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THE METHOD OF FUNDAMENTAL SOLUTIONS FOR THE INVERSE TIME-DEPENDENT PERFUSION COEFFICIENT PROBLEM

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ABSTRACT. This paper deals with an inverse problem associated with the bio-heat equation in living tissue in the human body. The inverse problem consists of the identification of time-dependent perfusion coefficient when the exact and noisy measurements of temperature at a fixed space point x^* are specified. The numerical method for the retrieval of the unknown perfusion coefficient is based on the method of fundamental solutions (MFS). By introducing the fundamental solution of the heat equation and theoretical properties of these solutions, the MFS is used in conjunction with the Tikhonov regularization method. The choice of the regularization parameter is based on L-curve criteria to obtain a stable solution. Our numerical approach for numerical differentiation of discrete noisy data is focused on the iterated Tikhonov method due to ill-posedness of problem. Numerical results show the efficiency and applicability of the proposed algorithm in approximation of unknown perfusion coefficient.

Keywords: Inverse Parabolic Problem, Ill-Posed Problem, Regularization Method, The MFS Method.

AMS Subject Classification: 35A08 ,35R30 , 65F22, 65M80.

1. Introduction

The mathematical model between tissue temperature and blood perfusion has been the point of interest for many studies in medicine and mathematics. During the past two decades, some numerical approaches have been applied for studying the heat transfer models in living tissues in a human body [1, 3, 5, 8, 16].

In the current investigation, the time-dependent coefficient identification problem including determination of the pairs $\big(p(x), u(x,t)\big)$ in inverse parabolic problem is considered. Estimation of the unknown time-dependent coefficient is based on inexact input data contain small noise. So noise in data may lead to a large error in the solution and so the problem is ill-posed. Therefore, regularization techniques like Tikhonov regularization method and other methods have been applied to remove the instability [15]. Our

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numerical method has been developed to deal with this inverse problem is the method of fundamental solution.

Blood perfusion is the local fluid flow through the capillary network and extracellular spaces of living tissue. It is characterized as the volumetric flow rate per volume of tissue. Blood perfusion is vital for normal tissue physiology and waste products. Blood perfusion is principle part of the thermal system of the body. Since the blood perfusion is so important in maintaining normal physiologic conditions, there is a vital need to be able easily and accurately measure it and so the measurement of this quantity is valuable. The mathematical modeling between the relation of temperature and blood perfusion has introduced by Pennes in 1948[12], which is defined by the following equation

$$u_{xx}(x,t) - p(t)u(x,t) = u_t(x,t), \qquad (x,t) \in (0,L) \times (0,T], \tag{1}$$

where T is a final time of interest, u is the temperature of the tissue, $(0,L)$ is the spatial solution domain and the perfusion coefficient

$$p(t) = \frac{\omega_b c_b l^2}{k_t}, \tag{2}$$

where ω_b is the blood perfusion rate, c_b is the specific heat of the blood, l is the reference length of the biological body and k_t is the thermal conductivity of the tissue.

In section 2, the mathematical formulation of the inverse time-dependent coefficient identification problem in tissue is introduced. Section 3 is devoted to a review of the method of fundamental solutions. In section 4, the fundamental solution method is introduced and applied to the inverse problem. Due to the large condition number of the resultant matrix A, the regularization technique should be applied. In our computations, the random errors to be considered in input data to investigate the role of noisy data in output time-dependent coefficient. In section 5, an efficient method is introduced for computing of numerical differentiation with discrete noisy data. In section 6, some test examples are considered to show the accuracy and validity of the presented method. Finally, conclusions are drawn in section 7.

