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PRACTICAL STABILITY AND BOUNDEDNESS CRITERIA OF IMPULSIVE DIFFERENTIAL SYSTEM WITH INITIAL TIME DIFFERENCE

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Abstract. In this paper, an impulsive differential system is investigated for the first time for practical stability and boundedness criteria with respect to initial time difference. The investigations are carried out by perturbing Lyapunov functions and by using comparison results. A generalized Lyapunov function has been used for the investigation. The present results indicate that the stability criteria significantly depend on the moment of impulses.

Keywords: Impulsive differential system, practical stability, boundedness, perturbed Lyapunov function, initial time difference

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1. Introduction

Impulsive differential systems have been emerging as an important area of investigation in many problems of practical significance and has a wider scope in many fields like population dynamics, ecology, control theory fields and many more [6, 12, 23, 24]. While studying the qualitative behavior of these impulsive differential systems, apart from stability, which is one of the most important feature, the investigation of systems within specified bounds is also of great significance. So far as stability is concerned, there are various stability criteria like asymptotic stability, exponential stability, weighted stability, eventual stability, practical stability etc. which are widely studied and available in the literature [5, 14, 18, 19, 2, 15, 16, 27]. In the study of physical as well as biological phenomenon, it is desired that sometimes the system may be unstable mathematically, but it is practically acceptable because it fluctuates near the equilibrium. In such cases, the notion of practical stability is very useful which stabilizes the system into certain subsets of phase space.

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In the classical stability theory, the solutions of differential equations are studied by keeping the starting time fixed, however, while considering the real world problems, it is sometimes not possible to investigate the stability of the solutions by taking the initial time same. Stability with respect to initial time difference (ITD) is a generalization of the basic concept of stability of solutions. The concept of stability with respect to ITD was initiated by Lakshmikantham and Vatsala [10] and Lakshmikantham et al. [8]. In the past, different types of stability criteria as well as boundedness properties are studied for ordinary differential equations relative to ITD without impulse effects [1, 22, 25]. The definition of practical stability for differential systems relative to ITD was proposed by Lakshmikantham and Vatsala [10]. Furthermore, the practical stability and boundedness criteria with respect to ITD are investigated by many researchers for nonlinear differential systems without impulses [11, 21, 26]. The investigation of impulsive differential equations with respect to ITD is at its initial stage and has not been explored much in the past. Nevertheless, so far to the best of our knowledge, Hristova [4] studied the stability properties of impulsive differential equations relative to ITD, but both practical stability and boundedness criteria has not been studied by any researcher for impulsive differential systems in terms of ITD.

In the present work, for the first time, we study the various practical stability criteria along with boundedness properties for a system of impulsive differential equation with ITD by perturbing the Lyapunov function. Lyapunov function is widely recognized as a tool for investigating the stability properties of nonlinear differential equations. When a Lyapunov function does not seem to satisfy all the required conditions to obtain the desired properties, then it becomes worth, perturbing the Lyapunov function rather than discarding it. The concept of perturbing Lyapunov function to study the nonuniform properties of solutions of differential equations was firstly given by Lakshmikantham [7] and further extended to investigate various stability criteria for impulsive differential equations [13, 28, 20]. The notion of perturbing Lyapunov function was used by McRae and Song et al. [17, 22] to investigate the stability properties of differential equations without impulse effect relative to ITD. Hristova [3] derived the practical stability criteria for delay differential equations with respect to ITD. Song and Li [20] investigated the practical stability and boundedness properties of nonlinear impulsive systems by using the perturbed Lyapunov function. Now, in this paper, we are using the technique of perturbing Lyapunov function for investigating the stability and boundedness criteria of impulsive differential equations with ITD. A generalized Lyapunov function is used to investigate the qualitative behavior of impulsive differential system.

The paper is organized as follows. In section 2, we introduce some basic definitions and notations. Some comparison results are presented in section 3. In section 4, we derive some criteria to bring the practical stability and boundedness of impulsive differential equation with respect to ITD. Finally, on the basis of these results, conclusion is drawn in section 5.

2. Preliminaries

Let R^n denotes the n dimensional Euclidean space and let $R_+ = [0, \infty)$.

