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SZÉKELY, CLARK & ENTRINGER'S AND TALENTI'S INEQUALITIES FOR SUGENO INTEGRAL

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Abstract. The purpose of this paper is to investigate the Székely, Clark & Entringer's and Talenti's inequalities for Sugeno integral. At the first, by an example, we show that Székely, Clark & Entringer's inequality is not valid for Sugeno integral. After that, we state and prove fuzzy version of this inequality. Finally, we state and prove Talenti's inequality for Sugeno integral.

Keywords: Székely, Clark & Entringer's inequality; Talenti's inequality.

AMS Subject Classification: 03E72, 26E50, 28E10.

1. Introduction

In 1974, M. Sugeno introduced fuzzy measures and Sugeno integral for the first time which was an important analytical method of uncertain information measuring [21]. Sugeno integral is applied in many fields such as management decision-making, medical decision-making, control engineering and so on. Many authors such as Ralescu and Adams considered equivalent definitions of Sugeno integral [18]. Román-Flores et al. examined level-continuity of Sugeno integral and H-continuity of fuzzy measures [19, 20]. For more details of Sugeno integral, we refer to [1, 2, 14, 15, 16, 17].

The study of fuzzy integral is attributed to Román-Flores et al. Many inequalities such as Markov's, Chebyshev's, Jensen's, Minkowski's, Hölder's and Hardy's inequalities have been studied by Flores-Franulič and Román-Flores for Sugeno integral (see [12, 13] and their references). Recently, B. Daraby et al. [4, 5, 6, 7, 8, 9, 10, 11] studied some inequalities for Sugeno integral.

In [3], Székely, Clark & Entringer's inequality is given as follows:

If $f \in \mathcal{L}([0,1])$, $f \geq 0$ and $p \geq 1$, then

$$\int_0^1 f(x) \left(\int_0^{1-x} f(t)dt \right)^p dx \leq \left(\int_0^1 f(x) \left(\int_0^{1-x} f(t)dt \right)^{1/p} dx \right)^p . \qquad (1)$$

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Again, in [3], Talenti's inequality is given as follows:
If $a > 0$ and f is positive and decreasing on $[a, b]$, then

$$\log\left(1 + \frac{1}{1 + af(a)} \int_a^b f(t)dt\right) \leq \int_a^b \frac{f(t)}{1 + tf(t)} dt. \tag{2}$$

In this paper, we intend to prove Székely, Clark & Entringer's and Talenti's inequalities for the Sugeno integral.
This paper is organized as follows: in Section 2, we fix the notations and collect all results and preliminaries. In Section 3, we propose the Székely, Clark & Entringer's and Talenti's inequalities for Sugeno integral. Finally, in the last section, a short conclusion is stated.

2. Preliminaries

In this section, we fix some notations and provide some definitions and concepts that are needed.

Throughout this paper, we let X be a non-empty set and Σ be a $\sigma-$algebra of subsets of X.

Definition 2.1. *A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:*

(1) $\mu(\emptyset) = 0$;
(2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ *(monotonicity);*
(3) $A_1 \subseteq A_2 \subseteq \ldots \Rightarrow \lim\limits_{i\to\infty} \mu(A_i) = \mu\left(\bigcup\limits_{i=1}^{\infty} A_i\right)$ *(continuity from below);*
(4) $A_1 \supseteq A_2 \supseteq \ldots$ *and* $\mu(A_1) < \infty \Rightarrow \lim\limits_{i\to\infty} \mu(A_i) = \mu\left(\bigcap\limits_{i=1}^{\infty} A_i\right)$ *(continuity from above).*

When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space.

If f is a non-negative real-valued function on X, we will denote $F_\alpha = \{x \in X \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$, the α-level of f, for $\alpha > 0$. The set $F_0 = \overline{\{x \in X \mid f(x) > 0\}} = \mathrm{supp}(f)$ is the support of f.
If μ is a fuzzy measure on X, we define the following:

$$\mathfrak{F}^\sigma(X) = \{f : X \to [0, \infty) | \quad f \text{ is } \mu - \text{measurable}\}.$$

Definition 2.2. *Let μ be a fuzzy measure on (X, Σ). If $f \in \mathfrak{F}^\sigma(X)$ and $A \in \Sigma$, then the Sugeno integral of f on A is defined by*

$$⨍_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$ and $\wedge$ denotes the operations sup and inf on $[0, \infty]$, respectively and μ is the Lebesgue measure. If $A = X$, the fuzzy integral may also be denoted by $⨍ f d\mu$.