2. Problem Formulation

In this section, we consider the following inverse problem of determining the temperature $u(x,t)$ and the time-dependent perfusion cofficient $p(t)$ in the parabolic heat equation

$$u_{xx}(x,t) - p(t)u(x,t) = u_t(x,t) \qquad 0 < x < 1, 0 < t \leq T. \tag{3}$$

We have to solve the Eq. (3) subject to the initial temperature

$$u(x,0) = u_0(x), \qquad 0 \leq x \leq 1, \tag{4}$$

and boundary conditions

$$u(0,t) = f_0(t), \qquad 0 \leq t \leq T, \tag{5}$$

$$u(1,t) = f_1(t), \qquad 0 \leq t \leq T. \tag{6}$$

and a permanent interior temperature measurement at a fixed space point $x^* \in (0,1)$:

$$u(x^*,t) = g(t), \qquad 0 \leq t \leq T. \tag{7}$$

Let define

$$r(t) = \exp(\int_0^t p(s)ds), \quad t \in [0,T]. \tag{8}$$

The change of variable

$$v(x,t) = r(t)u(x,t), \tag{9}$$

transforms the time-dependent problem (3)-(7) into a constant coefficient heat equation problem, as follows

$$v_{xx}(x,t) = v_t(x,t), \quad 0 < x < 1, 0 < t \leq T, \tag{10}$$

$$v(x,0) = u_0(x), \qquad 0 \leq x \leq 1, \tag{11}$$

$$v(0,t) = r(t)f_0(t), \qquad 0 \leq t \leq T, \tag{12}$$

$$v(l,t) = r(t)f_1(t), \qquad 0 \leq t \leq T, \tag{13}$$

$$v(x^*,t) = r(t)g(t), \qquad 0 \leq t \leq T. \tag{14}$$

We assume that all the functions appearing in heat Eq. (1) and initial and boundary conditions are measurable in order to obtain uniqueness solution for the inverse problem. In the following subsection, we will demonstrate the unique solvability of solution under suitable assumptions. To guarantee the existence and uniqueness of a solution to (3)–(7), we impose the following compatibility conditions of order zero

$$f_0(0) = u_0(0), \; f_1(0) = u_0(1), \; g(0) = u_0(x^*). \tag{15}$$

Further, we need compatibility conditions up to first-order which require condition (15) be satisfied and in addition

$$f_0'(0) = u_0''(0) + \frac{u_0(0)\big(g'(0) - u_0''(x^*)\big)}{u_0(x^*)},$$
$$f_1'(0) = u_0''(1) + \frac{u_0(1)\big(g'(0) - u_0''(x^*)\big)}{u_0(x^*)}, \; g(0) = u_0(x^*) > 0. \tag{16}$$

2.1. **Existence of Unique Solution for the Inverse Problem.** The solvability of the inverse problem (3)-(7) in spaces $\mathcal{L}^{k+\alpha}$, with α fixed in $(0,1)$ and $k \in \mathbb{N}$, of continuous functions with Holder continuous derivatives (see [4, 10]) has been established in [2, 13], as follows

Theorem 2.1. *If* $u_0 \in \mathcal{L}^{2+\alpha}([0,1]), f_0, f_1, g \in \mathcal{L}^{1+\frac{\alpha}{2}}([0,T]), u_0 \geq 0, f_0, f_1 \geq 0, g \geq 0,$ *and the compatibility conditions up to first order are satisfied, then there exists a unique solution* $u \in \mathcal{L}^{2+\alpha,1+\frac{\alpha}{2}}\big([0,1] \times [0,T]\big), p \in \mathcal{L}^{\frac{\alpha}{2}}([0,T])$ *of the inverse problem* (3)-(7) *which is continuously dependent upon data. Remark that the theorem does not guarantee that the solution for p is positive, hence only the uniqueness of the solution* $(u(x,t), p(t) > 0)$ *can be concluded.*

3. A Review of the method of fundamental solutions

Let $D = \{x : x \in (0,1)\}$, $\bar{D} = \{x : x \in [0,1]\}$, $D_T = \{(x,t) : (x,t) \in D \times [0,T]\}$, $\Gamma = \partial D = \{0,1\}$ is boundary of the domain D, D_E is open domain containing $\bar{D}$ and Γ_E is boundary of D_E on which source points are placed.

