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OPERATIONS ON TOTALLY WEIGHTED GRAPHS

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Abstract. The investigation and study of different operations on graphs leads many interesting results in various branches of science and technology. The operations of union, join, corona, Cartesian product and strong product in totally weighted graphs are introduced in this article. The types of edges in the resultant graphs are studied and strong connectivity indices of totally weighted graphs through these graph operations are discussed here. Weighted graph operations are relevant, as many complicated networks can be obtained through the operations of basic types of networks.

Keywords: Weighted graph, union, join, Cartesian product, composition, strong connectivity index.
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1. Introduction

Every network system can be modeled as a weighted or an unweighted graph. Unweighted graphs can be considered as weighted graphs with all edge weights equal to one. Graphs can also be used to model social structures depending on relationship between people. In such a model, vertices represent individuals and edges represent relationships between them. If we consider a communication network system as a weighted graph, where the points represent communication centers and connected lines (or edges) represents the communication links, then the weights of edges may represent the maintenance cost of various communication links.

Weighted graphs are considered as generalized structures of graphs. Any problems which can be solved using graph theory can also be solved using weighted graphs. A network of cities interconnected by roads can be represented by a weighted graph with nodes, edges and weights denoting cities, roads and travelling costs respectively. Likewise many real world problems can be designed using weighted or unweighted graphs. It helps to reduce the complexity of the problems and finds solutions easily. In all scientific areas, graph theory has wide applications. So nowadays it has become an interesting area of research in different areas of sciences.

A graph is said to be connected if there is a path joining every pair of vertices. Otherwise it is called disconnected. Connectivity concepts are the keys in network problems and

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graph clustering. As weighted graphs are generalized structures of graphs, connectivity concepts in weighted graphs generalizes the classic connectivity concepts. While classical parameters like vertex connectivity and edge connectivity in unweighted graphs deal with disconnection of the whole graph, the connectivity parameters in weighted graphs take care of reduction in flow or weight more than the total disconnection of the graph. It is considered that the earliest developments in weighted graph theory began with paths and cycles. Erdős and Gallai in 1959 studied on maximal paths, circuits and independent edges of weighted graphs.

After in 1990's there was a rapid growth in this field by J Bondy and his co-authors. Zhang and Broersma in 2002 mentioned that the weight of a cycle is the sum of the weights of its edges and an optimal cycle is one of maximum weight. Also they generalized a theorem of G Fan that describes existence of long cycles in unweighted to weighted graphs.

Connectivity has a great significance in graph theory and weighted graph theory. Dirac laid the foundation stone for the concept of connectivity in graph theory. Dirac in 1952 obtained relations between the vertices in a graph and the length of the circuits contained in it. Grőtschel gave a new point of view on graph connectivity that is based on geometric and physical intuition. Also he proved a theorem on geometric characterization of k-vertex connected graphs.

Sunil Mathew and Sunitha discussed some connectivity concepts in weighted graphs in 2010, where they made a classification of edges of graphs by means of strength of connectedness. They generalized many graph theoretic concepts in weighted graphs. Later these developments in weighted graphs helped to attract many researchers in this field.

2. Basic definitions of weighted graphs

As in graph theory, connectivity plays an important role in weighted graph theory also. The weight of a weakest arc in a path is defined as its strength. Some of the basic definitions by Sunil Mathew and Jicy in 2010 are given below.

Definition 2.1 Let $G(V, E)$ be a weighted graph. The strength of connectedness between two nodes u and v is denoted by $CONN_G(u, v)$, is defined as the maximum of the strengths of all paths between u and v.

Example 2.2 (Figure 1) Consider the following graph $G(V, E)$

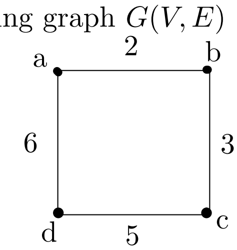

Figure 1: Strength of connectedness

In the above example, $CONN_G(a, b) = 3$, $CONN_G(a, c) = 5$, $CONN_G(a, d) = 6$, $CONN_G(b, c) = 3$, $CONN_G(b, d) = 3$ and $CONN_G(c, d) = 5$.

Definition 2.3 A $u - v$ path in a weighted graph G is called a strongest $u - v$ path if strength of the path $u - v$ is equal to the strength of connectivity between u and v.