The following proposition gives some of the most elementary properties of Sugeno integral.

Proposition 2.1. *([22]). Let (X, Σ, μ) be a fuzzy measure space, $A, B \in \sum$ and $f, g \in \mathfrak{F}^\sigma(X)$. We have*

(1) $⨍_A f d\mu \leq \mu(A)$;
(2) $⨍_A k d\mu = k \wedge \mu(A)$, *for any constant $k \in [0, \infty)$;*
(3) $⨍_A f d\mu < \alpha \Leftrightarrow$ *there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;*

(4) $⨍_A f d\mu > \alpha \Leftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.1. *Consider the distribution function F associated to f on A, that is to say,*

$$F(\alpha) = \mu(A \cap \{f \geq \alpha\}).$$

Then

$$F(\alpha) = \alpha \Rightarrow ⨍_A f d\mu = \alpha.$$

Thus, from a numerical (or computational) point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$ (if the solution exists).

3. Main results

In this section, we prove Székely, Clark & Entringer's and Talentie's inequalities for Sugeno integral.

3.1. Székely, Clark & Entringer type inequality for Sugeno integral.

At the first, by an example, we show that (1) is not valid for Sugeno integral.

Example 3.1. *Let $p = 2$ and the function $f : [0,1] \to [0,1]$ be defined by*

$$f(x) = \begin{cases} 0 & x = 0 \\ \dfrac{1}{10^4 x} & 0 \leq x \leq 1 \end{cases}$$

Then we have

$$\begin{aligned} ⨍_0^{1-x} f(x)dx &= ⨍_0^{1-x} \frac{1}{10^4 x} dx \\ &= \sup_{\alpha \in [0,1-x]} \left(\alpha \wedge \mu([0,1-x] \cap \left\{ x : \frac{1}{10^4 x} \geq \alpha \right\} \right) \\ &= \sup_{\alpha \in [0,1-x]} \left(\alpha \wedge \mu([0,1-x] \cap \left[0, \frac{1}{10^4 \alpha} \right] \right) \\ &= \sup_{\alpha \in [0,1-x]} \left(\alpha \wedge \frac{1}{10^4 \alpha} \right) \\ &= \frac{1}{10^2}, \end{aligned}$$

and

$$\begin{aligned} ⨍_0^1 \frac{1}{10^4 x} \times \left(\frac{1}{10^2} \right)^2 dx &= ⨍_0^1 \frac{1}{10^8 x} dx \\ &= \sup_{\alpha \in [0,1]} \left(\alpha \wedge \mu([0,1] \cap \left\{ x : \frac{1}{10^8 x} \geq \alpha \right\} \right) \\ &= \sup_{\alpha \in [0,1]} \left(\alpha \wedge \mu([0,1] \cap \left[0, \frac{1}{10^8 \alpha} \right] \right) \\ &= \sup_{\alpha \in [0,1]} \left(\alpha \wedge \frac{1}{10^8 \alpha} \right) \\ &= \frac{1}{10^4}. \end{aligned}$$

Also,

$$\begin{aligned}
⨍_0^1 \frac{1}{10^4 x} \times \left(\frac{1}{10^2}\right)^{\frac{1}{2}} dx &= ⨍_0^1 \frac{1}{10^5 x} dx \\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu([0,1] \cap \left\{x:\ \frac{1}{10^5 x} \geq \alpha\right\}\right) \\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu([0,1] \cap \left[0, \frac{1}{10^5 \alpha}\right]\right) \\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \frac{1}{10^5 \alpha}\right) \\
&= \sqrt{\frac{1}{10^5}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{10^4} &= ⨍_0^1 \frac{1}{10^4 x} \left(⨍_0^{1-x} \frac{1}{10^4 t} dt\right)^2 dx \\
&\geq \left(⨍_0^1 \frac{1}{10^4 x} \left(⨍_0^{1-x} \frac{1}{10^4 t} dt\right)^{1/2} dx\right)^2 = \frac{1}{10^5}.
\end{aligned}$$

Which means the Székely, Clark & Entringer's inequality does not hold for the function f and $p = 2$.