Consider the impulsive differential system:

$$\begin{cases} \dot{x} = f(t, x), & t \neq t_i \\ x(t^+) = x(t) + I_i(x), & t = t_i, \quad i = 1, 2, 3... \end{cases} \tag{1}$$

where $f : R_+ \times R^n \to R^n$ is piece-wise continuous function and $I_i : R^n \to R^n$ is continuous function for every i where $i = 1, 2, 3...$.

Let $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$ be two solutions of the system (1) through (t_0, x_0) and (τ_0, y_0) respectively, where $t_0, \tau_0 \in R_+$. Here both $x(t)$ and $y(t)$ are piecewise continuous having discontinuity of first kind at $t = t_i$ where $i = 1, 2, 3...$, having left continuity defined as $x(t_i^-) = x(t_i)$ and $x(t_i^+) = x(t_i) + I_i(x(t_i))$. In our investigation, we will study the stability and boundedness criteria with respect to the solution $x(t) = x(t; t_0, x_0)$.
Let $\gamma = \tau_0 - t_0 > 0$.
Denote $S(\rho) = \{x \in R^n : \|x\| < \rho\}$. Consider the complimentary sets of $S(\rho)$ as $S^c(\rho)$.
Also, consider the following function:

$$K = \{\phi \in C(R_+, R_+) : \phi \text{ is strictly increasing and } \phi(0) = 0\}$$

$$C_K = \left\{\omega \in C(R_+^2, R_+) : \omega(t, u) \in K \text{ for each } t \in R_+\right\}$$

In order to study the stability and boundedness properties of impulsive differential systems with respect to ITD, firstly we will discuss some of the definitions as given below:
Definition 2.1 [11]. Let $z(t) = x(t; t_0, x_0) - y(t + \gamma; \tau_0, y_0)$ such that $z_0 = x_0 - y_0$. Then, the solution $x(t) = x(t; t_0, x_0)$ of impulsive differential system (1) is:

(S1) practically equistable with respect to the solution $y(t) = y(t; \tau_0, y_0)$ of the system (1) through (τ_0, y_0), if for given (μ, A) with $0 < \mu < A$, there exists a $\sigma = \sigma(\mu, A) > 0$ such that $\|z_0\| < \mu$ and $|\gamma| < \sigma$ implies $\|z(t)\| < A,\ t \geq t_0$;
(S2) uniformly practically stable if (S1) holds for all $t_0 \in R_+$;
(S3) practically quasi- stable, with respect to the solution $y(t) = y(t; \tau_0, y_0)$ of the system (1) through (τ_0, y_0), if for given (μ, B, T), there exists a $\sigma^* = \sigma^*(\mu, B, T) > 0$ such that $\|z_0\| < \mu$ and $|\gamma| < \sigma^*$ implies $\|z(t)\| < B,\ t \geq t_0 + T$;
(S4) uniformly practically quasi- stable, if (S3) holds for all $t_0 \in R_+$;
(S5) strongly practically stable, if (S1) and (S3) both hold simultaneously ;
(S6) uniformly strongly practically stable, if (S2) and (S4) both hold simultaneously.

Definition 2.2 [11]. Let $z(t) = x(t; t_0, x_0) - y(t + \gamma; \tau_0, y_0)$ such that $z_0 = x_0 - y_0$. Then, the solution $x(t) = x(t; t_0, x_0)$ of impulsive differential system (1) is:

(B1) equibounded with respect to the solution $y(t) = y(t; \tau_0, y_0)$ of the system (1) through (τ_0, y_0), if for given $\alpha > 0$, there exist a $\sigma = \sigma(t_0, \alpha) > 0$ and $\beta = \beta(t_0, \alpha) > 0$ such that $\|z_0\| < \alpha$ and $|\gamma| < \sigma$ implies $\|z(t)\| < \beta,\ t \geq t_0$;
(B2) uniformly bounded if (B1) holds such that both σ and β are independent of $t_0 \in R_+$;
(B3) quasi- ultimately equibounded, with respect to the solution $y(t) = y(t; \tau_0, y_0)$ of the system (1) through (τ_0, y_0), if for given $\alpha > 0$, there exist $\beta > 0$, $\sigma = \sigma(t_0, \alpha) > 0$ and $T = T(t_0, \alpha)$ such that $\|z_0\| < \alpha$ and $|\gamma| < \sigma$ implies $\|z(t)\| < \beta,\ t \geq t_0 + T$;
(B4) quasi-uniformly ultimately bounded, if (B3) hold such that both σ and T are independent of $t_0 \in R_+$;
(B5) ultimately equibounded, if (B1) and (B3) both hold simultaneously;
(B6) uniformly ultimately bounded, if (B2) and (B4) both hold simultaneously.