Definition 2.4 Let G be a weighted graph. Then an arc $e = (u, v)$ is called α-strong if $CONN_{G-e}(u, v) < w(e)$, β-strong if $CONN_{G-e}(u, v) = w(e)$ and a δ-arc if $CONN_{G-e}(u, v) > w(e)$. A δ-arc e is called a δ^*-arc if e is not a weakest arc of G.

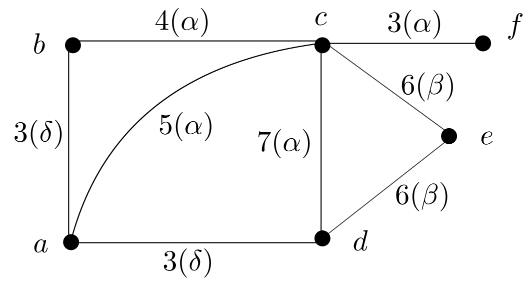

Figure 2: A weighted graph of all types of edges

Definition 2.5 A totally weighted graph $G(\delta, \omega)$ where $\delta : V \to R^+$ and $\omega : E \to R^+$ Such that $\omega(u, v) \leq \delta(u) \wedge \sigma(v)$ for any pair of nodes u, v of $V(G)$, where $\wedge$ denotes the minimum. If $\omega(u, v) = \delta(u) \wedge \delta(v)$, then the totally weighted graph $G : (\delta, \omega)$ is called precisely weighted graph.

Definition 2.6 Let G be a weighted graph. Then strong connectivity index of G denoted by $CI(G)$ is defined as $CI(G) = \sum_{u,v \in V(G)} w(u)w(v)CONN_G(u, v)$.

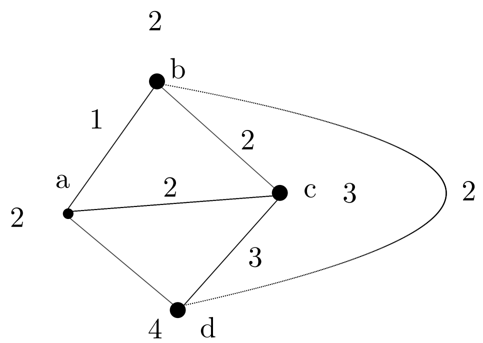

Figure 3: Weighted graph with a connectivity index

Example 2.7 (Figure 3) Let $G : (\delta, \omega)$ be a totally weighted graph with $V = \{a, b, c, d\}$ such that $w(a) = 2$, $w(b) = 2$, $w(c) = 3$, $w(d) = 4$, $w(a, b) = 1$, $w(b, c) = 2$, $w(a, c) = 2$ and $w(a, d) = 2$, $w(b, d) = 2$ and $w(c, d) = 3$. Then strong connectivity index of G is $CI(G) = 96$.

3. Operations on totally weighted graphs

Some important weighted graph operations are defined and the types of edges in such graphs are discussed here. Operation between graphs are important as many networks can be visualized as a graph obtained through the operations between some small graphs. For example, a cube is a product of different K_2's.

Definition 3.1 *Let $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be the two totally weighted graphs. Their union of G_1 and G_2 is a totally weighted graph and is defined as $G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ on $G^* : (V, E)$, where,*

$$\delta_(u) = \begin{cases} \delta_1(u), & if u \in V_1 - V_2 \\ \delta_2(u), & if u \in V_2 - V_1 \end{cases}$$

$$\omega_(u, v) = \begin{cases} \omega_1(u, v), & if (u, v) \in E_1 - E_2 \\ \omega_2(u, v), & if (u, v) \in E_2 - E_1 \end{cases}$$

Example 3.2

Let $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two totally weighted graphs given in Figure 4 Their union, $G_1 \cup G_2$ is given by the third graph.

