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ON SUBCLASSES OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this study, we introduce and investigate two new subclasses of the bi-univalent functions which both $f(z)$ and $f^{-1}(z)$ are m-fold symmetric analytic functions. Among other results, upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are found in this investigation.

Keywords: Univalent functions,Bi-univalent functions, m-fold symmetric functions, m-fold symmetric bi-univalent functions.

AMS Subject Classification: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \,\text{and}\, |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [6]).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

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A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1).

The problem of coefficient bounds of bi-univalent functions dates back to the 1967, when Lewin [10] investigated the class Σ. Following, Brannan and Taha studied with bi-univalent functions [3, 17]. Lewin, Brannan and Taha, thus pioneered the formation of the concept of cornerstone in bi-univalent functions theory. However, in these days a significat amount of theoretical work is done by outstanding mathematicians as Srivastava et al. [12, 13], Ali et al. [1], Çaglar et al. [4], Hamidi and Jahangiri [8], Hussain et al. [9], Şeker [15, 16], Zaprawa [20].

Let $m \in \mathbb{N}$. A domain E is said to be *m-fold symmetric* if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself. It follows that, a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric ($m \in \mathbb{N}$) if

$$f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z).$$

In particular every $f(z)$ is 1-fold symmetric and every odd $f(z)$ is 2-fold symmetric. We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U},\ m \in \mathbb{N}). \tag{3}$$

In [14] Srivastava et al. described the class of m-fold symmetric bi-univalent functions similar to the class of m-fold symmetric univalent functions (Also, see [7, 5, 18, 19, 2]). They obtained that each function $f \in \Sigma$, given by equations (3), constitue an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Also considering the normalized form of f is given by (3), they expressed the Maclaurin series for the inverse of a function as follows:

$$\begin{aligned} g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots \end{aligned} \tag{4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$.

In 1983, Salagean [11] has introduced the following differential operator :

$D^n : \mathcal{A} \to \mathcal{A}$

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1}f(z)) \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

For the functions given by (1.1), we can easily find that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0).$$

The object of the present paper is to introduce new subclasses of the function class bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and obtain coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

2. Coefficient Estimates for the function class $(\mathcal{T}^{t,n}_{\Sigma,m})$

We begin by introducing the function class $(\mathcal{T}^{t,n}_{\Sigma,m})$ by means of the following definition.

Definition 2.1. *A function $f(z)$ given by (3) is said to be in the class $(\mathcal{T}^{t,n}_{\Sigma,m})$ $(0 < \alpha \leq 1;\ n, t \in \mathbb{N}_0\,; t \geq n)$ if the following conditions are satisfied:*

$$f \in \Sigma_m \ \ and \ \ \left| arg\left(\frac{D^t f(z)}{D^n f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{5}$$

and

$$\left| arg\left(\frac{D^t g(w)}{D^n g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}) \tag{6}$$

where the function $g(w)$ is given by (4), D^t and D^n are Salagean differential operators and have the following forms

$$D^t f(z) = z + \sum_{k=1}^{\infty}(mk+1)^t a_{mk+1} z^{mk+1}$$

and

$$D^n g(w) = w + \sum_{k=1}^{\infty}(mk+1)^n b_{mk+1} w^{mk+1}.$$

Theorem 2.1. *Let $f \in (\mathcal{T}^{t,n}_{\Sigma,m})$ $(0 < \alpha \leq 1;\ n, t \in \mathbb{N}_0\,; t \geq n)$ be given by (3). Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha\mu[\lambda^t - \lambda^n] - 2\alpha[\mu^{n+t} - \mu^{2n}] - (\alpha-1)[\mu^t - \mu^n]^2}} \tag{7}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha}{\lambda^t - \lambda^n} + \frac{2\mu\alpha^2}{(\mu^t - \mu^n)^2}. \tag{8}$$

where $\lambda = 2m+1$ and $\mu = m+1$

Proof. From (5) and (6) we have

$$\frac{D^t f(z)}{D^n f(z)} = [p(z)]^{\alpha} \tag{9}$$

and for its inverse map, $g = f^{-1}$, we have

$$\frac{D^t g(w)}{D^n g(w)} = [q(w)]^{\alpha} \tag{10}$$

where $p(z)$ and $q(w)$ are in familiar Caratheodory Class $\mathcal{P}$ (see for details [6]) and have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{11}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{12}$$