Theorem 3.1. *Let $f : [0,1] \to [0,1]$ be a continuous function and $p \geq 1$. Then we have*

$$⨍_0^1 f(x) \left(⨍_0^{1-x} f(t)dt\right)^p dx \leq ⨍_0^1 f(x) \left(⨍_0^{1-x} f(t)dt\right)^{1/p} dx. \tag{3}$$

Proof. By applying the property on p, we can write

$$\left(⨍_0^{1-x} f(t)dt\right)^p \leq \left(⨍_0^{1-x} f(t)dt\right)^{1/p},$$

now, by multiplication $f(x)$ in both sides of above equation, we have

$$f(x) \left(⨍_0^{1-x} f(t)dt\right)^p \leq f(x) \left(⨍_0^{1-x} f(t)dt\right)^{1/p}.$$

By fuzzy integration from 0 to 1 of both sides, we get

$$⨍_0^1 f(x) \left(⨍_0^{1-x} f(t)dt\right)^p dx \leq ⨍_0^1 f(x) \left(⨍_0^{1-x} f(t)dt\right)^{1/p} dx.$$

□

The following example illustrate the above mentioned theorem.

Example 3.2. *Let $f : [0,1] \to [0,1]$ be defined by $f(x) = 1 - x$ and $p = 3$. We have*

$$\begin{aligned}
⨍_0^{1-x} (1-t)dt &= \sup_{\alpha\in[0,1-x]} \left(\alpha \wedge \mu\left([0, 1-x] \cap \{t:\ 1-t \geq \alpha\}\right)\right) \\
&= \sup_{\alpha\in[0,1-x]} \left(\alpha \wedge [0, 1-\alpha]\right) \\
&= \frac{1}{2},
\end{aligned}$$

hence

$$\begin{aligned}
\fint_0^1 (1-x)\left(\frac{1}{2}\right)^3 dx &= \fint_0^1 \frac{1-x}{8}dx \\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu\left([0,1-x]\cap\left\{x:\ \frac{1-x}{8}\geq\alpha\right\}\right)\right) \\
&= \sup_{\alpha\in[0,1]} (\alpha\wedge[0,1-8\alpha]) \\
&= \frac{1}{9}.
\end{aligned}$$

And

$$\begin{aligned}
\fint_0^1 (1-x)\left(\frac{1}{2}\right)^{1/3} dx &= \fint_0^1 (0.7937)(1-x)dx \\
&= \sup_{\alpha\in[0,1]} (\alpha\wedge\mu([0,1]\cap\{x:\ (0.7937)(1-x)\geq\alpha\})) \\
&= \sup_{\alpha\in[0,1]} \left(\alpha\wedge\left[0,1-\frac{\alpha}{0.7937}\right]\right) \\
&= 0.4424.
\end{aligned}$$

Therefore, we have

$$0.1111 = \frac{1}{9} \leq 0.4424.$$

In the following, we investigate the case of inequality, in which inner integrals are fuzzy integrals and external integrals are Rieman integrals.

Theorem 3.2. *Let $f:[0,1]\to[0,1]$ be a continuous function and $p\geq 1$. Then we have*

$$\int_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^p dx \leq \int_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^{1/p} dx. \tag{4}$$

Proof. By applying the property on p, we can write

$$\left(\fint_0^{1-x} f(t)dt\right)^p \leq \left(\fint_0^{1-x} f(t)dt\right)^{1/p},$$

now, by multiplication $f(x)$ in both sides of above equation, we have

$$f(x)\left(\fint_0^{1-x} f(t)dt\right)^p \leq f(x)\left(\fint_0^{1-x} f(t)dt\right)^{1/p}.$$

By integration from 0 to 1 of both sides, we get

$$\int_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^p dx \leq \int_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^{1/p} dx.$$

Which complete the proof. □

Now, with an example, we show that theorem 3.2 is valid.