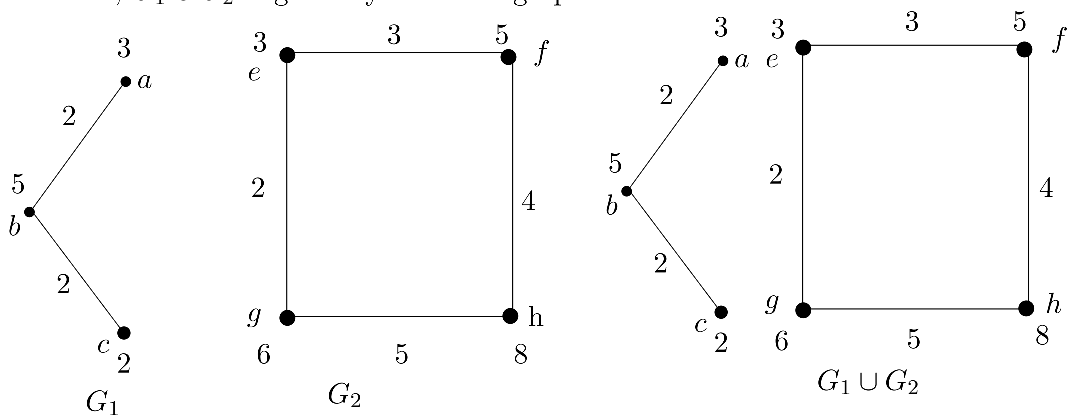

Figure 4: Union of two weighted graphs

Theorem 3.3 *Strong edges are preserved in the union of two totally weighted graphs.*

The connectivity and adjacencies of the vertices in the resultant graphs are preserved. Hence a formal proof is omitted.

Theorem 3.4 *Suppose $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two totally weighted graphs. Then the connectivity index of union of these graphs satisfies the relations $CI(G_1 \cup G_2) = Max\{CI(G_1), CI(G_2)\}$.*

Proof. Since all the edges and vertices are preserved in union. Then there is no change in strength of edges after the operations. Hence the maximum connectivity index among G_1 and G_2 gives the connectivity index of $G_1 \cup G_2$. □

Definition 3.5 *Let $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be the two totally weighted graphs. The join of G_1 and G_2 is defined as a totally weighted graph $G = G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ Where $E' = \{(u, v), u \in V_1 and\ v \in V_2\}$. Then $G_1 + G_2$ is the weighted graph $G : (\delta, \omega)$ with*

$$\delta(u) = (\delta_1 \cup \delta_2)(u), if u \in V_1 \cup V_2$$

$$\omega(u, v) = \begin{cases} \omega_1 \cup \omega_2(u, v), & if (u, v) \in E_1 \cap E_2 \\ \delta_1(u) \wedge \delta_2(v), & if (u, v) \in E' \end{cases}$$

Example 3.6 Let $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two totally weighted graphs given in Figure 5. Then their join, $G_1 + G_2$ is given by the third graph.

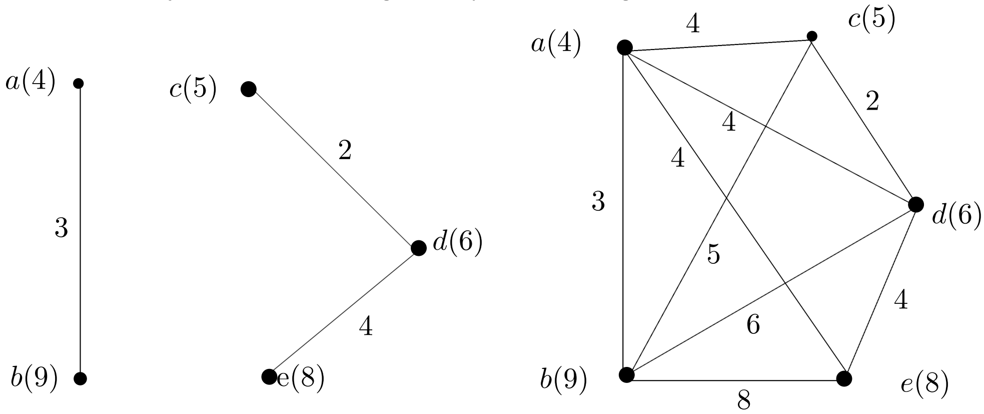

Figure 5: Join of two totally weighted graphs

Theorem 3.7 *If $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two precisely weighted graphs, then there are no weak edges in $G_1 + G_2$.*

Proof. A graph $G : (\delta, \omega)$ is precisely weighted, if $\omega(u, v) = \delta(u) \wedge \delta(v)$ for every $u \in V(G)$.

Case 1: $\delta_1(u) > \delta_2(u)$ $\forall u \in V(G_1)$.

The new edges in $G_1 + G_2$ have weight equal to $\delta_2(u)$ $\forall u \in V(G_1)$. But there already exists edges in G_2 with strength of edges in $G_1 + G_2$, all cycles contains more than one weak edges and all edges in $G_1 + G_2$ are strong edges.

Case 2: $\delta_2(u) > \delta_1(u)$ $\forall u \in V(G_1)$.