Comparing the corresponding coefficients of (9) and (10) yields

$$(\mu^t - \mu^n) a_{m+1} = \alpha p_m \tag{13}$$

$$(\lambda^t - \lambda^n) a_{2m+1} - (\mu^{t+n} - \mu^{2n}) a_{m+1}^2 = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2 \tag{14}$$

$$-(\mu^t - \mu^n) a_{m+1} = \alpha q_m \tag{15}$$

$$(\lambda^t - \lambda^n) \left[\mu a_{m+1}^2 - a_{2m+1}\right] - (\mu^{t+n} - \mu^{2n}) a_{m+1}^2 = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2. \tag{16}$$

From (13) and (15), we get

$$p_m = -q_m \tag{17}$$

and

$$2(\mu^t - \mu^n)^2 a_{m+1}^2 = \alpha^2 (p_m^2 + q_m^2). \tag{18}$$

Also from (14), (16) and (18), we get

$$a_{m+1}^2 = \frac{\alpha^2 (p_{2m} + q_{2m})}{\alpha\mu[\lambda^t - \lambda^n] - 2\alpha[\mu^{n+t} - \mu^{2n}] - (\alpha - 1)[\mu^t - \mu^n]^2}. \tag{19}$$

Note that, according to the Caratheodory Lemma (see [6]), $|p_m| \leq 2$ and $|q_m| \leq 2$ for $m \in \mathbb{N}$. Now taking the absolute value of (19) and applying the Caratheodory Lemma for coefficients p_{2m} and q_{2m} we obtain

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha\mu[\lambda^t - \lambda^n] - 2\alpha[\mu^{n+t} - \mu^{2n}] - (\alpha - 1)[\mu^t - \mu^n]^2}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted (7).

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (16) from (14), we get

$$(\lambda^t - \lambda^n) \left[2a_{2m+1} - \mu a_{m+1}^2\right] = \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2 - q_m^2).$$

Upon substituting the value of a_{m+1}^2 from (18) and observing that $p_m^2 = q_m^2$, it follows that

$$a_{2m+1} = \frac{\alpha(p_{2m} - q_{2m})}{2(\lambda^t - \lambda^n)} + \frac{\mu}{2} \frac{\alpha^2 (p_m^2 + q_m^2)}{2(\mu^t - \mu^n)^2}. \tag{20}$$

Thus, by applying the Caratheodory Lemma again for coefficients p_m, p_{2m} and q_{2m} we find that

$$|a_{2m+1}| \leq \frac{2\alpha}{\lambda^t - \lambda^n} + \frac{2\mu\alpha^2}{(\mu^t - \mu^n)^2}.$$

This completes the proof of the Theorem 2.1. □

3. Coefficient Estimates for the function class $\mathcal{T}_{\Sigma,m}^{t,n}(\beta)$

Definition 3.1. *A function $f(z)$ given by (3) is said to be in the class $\mathcal{T}_{\Sigma,m}^{t,n}(\beta)$ $(0 \leq \beta < 1;\ n, t \in \mathbb{N}_0\,;\ t \geq n)$ if the following conditions are satisfied.*

$$f \in \Sigma_m \text{ and } Re\left\{\frac{D^t f(z)}{D^n f(z)}\right\} > \beta \qquad (z \in \mathbb{U}) \tag{21}$$

and

$$Re\left\{\frac{D^t g(w)}{D^n g(w)}\right\} > \beta \qquad (w \in \mathbb{U}) \tag{22}$$

where the function $g(w)$ is given by (4).