Definition 2.3 [6]. Let $V : R_+ \times R^n \to R_+$. Then V is said to belong to class V_0 if

(i) V is continuous on each of the sets $(t_{i-1}, t_i] \times R^n$ and for each $x \in R^n$ and $i = 1, 2, 3...$, $\lim_{(t,y)\to(t_i^+,x)} V(t, y) = V(t_i^+, x)$ exists;
(ii) V is Locally Lipschitzian in x.
Define the derivative of the Lyapunov function $V \in V_0$ as:

$$D^+V(t, x) = \lim_{s\to 0^+} sup\frac{1}{s}\left\{V(t + s, x + sf(t, x)) - V(t, x)\right\}$$

Consider the nonlinear impulsive differential system (1), for $V \in PC[R_+ \times R^n \to R_+]$, we define the generalized derivative [9] depending upon the difference of solutions $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$ of the system (1) and a parameter γ, where $\gamma \in [0, \rho]$ for a fixed number $\rho > 0$ as follows:

$$D^+V(t, z; \gamma) = \lim_{s \to 0^+} sup \frac{1}{s} \left[V(t + s, z + s\tilde{f}(t, z; \gamma) - V(t, z)) \right] \tag{2}$$

where $\tilde{f}(t, z; \gamma) = f(t + \gamma, y(t + \gamma)) - f(t, x(t))$ where $z = z(t) = y(t + \gamma) - x(t)$ such that $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$ are two solutions of system (1).

3. Comparison Results

First of all, we will present a comparison Theorem [4] which will be very helpful in our further investigations.

Lemma 3.1. Let the following conditions holds for $t_0, \tau_0, T \in R_+$ where $T > t_0$ and $\gamma = \tau_0 - t_0 \in (0, \rho)$, for a fixed number ρ:

(1) Let $V : [t_0, T] \times R^n \to R_+$ and $V \in V_0$ such that

$$\begin{cases} D^+V(t, z; \gamma) \leq g(t, V(t, z(t)), \gamma) \\ V(t_i^+, z(t_i^+)) \leq J_i(V(t_i, z(t_i)) \end{cases} \tag{3}$$

where $g : [t_0, T] \times R_+ \times [0, \rho] \to R$ is continuous and $J_i : R \to R$ is continuous and non-decreasing.

(2) Let $r(t) = r(t; t_0, w_0, \gamma)$ be the maximal solution of the scaler differential equation such that $w \in R$ and

$$\begin{cases} \dot{w} = g(t, w, \gamma), & t \neq t_i, \\ w(t_i^+) = J_i(w(t_i)), & t = t_i, \quad i = 1, 2, 3... \\ w(t_0) = w_0, & t > t_0 \end{cases} \tag{4}$$

existing for $t \in [t_0, T]$.

Then, $V(t_0, z_0) \leq w_0$ implies that

$$V(t, z(t)) \leq r(t, t_0, w_0, \gamma), \; t \in [t_0, T].$$

Comparison systems: For our investigation, we will use the following two comparison systems:

$$\begin{cases} \dot{v_1} = g_1(t, v_1, \gamma), & t \neq t_i, \\ v_1(t_i^+) = J_i(v_1(t_i)), & t = t_i, \quad i = 1, 2, 3... \\ v_1(t_0) = v_{10} \geq 0 \end{cases} \tag{5}$$

where $v_1(t; t_0, v_{10}, \gamma)$ is any solution of the system (5) whose initial solution at t_0 is v_{10} and

$$\begin{cases} \dot{v_2} = g_2(t, v_2, \gamma), & t \neq t_i, \\ v_2(t_i^+) = F_i(v_2(t_i)), & t = t_i, \quad i = 1, 2, 3... \\ v_2(t_0) = v_{20} \geq 0 \end{cases} \tag{6}$$

where $v_2(t; t_0, v_{20}, \gamma)$ is any solution of the system (6) whose initial solution at t_0 is v_{20}.
Let the comparison systems (5) and (6) satisfy the following assumptions for a fixed number $\rho > 0$:

(C1) $g_1 : R_+ \times R_+ \times [0, \rho] \to R_+$ is continuous function and $J_i : R_+ \to R_+$ is continuous and non-decreasing.