The new edges in $G_1 + G_2$ have strength equal to $\delta_1(u)$ $\forall u \in V(G_1)$. But there already exists edges in G_1 with strength of edges as δ_1. Hence in $G_1 + G_2$, all cycles contains more than one weak edges. And all edges in $G_1 + G_2$ are strong edges.

Case 3: $\delta_1(u) = \delta_2(u)$ $\forall u \in V(G_1)$.

Then clearly in $G_1 + G_2$, all cycles contains more than one weak edges. Hence all edges in $G_1 + G_2$ are strong edges. And there exists no weak edge in $G_1 + G_2$. Hence the proof. □

Definition 3.8 *Let $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two totally weighted graphs. Then the corona graph $G : (V, E \cup E')$ is$G_1[G_2]$ of two totally weighted graph obtained by taking one copy of G_1 having n_1 vertices and n_1 copies of G_2. Then join the vertex i of G_1 to every vertex in the i^{th} copy of G_2. Where,*

$$\delta_(u) = \begin{cases} \delta_1(u), & if u \in V_1 - V_2 \\ \delta_2(u), & if u \in V_2 - V_1 \end{cases}$$

$$\omega_(u, v) = \begin{cases} \omega_1(u, v), & if (u, v) \in E_1 - E_2 \\ \omega_2(u, v), & if (u, v) \in E_2 - E_1 \\ \delta_1(u) \wedge \delta_2(v), & if (u, v) \in E' \end{cases}$$

Example 3.9 Two totally weighted graphs and their corona are given in figure 9. Their corona $G_1[G_2]$, is given by the third graph.

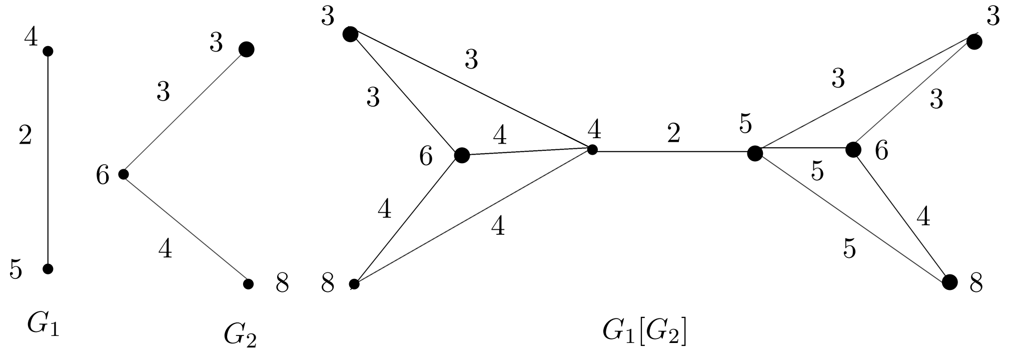

Figure 6: Corona of two totally weighted graphs

Theorem 3.10 *Let $G_1 : (\delta_1, \omega_1)$ be any weighted graph and $G_2 : (\delta_2, \omega_2)$ be a precisely weighted graph, then all edges of $G_1[G_2]$ are strong.*

Proof. A graph $G : (\delta, \omega)$ is precisely weighted, if $\omega(u, v) = \delta(u) \wedge \delta(v)$ for every $u, v \in V(G)$. Where $\wedge$ represents the minimum operator.

Case 1: $\delta_1(u) > \delta_2(u)\ \forall\ u \in V(G)$.

The new edges in $G_1[G_2]$ have weight equal to $\delta_2(u)\ \forall\ u \in V(G)$. The new edges together with the copies of G_2 form cycles which contains more than one weak edges. Hence all edges in $G_1[G_2]$ are strong edges. And there exists no weak edge in $G_1[G_2]$.

Case 2: $\delta_2(u) > \delta_1(u)\ \forall\ u \in V(G)$.

The new edges in $G_1[G_2]$ have strength equal to $\delta_1(u)\ \forall\ u \in V(G)$. The new edges together with the copies of G_2 form cycles which contains more than one weak edges. Hence all edges in $G_1[G_2]$ are strong edges. And there exists no weak edge in $G_1[G_2]$.

Case 3: $\delta_1(u) = \delta_2(u)\ \forall\ u \in V(G)$.

Then clearly the new edges together with the copies of G_2 form cycles which contains more than one weak edges. Hence all edges in $G_1[G_2]$ are strong edges. And there exists no weak edge in $G_1[G_2]$.