The fundamental solution of Eq. (10) in one-dimensional case is given by

$$F(x,t;y,\tau) = \frac{H(t-\tau)}{\sqrt{4\pi(t-\tau)}} \exp(-\frac{(x-y)^2}{4(t-\tau)}), \tag{17}$$

where H is the Heaviside function, which is necessary in order to emphasize that the fundamental solution is zero for $t \leq \tau$. We begin by constructing a set of source points placed outside the region $\bar{D}$. Let $\{y_j, \tau_m\}_{j,m=1,2,\ldots}$be a denumerable, everywhere dense set of points in $\Gamma_E \times [-T,T], (\tau_m \neq 0)$. In this paper, we show the source points have

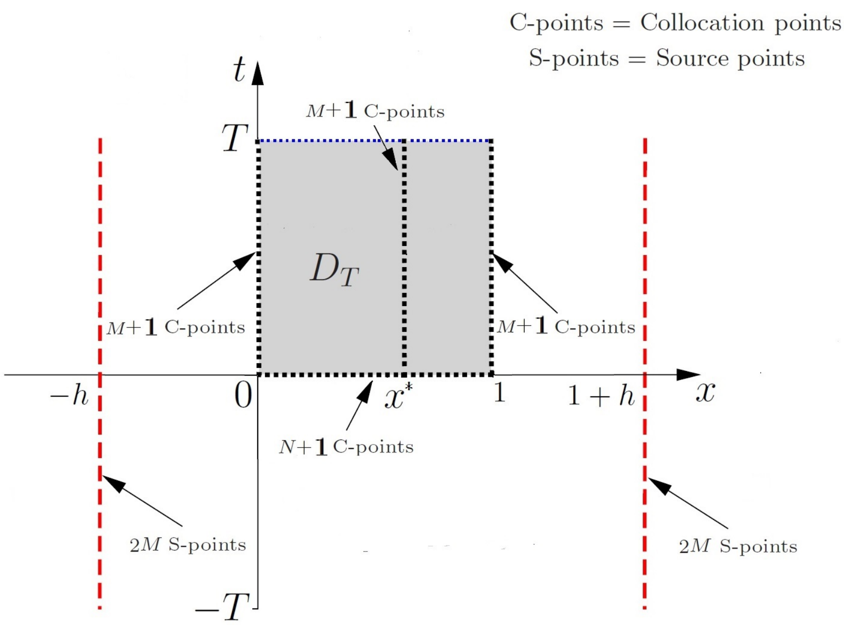

FIGURE 1. The representation of the placement of source and collocation points.

placed around a domain D. We approximate the solution of Eqs. (10)-(14) by a linear combination of fundamental solutions in the form

$$v(x,t) \approx v_M(x,t) = \sum_{j=1}^{2}\sum_{m=1}^{2M} c_m^{(j)} F(x,t;y_j,\tau_m), \quad (x,t) \in \bar{D}, \tag{18}$$

where $c_m^{(j)}$ are unknown coefficients.

We shall investigate some properties of linear combinations of such functions for various source points y_j. The source points will be placed on $y_1 = -h$ and $y_2 = 1 + h$ at time points $\{\tau_m\}_{m=1,\cdots,2M,} \in (-T, T)$ given by

$$\tau_m = \frac{2(m-M)-1}{(2M)}T, \quad m = 1,\cdots,2M, \tag{19}$$

We have $4M$ source points on the external boundary Γ_E, and we place the other number of collocation points on the any fixed space point x^*, lateral and base surfaces $\big(x^* \times [0,T]\big) \cup \big(\Gamma \times [0,T]\big) \cup \big(D \times \{0\}\big)$.
Letting

$$t_i = \frac{i}{M}T, \ i = 0,\cdots,M, \quad x_j = \frac{j}{N}, \ j = 0,\cdots,N. \tag{20}$$

Fig. 1 displays placement of source and collocation points for a one-dimensional domain in time when D is a rectangular domain. Denseness results are given for both the lateral and base surfaces, justifying the use of this MFS. The proofs of the following theorems are similar to the proof of Theorem 2.2 in [9], hence omitted.

Theorem 3.1. ***Denseness on the lateral surface*** *The set of functions* $\{F(x,t;y_j,\tau_m)\}_{m=1}^{\infty}$, $j = 1, 2$ *defined on* $\Gamma \times (-T,T)$ *form a linearly independent and dense set in* $L^2\big(\Gamma \times (-T,T)\big)$.

Theorem 3.2. ***Denseness on the base surface*** *The set of functions* $F(x, 0; y_j, \tau_m)_{m=1}^{\infty}$, $j = 1, 2$, *with* $\tau_m < 0$, *forms a linearly independent and dense set in* $L^2(D)$.

