacent to some vertex in* S_0. *This gives that* S_0 *is a triple connected dominating set in* Y_3. *Now* $S_1 = S_0 - \{v_1\} \cup \{v_2\}$ *is a triple connected dominating set in* Y_3. *Also* $S_2 = S_1 - \{v_2\} \cup \{v_3\}$ *and* $S_3 = S_2 - \{v_3\} \cup \{v_1\} = S_0$ *are all triple connected dominating set in* Y_3. *Therefore* S_0 *is a triple connected eternal dominating set in* Y_3, *which is minimum. This gives that* $\gamma_{tc,\infty}(Y_3) = 4 = ((2 \times 3) - 2)$.
Also $\gamma_{tc,\infty}(Y_4) = 6 = ((2 \times 4) - 2)$.
$\gamma_{tc,\infty}(Y_7) = 12 = ((2 \times 7) - 2)$.

Theorem 2.6. *For any Ladder graph* L_n, $n \geq 2$, $\gamma_{tc,\infty}(L_n) = 2n - 1$.

Proof. Consider the prism graph L_n, $n \geq 2$. Here $V(L_n) = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots v_n\}$ has $2n$ vertices. Consider the set $S = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots v_{n-1}\}$.
Claim: S is a triple connected eternal dominating set in L_n.

Here every vertex in $V - S$ is adjacent to some vertices in S. This gives that S is a dominating set in L_n. Also $< S >$ is triple connected. Now $S_1 = S - \{v_{n-1}\} \cup \{v_n\}$ is also a triple connected dominating set in Y_n. In this way we find $S_i = S_{i-1} - \{v_{i+1}\} \cup \{v_i\}$ & $S_n = S$. All the sets S_i, $1 \leq i \leq n$ are triple connected dominating sets in L_n. This implies S is a triple connected eternal dominating set in L_n. Therefore

$$\gamma_{tc,\infty}(L_n) \leq 2n - 1. \tag{11}$$

Consider the set S' with $|S'| < 2n-1$. That is $|S'| \leq 2n-2$. Now suppose $|S'| = 2n-2$. That is $S' = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots v_{n-2}\}$. Since every vertex in $V - S'$ is adjacent to some vertices in S' which implies S' is a dominating set in L_n. Also $< S' >$ is triple connected. Now $S'_1 = S' - \{v_{n-2}\} \cup \{v_{n-1}\}$ is a dominating set but $< S'_1 >$ is not triple connected. This gives that S'_1 is not a triple connected set in L_n. Therefore S' is not a triple connected eternal dominating set in L_n. Therefore $\gamma_{tc,\infty}(L_n) \neq 2n - 1$. Also any set with cardinality less then $2n - 2$ is not a triple connected eternal dominating set in L_n. This gives that $\gamma_{tc,\infty}(L_n) \not\leq 2n - 2$ which implies

$$\gamma_{tc,\infty}(L_n) > 2n - 2 \Rightarrow \gamma_{tc,\infty}(L_n) \geq 2n - 1 \tag{12}$$

From (11) and (12) $\gamma_{tc,\infty}(L_n) = 2n - 1$. □

Example 2.3. *Consider the Ladder graph* L_2

Here $V(L_2) = \{u_1, u_2, v_1, v_2\}$ *is a vertex set in* L_2. *Consider the set* $S_0 = \{u_1, u_2, v_1\}$. $< S_0 >= P_3$ *is triple connected. Also for every vertex in* $V - S_0$ *is adjacent to some vertex in* S_0. *This gives that* S_0 *is a triple connected dominating set in* L_2. *Now* $S_1 = S_0 - \{v_1\} \cup \{v_2\}$ *is a triple connected dominating set in* L_2. *Also* $S_2 = S_1 - \{v_2\} \cup \{v_1\} = S_0$ *is all triple*

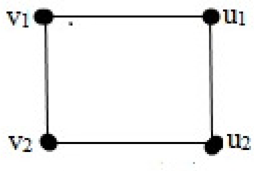

FIGURE 3

connected dominating set in L_2. Therefore S_0 is a triple connected eternal dominating set in L_2, which is minimum. This gives that $\gamma_{tc,\infty}(L_2) = 3 = ((2 \times 2) - 1)$.
Also $\gamma_{tc,\infty}(L_3) = 5 = ((2 \times 3) - 1)$.
$\gamma_{tc,\infty}(L_8) = 15 = ((2 \times 8) - 1)$.