(C2) $g_2 : R_+ \times R_+ \times [0,\rho] \to R_+$ is continuous and $F_i : R_+ \to R_+$ is continuous and non-decreasing.

Next, will obtain a comparison result for investigating the stability and boundedness properties of impulsive differential system(1).

Theorem 3.1. *Let $V_1 : R_+ \times S(\rho) \to R_+$ and $V_1(t,x) \in V_0$ such that*

$$\begin{cases} D^+V_1(t,z;\gamma) + h(t,z(t)) \le g_1(t, V_1(t,z(t)),\gamma), & t \ne t_i \\ V_1(t_i^+, z(t_i^+)) + \int_{t_0}^{t_i} h(s,z(s))ds \le J_i(V_1(t_i, z(t_i))), & t = t_i, i = 1,2,3... \end{cases} \tag{7}$$

where g_1 and J_i satisfy (C1), $h(t,x) : R_+ \times S(\rho) \to R_+$ is piecewise continuous and integrable such that $h(t,x) \ge b_0(\|x\|)$ where $b_0 \in K$. Let $r_1(t;t_0, V_1(t_0,z_0),\gamma)$ be the maximal solution of the comparison system (5).
Then, $V_1(t_0,z_0) \le v_{10}$ implies that $V_1(t,z(t)) + \int_{t_0}^{t} h(s,z(s))ds \le r_1(t)$, $t \ge t_0$.

Proof. Let $x(t) = x(t;t_0,x_0)$ and $y(t) = y(t;\tau_0,y_0)$ be the solutions of the system (1) such that $V(t_0,z_0) \le v_{10}$.

$$\text{Let } G(t,z(t)) = V_1(t,z(t)) + \int_{t_0}^{t} h(s,z(s))ds \tag{8}$$

Then, by using inequalities defined in (7), we get

$$D^+G(t,z(t);\gamma) \le g_1(t,V_1(t,z(t)),\gamma) \quad \text{and}$$

$$G(t_i^+, z(t_i^+)) = V_1(t_i^+, z(t_i^+)) + \int_{t_0}^{t_i} h(s,z(s))ds$$
$$\le J_i(V_1(t_i,z(t_i))$$

Therefore, by Lemma 3.1, we have

$$G(t,z(t)) \le r_1(t) \quad \text{for } t \ge t_0$$

Hence, we get the desired result. □

4. Main Results

Theorem 4.1. *Assume that the following conditions are fulfilled:*
(i) $0 < \mu < A < \rho$;
(ii) Let $V_1 : R_+ \times S(\rho) \to R_+, V_1(t,x) \in V_0$ and $V_1(t,x) \le p_1(t,\|x\|)$ where $p_1 \in C_K$ such that

$$\begin{cases} D^+V_1(t,z(t);\gamma) \le g_1(t,V_1(t,z(t)),\gamma), & t \ne t_i \\ V_1(t_i^+, z(t_i^+)) \le J_i(V_1(t_i,z(t_i)), & t = t_i, i = 1,2,3... \end{cases} \tag{9}$$

where g_1 and J_i satisfy (C1).
(iii) For $0 < \eta < \rho$, there exist a $V_2 \in PC(R_+ \times S(\rho) \bigcap S^c(\eta), R_+)$, where $V_2(t,x) \in V_0$ such that:

$$q(\|x\|) \le V_2(t,x) \le p_2(\|x\|), \quad p_2, q \in K$$

and

$$\begin{cases} D^+[V_1(t,z(t);\gamma) + V_2(t,z(t),\gamma)] \le g_2(t, V_1(t,z(t)) + V_2(t,z(t)),\gamma), & t \ne t_i \\ V_1(t_i^+, z(t_i^+)) + V_2(t_i^+, z(t_i^+)) \le F_i((V_1(t_i,z(t_i)) + V_2(t_i,z(t_i))), & i = 1,2,3... \end{cases}$$

where g_2 and F_i satisfy (C2).
(iv) $p_1(t_0,\mu) + p_2(\mu) < q(A)$, for some $t_0 \in R$

(v) For the comparison system (5),
$v_{10} < p_1(t_0, \mu)$ *implies* $v_1(t; t_0, v_{10}, \gamma) < p_1(t_0, \mu)$
and for the comparison system (6),
$v_{20} < p_1(t_0, \mu) + p_2(\mu)$ *implies* $v_2(t; t_0, v_{20}, \gamma) < q(A)$
Then, the solution $x(t) = x(t; t_0, x_0)$ *of system (1) is practically equistable.*