Example 3.3. *Let* $f(x) = \dfrac{1}{1+x^2}$ *and* $p = 3$. *We have*

$$\begin{aligned}\int_0^{1-x} \frac{1}{1+t^2} dt &= \sup_{\alpha \in [0,1-x]} \left(\alpha \wedge \mu \left([0, 1-x] \cap \left\{ x : \frac{1}{1+t^2} \geq \alpha \right\} \right) \right) \\ &= \sup_{\alpha \in [0,1-x]} \left(\alpha \wedge \mu \left([0, 1-x] \cap \left[0, \sqrt{\frac{1-\alpha}{\alpha}} \right] \right) \right) \\ &= 0.6823,\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2} (0.6823)^3 \, dx &= (0.6823)^3 \int_0^1 \frac{1}{1+x^2} dx \\ &= 0.3176 \left(\tan^{-1}(x)\Big|_0^1 \right) \\ &= 0.3176 \times \frac{\pi}{4} = 0.2494.\end{aligned}$$

On the other hand, we get

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2} (0.6823)^{1/3} \, dx &= (0.6823)^{1/3} \left(\tan^{-1}(x)\Big|_0^1 \right) \\ &= 0.8803 \times 0.7853 = 0.6913.\end{aligned}$$

By replacing in relation (4), *we have*

$$0.2494 \leq 0.6912.$$

(Notice that, in this example we suppose $\dfrac{\pi}{4} \cong 0.7853$*).*

In the following, we are going to state and prove Talenti's inequality for Sugeno integral.

3.2. **Talenti type inequality for Sugeno integral.**

Theorem 3.3. *Let* $f : [a, b] \to [0, \infty]$ *be decreasing and positive function. Then*

$$\log \left(1 + \frac{1}{1 + af(a)} \wedge \int_a^b f(t) dt \right) \leq \int_a^b \frac{f(t)}{1 + tf(t)} dt, \tag{5}$$

holds, where $a > 0$.

Proof. Since function f is decresing, we have $f(a) \geq f(t)$. We can easily see that $1 + af(a) \geq 1 + tf(t)$. Now, by reversing the two sides, we get

$$\frac{f(t)}{1 + af(a)} \leq \frac{f(t)}{1 + tf(t)}.$$

Now, by fuzzy integration of both sides, we have

$$\int_a^b \frac{f(t)}{1 + af(a)} dt \leq \int_a^b \frac{f(t)}{1 + tf(t)} dt.$$

It can be written according to the fuzzy integral properties

$$\frac{1}{1 + af(a)} \wedge \int_a^b f(t) dt \leq \int_a^b \frac{f(t)}{1 + tf(t)} dt.$$

From properties of log function, we get

$$\log \left(\frac{1}{1 + af(a)} \wedge \int_a^b f(t) dt \right) \leq \frac{1}{1 + af(a)} \wedge \int_a^b f(t) dt.$$

Finally,

$$\log\left(\frac{1}{1+af(a)} \wedge \fint_a^b f(t)dt\right) \leq \fint_a^b \frac{f(t)}{1+tf(t)}dt.$$

□

Now, by an example, we show the validity of Theorem 3.3.

Example 3.4. *Let* $f:[1,2]\to[0,1]$ *be defined by* $f(t)=\dfrac{1}{3-t}$. *Then*

$$\begin{aligned}\fint_1^2 \frac{1}{3-t}dt &= \sup_{\alpha\in[1,2]}\left(\alpha\wedge\mu\left([1,2]\cap\left\{t:\ \frac{1}{3-t}\geq\alpha\right\}\right)\right)\\ &= \sup_{\alpha\in[1,2]}\left(\alpha\wedge\mu\left([1,2]\cap\left[\frac{3\alpha-1}{3},2\right]\right)\right)\\ &= \frac{7}{6}\cong 1.16666,\end{aligned}$$

and

$$\begin{aligned}\fint_1^2 \frac{f(t)}{1+tf(t)}dt &= \fint_1^2 \frac{\frac{1}{3-t}}{1+\frac{t}{3-t}}dt\\ &= \fint_1^2 \frac{1}{3}dt\\ &= \frac{1}{3}\cong 0.3333.\end{aligned}$$

Now, by replacing in (5), *we get*

$$\log(0.6666\wedge 1.16666)\leq 0.3333$$

$$-0.1761=\log(0.6666)\leq 0.3333.$$

4. Conclusion

In this paper, we proved the Székely, Clark & Entringer's and Talenti's inequalities for Sugeno integral. By considering the different initial conditions for the Székely, Clark & Entringer's inequality, we proved this inequality for different forms. Indeed, we showed that:

$$\fint_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^p dx \leq \fint_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^{1/p} dx.,$$

holds, where $f:[0,1]\to[0,1]$ is continuous function and $p\geq 1$. Also the above mentioned inequality is proved when the inequality consists of inner integrals as fuzzy integrals and the external integrals are Riemman integrals as following:

$$\int_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^p dx \leq \int_0^1 f(x)\left(\fint_0^{1-x} f(t)dt\right)^{1/p} dx,$$

where $f:[0,1]\to[0,1]$ and $p\geq 1$.