Hence the proof. □

Definition 3.11 *The Cartesian product of two totally weighted graphs G_1 and G_2 is defined as a totally weighted graph $G = G_1 \times G_2$ on $G^* : (V, E)$, where $V = V_1 \times V_2$ and $E = \{(u, u_1), (u, v_1) | u \in V_1, (u_1, v_1) \in E_2\} \cup \{(u_1, v), (v_1, v) | v \in V_2, (u_1, v_1) \in E_1\}$ with*

$\delta_(uv) = \{\delta_1(u) \wedge \delta_2(v)$, for all $uv \in V_1 \times V_2$

$\omega(uu_2, uv_2) = \{\delta_1(u) \wedge \omega_2(u_2, v_2)\}$, if $u \in V_1$, $(u_2, v_2) \in E_2$

$\omega(u_1v, v_1v) = \{\delta_2(v) \wedge \omega_1(u_1, v_1)\}$, if $v \in V_2$, $(u_1, v_1) \in E_1$

Example 3.12 Two totally weighted graphs $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ their Cartesian product, $G_1 \times G_2$ are given in Figure 7.

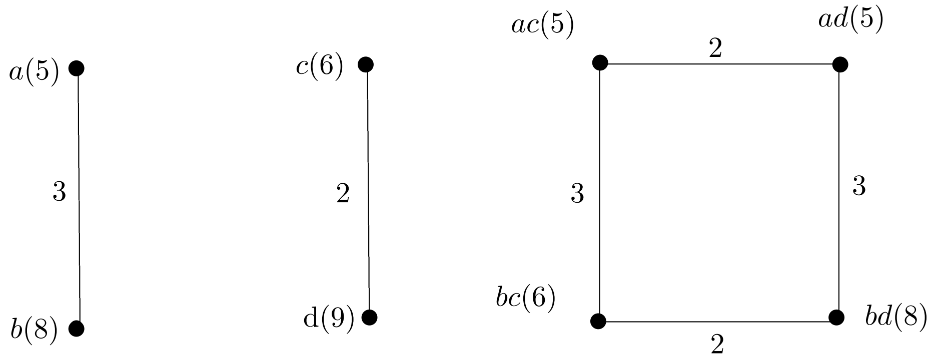

Figure 7: Cartesian product of two weighted graphs

Theorem 3.13 *If $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two totally weighted graphs. Then there are no $\delta-$ edges in $G_1 \times G_2$.*

Proof. Case 1: Suppose that all the edges and vertices in G_1 have weights greater than $\delta_2(v)$ for every $v \in V_2$. Then the edge weight and vertex weight of $G_1 \times G_2$ will be $\delta_2(v)$ or $\omega_2(v_i, v_j)$. Then in $G_1 \times G_2$, for any path joining $u_i v_i$ and $u_j v_j$ there is another path with same minimum weight as $\omega_2(v_i, v_j)$ or $\delta_2(v)$ for every $v \in V_2$. Therefore there exists cycles with more than one weakest edges. Hence there is no δ- edges in $G_1 \times G_2$.

Case 2: By a similar argument, That is, if all the edges and vertices in G_2 have strength greater than $\delta_1(v)$ for every $v \in V_2$. In this case also there is no δ- edges in $G_1 \times G_2$. □

Definition 3.14 *The strong product of two weighted graphs G_1 and G_2 is defined as a totally weighted graph $G = G_1 \otimes G_2$ on $G^* : (V, E)$. Where $V = V_1 \times V_2$ and $E = \{(u, u_1), (u, v_1) | u \in V_1, (u_1, v_1) \in E_2\} \cup \{(u_1, v), (v_1, v) | v \in V_2, (u_1, v_1) \in E_1\} \cup \{(u_1, u_2), (v_1, v_2) | (u_1, v_1) \in E_1, (u_2, v_2) \in E_2\}$ with*

$\delta(uv) = \{\delta_1(u) \wedge \delta_2(v)$, for all $uv \in V_1 \times V_2$

$\omega(uu_2, uv_2) = \{\delta_1(u) \wedge \omega_2(u_2, v_2)\}$, if $u \in V_1$, $(u_2, v_2) \in E_2$

$\omega(u_1u_2, v_1u_2) = \{\omega_1(u_1, v_1) \wedge \omega_2(u_2, v_2)\}$, if $(u_1, v_1) \in E_1, (u_2, v_2) \in E_2$.