Proof. Let $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$ be two solutions of the system (1) at different initial times t_0 and τ_0 respectively.
Let for given $0 < \mu < A$, there exists a $\sigma(\mu, A) \geq 0$, such that $\|z_0\| < \mu$ and $|\gamma| \leq \sigma$ are satisfied.
We claim that for above conditions, $\|z(t)\| < A$ holds for $t \geq t_0$.
If it is not true, then for the given solutions of the system (1) we have $t_2 > t_1 > t_0$ such that

$$\|z(t_1)\| = \mu; \;\; \|z(t_2)\| = A \text{ and } \mu < \|z(t)\| < A \text{ for } t_1 < t < t_2 \tag{10}$$

Set $v_{10} = V_1(t_0, z_0)$.
Since $\|z_0\| < \mu$, hence by condition (ii) of Theorem 4.1, we have
$v_{10} = V_1(t_0, z_0) \leq p_1(t_0, \|z_0\|) \leq p_1(t_0, \mu)$ holds.
As all the conditions of Lemma 3.1 are satisfied by using condition ii of Theorem 4.1, we have

$$V_1(t, z(t)) \leq r_1(t), \;\; t \in [t_0, t_1] \tag{11}$$

where $r_1(t; t_0, V_1(t_0, z_0), \gamma)$ is the maximal solution of the comparison system (5).
Thus, from condition (v) of Theorem 4.1 and inequality (11), we have

$$V_1(t_0, z_0) \leq p_1(t_0, \mu) \text{ implies } V_1(t_1, z(t_1)) \leq p_1(t_0, \mu)$$

Also, from condition (iii) and (10), we get $V_2(t_1, z(t_1)) \leq p_2(\|z(t_1)\|) = p_2(\mu)$.
Hence, again by using condition (iii) of Theorem 4.1 and Lemma 3.1, $V_1(t_1, z(t_1)) + V_2(t_1, z(t_1)) \leq p_1(t_0, \mu) + p_2(\mu)$, implies

$$V_1(t, z(t)) + V_2(t, z(t)) \leq r_2(t), \;\; t \in [t_1, t_2] \tag{12}$$

where $r_2(t; t_1, V_1(t_1, z(t_1)) + V_2(t_1, z(t_1)), \gamma)$ is the maximal solution of the comparison system (6).
Thus, from condition (v) of Theorem 4.1 and inequality (12), we have
$V_1(t_2, z(t_2)) + V_2(t_2, z(t_2)) < q(A)$.
But, as $V_1(t_2, z(t_2)) \geq 0$, by condition (iii) and (10), we have
$V_1(t_2, z(t_2)) + V_2(t_2, z(t_2)) \geq V_2(t_2, z(t_2)) \geq q(\|z(t_2)\|) = q(A)$
which is a contradiction.
Hence, $\|z(t)\| < A, \;\; t \geq t_0$ holds, which proves that the system (1) is practically equistable.
□

Remark 4.1: If in condition (i) of Theorem 4.1, $p_1 \in C_k$ is replaced as $p_1 \in K$ and accordingly take $p_1 = p_1(\mu)$ in condition (iv) and (v), then the system (1) is uniformly practically stable.

Theorem 4.2. *Assume that all the conditions of Theorem 4.1 holds except condition (ii), which is replaced as follows:*
Let $V_1 : R_+ \times S(\rho) \to R_+, V_1(t, x) \in V_0$ *and* $V_1(t, x) \leq p_1(t, \|x\|)$ *where* $p_1 \in C_K$ *such that*

$$\begin{cases} D^+V_1(t, z(t); \gamma) + h(t, z(t)) \leq g_1(t, V_1(t, z(t)), \gamma), & t \neq t_i \\ V_1(t_i^+, z(t_i^+)) + \displaystyle\int_{t_0}^{t_i} h(s, z(s))ds \leq J_i(V_1(t_i, z(t_i))), & t = t_i, i = 1, 2, 3... \end{cases} \tag{13}$$

where g_1 and and J_i satisfy (C1) , $h(t,x) : R_+ \times S(\rho) \to R_+$ is piecewise continuous and integrable such that $h(t,x) \geq b_0(\|x\|)$ where $b_0 \in K$. Then, the solution $x(t) = x(t;t_0,x_0)$ of system (1) is strongly practically stable.