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References

[1] Agahi, H., Mesiar, R., Ouyang, Y., Pape, E. and Štrboja, M., (2010), Berwald type inequality for Sugeno integral, Applied Mathematics and Computation., 217 , pp. 4100-4108.

[2] Agahi, H., Mesiar, R., Ouyang, Y., Pape, E. and Štrboja, M., (2014), On Stolarsky inequality for Sugeno and Choquet integrals, Information Sciences., 266, pp. 134-139.

[3] Bullen, P. S., (1998), A dictionary of inequalities, Addison Wesley Longman Inc.

[4] Daraby, B., and Arabi L., Related Fritz Carlson type inequality for Sugeno integrals, Soft Comput., 17 (2013), pp. 1745-1750.

[5] Daraby, B., Rostampour, F., Khodadadi, A. R., Rahimi, A. and Mesiar, R., Pacuteolya-Knopp and Hardy-Knopp type inequalities for Sugeno integral, arXiv:1910.03812v1.

[6] Daraby, B., General Related Jensen type Inequalities for fuzzy integrals, TWMS J. Pure Appl. Math. , 8 (2018), pp. 1-7.

[7] Sadeqi, I., Ghazanfary Asll, H. and Daraby, B., Gauss type inequality for Sugeno integral, J. Adv. Math. Stud. , 10 (2017), pp. 167-173.

[8] Daraby, B., Shafiloo, A. and Rahimi, A., General Lyapunov type inequality for Sugeno integral, J. Adv. Math. Stud. , 11 (2018), pp. 37-46.

[9] Daraby, B., Ghazanfary Asll, H. and Sadeqi, I., General related inequalities to Carlson-type inequality for the Sugeno integral, Appl. Math. Comput., 305 (2017), pp. 323-329.

[10] Daraby, B., Ghazanfary Asll, H. and Sadeqi, I., Favard's inequality for seminormed fuzzy integral and semiconormed fuzzy integral, Mathematica, 58 (2016), pp. 39-45.

[11] Daraby B. and Rahimi, A. Jensen type inequality for seminormed fuzzy integrals, Acta Univ. Apulensis, Math. Inform. , (2016), pp. 1-8.

[12] Flores-Franulič, A. and Román-Flores, A., (2007), A Chebyshev type inequality for fuzzy integrals, Applied Mathematics and Computation., 190, pp. 1178-1184.

[13] Flores-Franulič, A., Román-Flores, H. and Chalco-Cano, Y., (2009), Markov type inequalities for fuzzy integrals, Applied Mathematics and Computation., 207, 242-247.

[14] Grabisch, M., (2016), Set functions, Games and Capacities in Decision Making, Springer.

[15] Pap, E., (1995), Null-additive Set Functions, Kluwer, Dordrecht.

[16] Pap, E., (2002), Pseudo-additive measures and their applications, Handbook of Neasure Theory (Editor E. Pap), Volume II, Elsevier, North-Holland., 1403-1465.

[17] Pap, E. and Štrboja, M., (2013), Generalization of inegrals based on nonadditive measures, Topics in Intelligent Engineering and Informations, Intelligent Systems: Models and Applications (Ed. E. Pap), Springer., 3-22.

[18] Ralescu, D. and Adams, J., (1980), The fuzzy integral, Journal of Mathematical Analysis and Applications., 75, 562-570.

[19] Román-Flores, H. and Chalco-Cano, H., (2006), H-continuity of fuzzy measures and set defuzzification, Fuzzy Sets and Systems., 157, 230-242.

[20] Román-Flores, H., Flores-Franulič, A., Bassanezi, R. and Rojas-Medar, M., (1996), On the level-continuity of fuzzy integrals, Fuzzy Sets and Systems., 80, 339-344.

[21] Sugeno, M., (1974), Theory of Fuzzy Integrals and its Applications, (Ph. D. dissertation), Tokyo Institute of Technology.

[22] Wang, Z. and Klir, G. K., (1992), Fuzzy Measure Theory, Plenum Press, New York.

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