Theorem 2.7. *For any Wheel graph W_N, $N = n+1$, $n \geq 4$, $\gamma_{tc,\infty}(W_N) = N - 2 = n - 1$.*

Proof. Consider the prism graph W_N, $n \geq 3$. Here $V(W_N) = \{u, v_1, v_2, v_3, \ldots v_n\}$ has $n + 1$ vertices. Consider the set $S = \{u, v_1, v_2, v_3, \ldots v_{n-2}\}$.
Claim: S is a triple connected eternal dominating set in W_N.

Here every vertex in $V - S$ is adjacent to some vertices in S. This gives that S is a dominating set in W_N. Also $< S >$ is triple connected. Now $S_1 = S - \{v_{n-2}\} \cup \{v_{n-1}\}$ is also a triple connected dominating set in W_N. In this way we find $S_i = S_{i-1} - \{v_{i+1}\} \cup \{v_i\}$ & $S_n = S$. All the sets S_i, $1 \leq i \leq n$ are triple connected dominating sets in W_N. This implies S is a triple connected eternal dominating set in W_N. Therefore

$$\gamma_{tc,\infty}(W_N) \leq n - 1 \tag{13}$$

Consider the set S' with $|S'| < n - 1$. That is $|S'| \leq n - 2$. Now suppose $|S'| = n - 2$. That is $S' = \{u, v_1, v_2, v_3, \ldots v_{n-3}\}$. Since every vertex in $V - S'$ is adjacent to some vertices in S' which implies S' is a dominating set in W_N. Also $< S' >$ is triple connected. Now $S'_1 = S' - \{u\} \cup \{v_{n-1}\}$ is a dominating set but $< S'_1 >= P_{n-3} \cup K_1$ is disconnected. This gives that S'_1 is not a triple connected set in W_N. Therefore S' is not a triple connected eternal dominating set in W_N. Therefore $\gamma_{tc,\infty}(W_N) \neq n - 2$. Also any set with cardinality less then $2n - 2$ is not a triple connected eternal dominating set in W_N. This gives that $\gamma_{tc,\infty}(W_N) \neq n - 2$ which implies

$$\gamma_{tc,\infty}(W_N) > n - 2 \Rightarrow \gamma_{tc,\infty}(W_N) \geq n - 1 \tag{14}$$

From (13) and (14) $\gamma_{tc,\infty}(W_N) = n - 1 = N - 2$. □

Example 2.4. *Consider the wheel graph W_5*

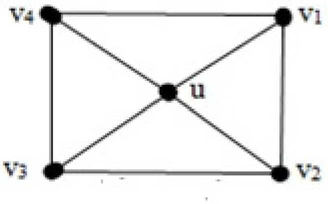

FIGURE 4

Here $V(W_5) = \{u, v_1, v_2, v_3, v_4\}$ is a vertex set in W_5. Consider the set $S_0 = \{u, v_1, v_2\}$. $< S_0 >= C_3$ is triple connected. Also for every vertex in $V - S_0$ is adjacent to some vertex in S_0. This gives that S_0 is a triple connected dominating set in W_5. Now $S_1 =$

$S_0 - \{u\} \cup \{v_3\}$ is a triple connected dominating set in W_5. Also $S_2 = S_1 - \{v_3\} \cup \{v_4\}$, $S_3 = S_2 - \{v_4\} \cup \{u\} = S_0$ are all triple connected dominating sets in W_5. Therefore S_0 is a triple connected eternal dominating set in W_5, which is minimum. This gives that $\gamma_{tc,\infty}(W_5) = 3 = 5 - 2$.
Also $\gamma_{tc,\infty}(W_6) = 4 = 6 - 2$.
$\gamma_{tc,\infty}(W_{10}) = 8 = 10 - 2$.

Note 2.1. *Consider the wheel graph W_4, Here the triple connected eternal domination number is 3. That is $\gamma_{tc,\infty}(W_4) = 3$. Thus $\gamma_{tc,\infty}(W_4) = 3 \neq 4 - 2$.*

Remark 2.1. *Triple connected eternal domination number does not exists for the following graphs.*
1) Star graph
2) Butterfly graph
3) Helm graph
4) Friendship graph
5) Fan graph

Observation 2.1. *Every triple connected eternal dominating set is triple connected dominating set but the converse is not true.*

Proof. Let S_0 be a triple connected eternal dominating set in the graph G. Then each set S_i, $0 \leq i \leq n$ is triple connected dominating set in G. This gives that S_0 is a triple connected dominating set in the graph G. Therefore every triple connected eternal dominating set is triple connected dominating set. But the converse is not true. □