Proof. Since, the system (1) is practically stable by Theorem 4.1. In order to prove that the system (1) is strongly practically stable, it is sufficient to prove its quasi- practical stability.
Let $0 < \mu < B$ be given and choose $T > \frac{p_1(t_0,\mu)}{b_0(B)}$.
We claim that for given (μ, B, T) and $\sigma^* = \sigma^*(\mu, B, T) > 0$, $\|z_0\| < \mu$ and $|\gamma| < \sigma^*$ implies

$$\|z(t)\| < B, \ \ t \geq t_0 + T \tag{14}$$

In order to obtain (14), we claim that there exists a $t^* \geq t_0 + T$, such that $h(t,x) < b_0(B)$ holds.
If it doesn't holds, then $h(t,x) \geq b_0(B)$ for all $t \in [t_0, t^*]$.
As all the conditions of Theorem 3.1 are satisfied by using inequalities (13), we have

$$\begin{aligned} 0 \leq V_1\left(t^*, z(t^*)\right) &\leq r_1(t^*, t_0, z_0) - \int_{t_0}^{t^*} h(s, z(s)ds \\ &\leq p_1(t_0,\mu) - \int_{t_0}^{t_0+T} h(s, z(s)ds \\ &\leq p_1(t_0,\mu) - b_0(B)T \\ &\leq p_1(t_0,\mu) - b_0(B)\frac{p_1(t_0,\mu)}{b_0(B)} \\ &< 0 \end{aligned}$$

which is a contradiction.
Hence, $h(t,x) < b_0(B)$ holds for $t \geq t_0 + T$.
Thus, $b_0(\|z(t)\|) \leq h(t, z(t)) < b_0(B)$ implies

$$\|z(t)\| < B \text{ for } t \geq t_0 + T$$

Hence, the system (1) is strongly practically stable. □

Remarks 4.2: In Theorem 4.2, if $p_1 \in C_k$ is replaced as $p_1 \in K$, then the system (1) is uniformly strongly practically stable.

Boundedness

Next, we will discuss the boundedness properties of the impulsive differential system (1).

Theorem 4.3. *Assume that the following conditions are fulfilled:*
(I) Let there exists a $0 < \rho_0 < \rho$, such that $V_1 : R_+ \times S^c(\rho_0) \to R_+$, where $V_1(t,x) \in V_0$ is bounded on $R_+ \times S^c(\rho_0)$ and

$$\begin{cases} D^+V_1(t, z(t); \gamma) \leq g_1(t, V_1(t, z(t)), \gamma), & t \neq t_i \\ \quad V_1(t_i^+, z(t_i^+)) \leq J_i(V_1(t_i, z(t_i)), & t = t_i, i = 1, 2, 3... \end{cases} \tag{15}$$

where g_1 and J_i satisfy (C1).
(II) Let there exists $V_2 : R_+ \times S^c(\rho) \to R_+$, where $V_2(t,x) \in V_0$ such that:

$$q(\|x\|) \leq V_2(t,x) \leq p(\|x\|)$$

where $p, q \in K$ and $q(u) \to \infty$ as $u \to \infty$ and

$$\begin{cases} D^+[V_1(t, z(t); \gamma) + V_2(t, z(t), \gamma)] \leq g_2(t, V_1(t, z(t)) + V_2(t, z(t)), \gamma), & t \neq t_i \\ V_1(t_i^+, z(t_i^+)) + V_2(t_i^+, z(t_i^+)) \leq F_i(V_1(t_i, z(t_i) + V_2(t_i, z(t_i)), & t = t_i, i = 1, 2, 3... \end{cases}$$

where g_2 and F_i satisfy (C2).
(III) The comparison system (5) and (6) are equibounded and uniformly equibounded respectively.
Then, the solution $x(t) = x(t; t_0, x_0)$ of system (1) equibounded.