Theorem 3.1. *Let $f \in \mathcal{T}_{\Sigma,m}^{t,n}(\beta)$ $(\ 0 \leq \beta < 1\,;\ n, t \in \mathbb{N}_0\,;\ t \geq n)$ be given by (3). Then*

$$|a_{m+1}| \leq 2\sqrt{\frac{1-\beta}{\mu[\lambda^t - \lambda^n] - 2(\mu^{n+t} - \mu^{2n})}} \tag{23}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{\lambda^t - \lambda^n} + \frac{\mu(1-\beta)^2}{(\mu^t - \mu^n)^2}. \tag{24}$$

where $\lambda = 2m+1$ and $\mu = m+1$

Proof. It follows from (21) and (22) that

$$\frac{D^t f(z)}{D^n f(z)} = \beta + (1-\beta)p(z) \tag{25}$$

and

$$\frac{D^t g(w)}{D^n g(w)} = \beta + (1-\beta)q(w) \tag{26}$$

where $p(z)$ and $q(w)$ have the forms (11) and (12), respectively. Equating coefficients (25) and (26) yields

$$(\mu^t - \mu^n)a_{m+1} = (1-\beta)p_m \tag{27}$$

$$(\lambda^t - \lambda^n)a_{2m+1} - (\mu^{t+n} - \mu^{2n})a_{m+1}^2 = (1-\beta)p_{2m} \tag{28}$$

$$-(\mu^t - \mu^n)a_{m+1} = (1-\beta)q_m \tag{29}$$

$$(\lambda^t - \lambda^n)\left[\mu a_{m+1}^2 - a_{2m+1}\right] - (\mu^{t+n} - \mu^{2n})a_{m+1}^2 = (1-\beta)q_{2m}. \tag{30}$$

From (27) and (29) we get

$$p_m = -q_m \tag{31}$$

and

$$2(\mu^t - \mu^n)^2 a_{m+1}^2 = (1-\beta)^2(p_m^2 + q_m^2). \tag{32}$$

Also from (28) and (30), we obtain

$$\left[(\lambda^t - \lambda^n)\mu - 2(\mu^{t+n} - \mu^{2n})\right] a_{m+1}^2 = (1-\beta)(p_{2m} + q_{2m}). \tag{33}$$

Thus we have

$$\left|a_{m+1}^2\right| \leq \frac{(1-\beta)}{(\lambda^t - \lambda^n)\mu - 2(\mu^{t+n} - \mu^{2n})} \left(|p_{2m}| + |q_{2m}|\right).$$
$$= \frac{4(1-\beta)}{(\lambda^t - \lambda^n)\mu - 2(\mu^{t+n} - \mu^{2n})},$$

which is the bound on $|a_{m+1}|$ as given in the Theorem 3.1.

In order to find the bound on $|a_{2m+1}|$, by subtracting (30) from (28), we get

$$(\lambda^t - \lambda^n)\left(2a_{2m+1} - \mu a_{m+1}^2\right) = (1-\beta)(p_{2m} - q_{2m}) + (1 + 2m\lambda)(m+1)a_{m+1}^2$$

or equivalently

$$a_{2m+1} = \frac{(1-\beta)(p_{2m} - q_{2m})}{2(\lambda^t - \lambda^n)} + \frac{\mu(1-\beta)^2(p_m^2 + q_m^2)}{4(\mu^t - \mu^n)^2}.$$

Applying the Caratheodory Lemma for the coefficients p_m, q_m, p_{2m} and q_{2m}, we find

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{\lambda^t - \lambda^n} + \frac{\mu(1-\beta)^2}{(\mu^t - \mu^n)^2}.$$

which is the bound on $|a_{2m+1}|$ as asserted in Theorem 3.2. □

Remark 3.1. *For 1-fold symmetric bi-univalent functions, if we put $t = 1$ and $n = 0$ in Theorem 2.1 and Theorem 3.1 , we obtain to results which were given by* [3]. *Furthermore, for one-fold symmetric bi-univalent functions in Theorem 2.1 and Theorem 3.1, we obtain to results which were given by* [15].

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