Example 3.15 Two totally weighted graphs $G_1 : (\delta_1, \omega_1)$, $G_2 : (\delta_2, \omega_2)$ and their strong product, $G_1 \otimes G_2$ are given in the following figure (Figure 8)

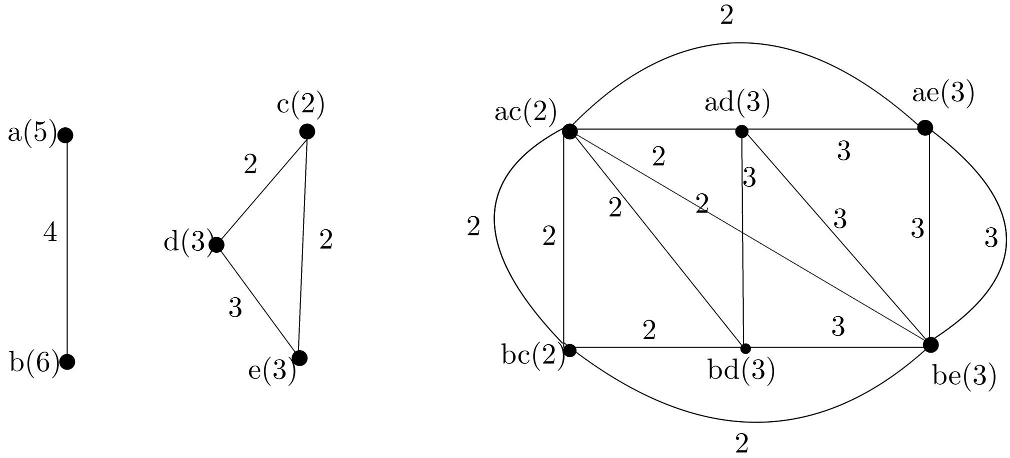

Figure 8: Strong product of two weighted graphs

Theorem 3.16 *If $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ are two totally weighted graphs, then there are no weak edges in $G_1 \otimes G_2$.*

Proof. Case 1: Suppose that all the edges and vertices in G_1 have weights greater than $\delta_2(v)$ for every $v \in V_2$. Then the edge weight and vertex weight of $G_1 \times G_2$ will be $\delta_2(v)$ or $\omega_2(v_i, v_j)$, for every $(v_i, v_j) \in E_2$ respectively. Then in $G_1 \otimes G_2$, there are many cycles joining the edges for $u_i v_i$ and $u_j v_j$ which contains at least one edge of minimum weight as $\omega_2(v_i, v_j)$ for every $(v_i, v_j) \in E_2$ or $\delta_2(v)$ for every $v \in V_2$. These edges together with edges in G_2 form cycles with more than one weakest edges. Hence there is no weak edges in $G_1 \otimes G_2$.

Case 2: If all the edges and vertices in G_2 have weights greater than $\delta_1(v)$ for every $v \in V_2$, by a similar argument as above it is seen that there are no weak edges in $G_1 \otimes G_2$. □

Theorem 3.17 Suppose $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two non trivial connected totally weighted graphs with cycle connectivity greater than zero. Then the following relations on strong connectivity index are true.

1) $CI(G_1 + G_2) > CI(G_1) + CI(G_2)$.
2) $CI(G_1[G_2)] > CI(G_1) + CC(G_2)$.
3) $CI(G_1 \times G_2) > CI(G_1) + CC(G_2)$.
4) $CI(G_1 \otimes G_2) > CI(G_1) + CI(G_2)$.

Proof. Let $G_1 : (\delta_1, \omega_1)$ and $G_2 : (\delta_2, \omega_2)$ be two non trivial connected totally weighted graphs with cycle connectivity greater than zero. Suppose H is a new graph obtained after the operations. There are new edges in H in each case, whose strength is more or equal to at least one of end vertex of G_1 or G_2. The distance of newly formed pairs are also included while finding the strong connectivity index of H other than the sum of distances of vertices in G_1 and G_2. Therefore in all these cases the strong connectivity index of

$G_1 + G_2, G_1[G_2], G_1 \times G_2$ and $G_1 \otimes G_2$ is greater than the sum of strong connectivity index of G_1 and G_2. □

4. Conclusions

Different graph operations on totally weighted graphs are introduced and the types of edges in the resultant graphs are studied. As in network analysis it is important to identify the nature of a weighted network. As applying weighted graph operations, many small graphs can be extended to large weighted networks so that solutions of verity network problems like traffic network problems can be done using different weighted graph models.

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