Example 2.5. *Consider the graph P_4. Here $S = \{v_1, v_2, v_3\}$ is a triple connected dominating set but it is not triple connected eternal dominating set.*

Observation 2.2. *Every triple connected eternal dominating set is eternal dominating set but the converse is not true.*

Proof. Let S_0 be a triple connected eternal dominating set in the graph G. Then each set S_i, $0 \leq i \leq n$ is an eternal dominating set in G. This gives that S_0 is an eternal dominating set in the graph G. Therefore every triple connected eternal dominating set is eternal dominating set. But the converse is not true. □

Example 2.6. *Consider the graph P_4. Here $S = \{v_1, v_2, v_3\}$ is an eternal dominating set but it is not tripe connected eternal dominating set.*

Observation 2.3. *For any graph G, $\gamma_{tc}(G) \leq \gamma_{tc,\infty}(G)$ and the bound is sharp if $G \cong K_n$*

Observation 2.4. *For any graph G, $\gamma_{\infty}(G) \leq \gamma_{tc,\infty}(G)$ and the bound is sharp if $G \cong K_{m,n}$.*

Observation 2.5. *For any graph G, $\gamma_{nsptc}(G) \leq \gamma_{tc,\infty}(G)$ and the bound is sharp if $G \cong K_n$, n is odd.*

References

[1] Bondy, J.A. and Moorthy, U.S.R. (1976), Graph Theory with applications, The Macmillan Press Ltd Great Britain.

[2] Finbow, S. and Messinger, M.E. (2015), Eternal Domination on $3 \times n$ Grid graphs, Australasian Journal of Combinatorics, 61, (2), pp. 156-174.

[3] Goddard, W., Hedetniemi, S.M. and Hedetniemi, S.T. (2008), Eternal Security in Graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 52, pp. 2589-2593.

[4] Gross, J.L. and Yellen, J. (2003), Handbook of Graph Theory, CRC Press Ltd New York.

[5] Harary, G. (1969), Graph theory, Addison Wesley Publishing Company pvt Ltd USA.

[6] Mahadevan, G., Iravithul Basira, A. and Sivagnanam, C. (2016), Non-Split Perfect Triple Connected Domination Number of a Graph, Asian Journal of Research in Social Sciences and Humanities, 6, pp. 1954-1966.

[7] Mahadevan, G., Ponnuchamy, T. and Selvam, A. (2018), Non-Split Perfect Triple connected Domination number on Different Product of Paths, International Journal of Mathematical Combinatorics, pp. 118-131.

[8] Mahadevan, G., Ponnuchamy, T. and Selvam, A. (2019), Investigation of Non-split perfect triple connected domination number of some semi product related graphs, The International Journal of Analytical and Experimental modal Analysis, 2, (10), pp. 191-196.

[9] Paulraj Joseph, J., Angel Jebitha, M.K., Chithra Devi, P. and Sudhana, G. (2012), Triple Connected Graphs, Indian Journal Mathematics and Mathematical Sciences, 8, (1), pp. 61-75.

[10] Paulraj Joseph, J., Mahadevan, G. and Selvam, A. (2012), Triple Connected Domination number of a Graph, International Journal of Mathematical Combinatorics, 3, pp. 93-104.

Dr. G. Mahadevan M.Sc., M.Phil., M.Tech., Ph.D., is working as Asst. Professor, Dept. of Mathematics, Gandhigram Rural Institute-Deemed to be University, Gandhigram. He is the Associate Editor of International Journal of Applied Graph theory. He reviewed many papers in reputed international and national journals.

T. Ponnuchamy M.Sc., M.Phil., received his Bachelor of Science in Mathematics (2013) Master of Science in Mathematics (2015) and Master of Philosophy in Mathematics (2016) degree from Ayya Nadar Janaki Ammal Collage, Sivakasi, India. He is currently doing research as full time research scholar under the guidance of Dr. G. Mahadevan, Dept. of Mathematics, Gandhigram Rural Insitute, Gandhigram.

Dr. Selvam Avadayappan M.Sc., M.Phil., Ph.D., Associate Professor, Research Centre of Mathematics, VHN Senthikumara Nadar College (Autonomous), Virudhunagar has 24 years of research experience and 29 years of post graduate teaching experience. He has successfully guided 7 Ph. D scholars and presently guiding 2 scholars.