Proof. Let $\alpha \geq \rho$ be given.
Let $\alpha^* =$ Sup $\{V_1(t_0^+, z_0) : \rho_0 \leq z_0 \leq \alpha\}$ and
$\alpha^{**} =$ Sup $\{V_1(t, z(t) : (t, z) \in (R_+ \times \partial S(\rho_0))\}$
Take $\alpha_1 =$ max $\{\alpha^*, \alpha^{**}\}$.
Since the comparison system (5) is equibounded, therefore for given $\alpha_1 > 0$, there exists a $\beta_1 = \beta_1(t_0, \alpha_1)$ and $\sigma_1 = \sigma_1(t_0, \alpha_1)$, such that $v_{10} \leq \alpha_1$ and $|\gamma| \leq \sigma_1$ implies

$$v_1(t; t_0, v_{10}, \gamma) < \beta_1, \quad t \geq t_0 \tag{16}$$

As, the comparison system (6) is uniformly equibounded, hence for $\alpha_2 = p(\alpha) + \beta_1 > 0$, there exists $\beta_2(\alpha_2) > 0$ and $\sigma_2 = \sigma_2(\alpha_2) > 0$ such that $v_{20} \leq \alpha_2$ implies

$$v_2(t; t_0, v_{20}, \gamma) < \beta_2(\alpha_2), \quad t \geq t_0 \tag{17}$$

Also, $q(u) \to \infty$ as $u \to \infty$, choose $\beta = \beta(t_0, \alpha)$ such that

$$q(\beta) > \beta_2 \tag{18}$$

In order to prove that the solution $x(t) = x(t; t_0, x_0)$ of system (1) is equibounded, we need to prove that for $\|z_0\| \leq \alpha$ and $|\gamma| \leq \sigma$, implies $\|z(t)\| < \beta, \ t \geq t_0$.
If it is not true, let there exists a solution $x(t) = x(t; t_0, x_0)$ of system (1) such that $\|z_0\| \leq \alpha$ and $|\gamma| \leq \sigma$ holds for some $t_2 > t_1 > t_0$ with the following conditions:

$$\{\|z(t_1)\| = \alpha; \ \|z(t_2)\| = \beta \text{ and } \alpha < \|z(t)\| < \beta \quad \text{for } t_1 < t < t_2 \tag{19}$$

Set $v_{10} = V_1(t_0, z_0)$ and $v_{20} = V_1(t_1, z(t_1)) + V_2(t_1, z(t_1))$
As, $v_{10} \leq V_1(t_0, z_0)$, by using condition (I) of Theorem 4.3 and Lemma 3.1, we get

$$V_1(t, z(t)) \leq r_1(t), \ \ t \in [t_0, t_1] \tag{20}$$

where $r_1(t; t_0, V_1(t_0, z_0), \gamma)$ is the maximal solution of the comparison system (5).
Hence by using (16) and (20), we have

$$V_1(t_1, z(t_1)) \leq \beta_1 \tag{21}$$

Also, by condition (II) of Theorem 4.3 and (19), we have

$$V_2(t_1, z(t_1)) \leq p(\|z(t_1)\| = p(\alpha)$$

$$\text{Therefore, } v_{20} = V_1(t_1, z(t_1)) + V_2(t_1, z(t_1)) < \beta_1 + p(\alpha) = \alpha_2 \tag{22}$$

Again, by using (22), condition (II) of Theorem 4.3 and Lemma 3.1, we get

$$V_1(t, z(t)) + V_2(t, z(t)) \leq r_2(t), \ \ t \in [t_1, t_2] \tag{23}$$

where $r_2(t; t_1, V_1(t_1, z(t_1)) + V_2(t_1, z(t_1)), \gamma)$ is the maximal solution of the comparison system (6).
Hence, by using inequalities (17) and (23), we have

$$V_1(t_2, z(t_2)) + V_2(t_2, z(t_2)) < \beta_2 < q(\beta)$$

But, as $V_1(t_1, z(t_1)) \geq 0$, by condition (III) and (19), we have
$V_1(t_2, z(t_2)) + V_2(t_2, z(t_2)) \geq V_2(t_2, z(t_2)) \geq q(\|z(t_2)\|) > q(\beta)$

which gives a contradiction.
Hence, $\|z(t)\| < \beta, \quad$ for $t \geq t_0$ holds provided $\|z_0\| \leq \alpha$ and $|\gamma| \leq \sigma$.
If $\alpha < \rho$, in that case we can choose $\beta = \beta(t_0, \alpha) = \beta(t_0, \rho)$ and we get the desired result.
This proves that the system (1) is equibounded. □

Theorem 4.4. *Assume that all the conditions of Theorem 4.3 are satisfied except condition (I), which is replaced as follows:*
Let there exists a $0 < \rho_0 < \rho$, *such that* $V_1 : R_+ \times S^c(\rho_0) \to R_+$, *where* $V_1(t,x) \in V_0$ *is bounded on* $R_+ \times S^c(\rho_0)$ *and*

$$\begin{cases} D^+V_1(t, z(t); \gamma) + h(t, z(t)) \leq g_1(t, V_1(t, z(t)), \gamma), & t \neq t_i \\ V_1(t_i^+, z(t_i^+)) + \displaystyle\int_{t_0}^{t_i} h(s, z(s))ds \leq J_i(V_1(t_i, z(t_i))), & t = t_i, i = 1, 2, 3... \end{cases} \tag{24}$$

where g_1 *and* J_i *satisfy (C1),* $h(t,x) : R_+ \times S(\rho) \to R_+$ *is piecewise continuous and integrable such that* $h(t,x) \geq b_0(\|x\|)$ *where* $b_0 \in K$.
Then, the solution $x(t) = x(t; t_0, x_0)$ *of system (1) is ultimately equibounded.*

Proof. Let $\alpha \geq \rho$
Since the system (1) is equibounded by Theorem 4.3.
Hence for given $\alpha > 0$, let there exists $\sigma = \sigma(t_0, \alpha)$ and $\beta = \beta(t_0, \alpha)$ such that $|\gamma| < \sigma$ and $\|z_0\| < \alpha$ implies

$$\|z(t)\| < \beta; \;\; t \geq t_0 \tag{25}$$

In order to prove that the system (1) is ultimately equibounded, it is sufficient to prove that the system (1) is quasi- ultimately equibounded.
Choose $T > \frac{\beta_1(t_0,\alpha_1)}{b_0(\beta)}$
In order to prove the quasi- ultimately equiboundedness of system (1), we claim that for given $|\gamma| < \sigma$ and $\|z_0\| < \alpha$ implies

$$\|z(t)\| < \beta; \;\; t \geq t_0 + T \tag{26}$$

To obtain (26), firstly we claim that there exist a $t' \geq t_0 + T$, such that $h(t,x) < b_0(\beta)$ holds.
If it doesn't holds, then $h(t,x) \geq b_0(\beta)$ for all $t \in [t_0, t']$.
As all the conditions of Theorem (3.1) are satisfied by using inequalities in (24), we have

$$\begin{aligned} 0 \leq V_1\left(t', z(t')\right) &\leq r_1(t'; t_0, z_0) - \int_{t_0}^{t'} h(s, z(s)ds \\ &\leq \beta_1(t_0, \alpha_1) - \int_{t_0}^{t_0+T} h(s, z(s)ds \\ &\leq \beta_1(t_0, \alpha_1) - b_0(\beta)T \\ &\leq \beta_1(t_0, \alpha_1) - b_0(\beta)\frac{\beta_1(t_0, \alpha_1)}{b_0(\beta)} \\ &< 0 \end{aligned}$$

which is a contradiction.
Hence, $h(t,x) < b_0(\beta)$ holds for $t \geq t_0 + T$.
Thus, $b_0(\|z(t)\|) \leq h(t, z(t)) < b_0(\beta)$ implies

$$\|z(t)\| < \beta \text{ for } t \geq t_0 + T$$

If $\alpha < \rho$, in that case we can choose $\beta = \beta(t_0, \alpha) = \beta(t_0, \rho)$ and we get the desired result.
Hence the system (1) is ultimately equibounded. □

Remark 4.3: In both Theorems 4.3 and 4.4, if the comparison system (5) is uniformly equibounded, then the system (1) is uniformly bounded and uniformly ultimately bounded respectively.

5. Conclusions

In this paper, for the first time, we investigated the practical stability and boundedness criteria for impulsive differential equations with respect to ITD. The stability criteria for impulsive differential equation relative to ITD is initiated by Hristova [4] in the past. Stability and Boundedness criteria are investigated by Li et al. and Song et al. [11, 21] for differential system in relative to ITD. Song and Li [20] and Song et al. [22] studied differential system without impulse effect for various stability and boundedness conditions in relative to ITD by using perturbed Lyapunov function. We have generalized the technique of perturbing Lyapunov function to obtain the sufficient conditions for practical stability and boundedness of impulsive differential system with respect to ITD.

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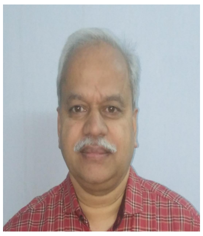

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