4. Implementation of the MFS for the Inverse problem

From Eqs. (11)-(14) and Eq. (18) we obtain the following system of equations

$$v_M(0, t_i) = r(t_i)f_0(t_i), \quad i = 0, \cdots, M, \tag{21}$$

$$v_M(1, t_i) = r(t_i)f_1(t_i), \quad i = 0, \cdots, M, \tag{22}$$

$$v_M(x^*, t_i) = r(t_i)g(t_i), \quad i = 0, \cdots, M, \tag{23}$$

$$v_M(x_j, 0) = u_0(x_j), \quad j = 0, \cdots, N. \tag{24}$$

The resultant linear algebraic equations can be represented by

$$\mathbf{A}_{(3M+N+4)\times(5M+1)}\mathbf{C}_{(5M+1)\times 1} = \mathbf{b}_{(3M+N+4)\times 1}, \tag{25}$$

where the vectors $\mathbf{C}$ and $\mathbf{b}$ denote the vectors of unknown constant coefficients and known right hand side respectively, as follows

$$\mathbf{C} = \begin{bmatrix} c_1^{(1)} & \cdots & c_{2M}^{(1)} & c_1^{(2)} & \cdots & c_{2M}^{(2)} & r(t_0) & \cdots & r(t_M) \end{bmatrix}^T,$$

and

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & \cdots & 0 & u_0(x_0) & u_0(x_1) & \cdots & u_0(x_N) \end{bmatrix}^T,$$

and $\mathbf{A}$ is a known coefficients Block matrix of order $(3M + N + 4) \times (5M + 1)$ which is defined by

$$\mathbf{A} = \left[\begin{array}{c|c|c} \big(F(0, t_i; y_1, \tau_m)\big)_{m=1,\cdots,2M}^{i=0,\cdots,M} & \big(F(0, t_i; y_2, \tau_m)\big)_{m=1,\cdots,2M}^{i=0,\cdots,M} & d_{i,j} = \begin{cases} 0 & i \neq j \\ -f_0(t_i) & i = j \end{cases} \\ \hline \big(F(1, t_i; y_1, \tau_m)\big)_{m=1,\cdots,2M}^{i=0,\cdots,M} & \big(F(1, t_i; y_2, \tau_m)\big)_{m=1,\cdots,2M}^{i=0,\cdots,M} & d_{i,j} = \begin{cases} 0 & i \neq j \\ -f_1(t_i) & i = j \end{cases} \\ \hline \big(F(x^*, t_i; y_1, \tau_m)\big)_{m=1,\cdots,2M}^{i=0,\cdots,M} & \big(F(x^*, t_i; y_2, \tau_m)\big)_{m=1,\cdots,2M}^{i=0,\cdots,M} & d_{i,j} = \begin{cases} 0 & i \neq j \\ -g(t_i) & i = j \end{cases} \\ \hline \big(F(x_j, 0; y_1, \tau_m)\big)_{m=1,\cdots,2M}^{j=0,\cdots,N} & \big(F(x_j, 0; y_2, \tau_m)\big)_{m=1,\cdots,2M}^{j=0,\cdots,N} & (d_{i,j})_{i=0,\cdots,M}^{j=0,\cdots,N} = 0 \end{array}\right]$$

This system contains $3M + N + 4$ equations and $5M + 1$ unknowns. In order to obtain a unique solution, we require $N \geq 2M - 3$. Since the resulting matrix is ill-conditioned so applying the regularization technique is necessary. We will apply Tikhonov regularization, with L-curve criterion.

By employing the Tikhonov regularization method, the ill-conditioned linear system (25) is replaced by the new system of linear equations $(A^TA + \lambda\mathbf{I})\mathbf{C} = A^T\mathbf{b}$, where A^T is the transpose of $\mathbf{A}$ and λ is found by L-curve criterion, as suggested by Hansen [6, 7]. By solving Eq. (25) the numerical approximations of $r(t_i)$ are found.

We implement the proposed method with Matlab 2018a software in a personal computer. In next section, we introduce an efficient method for computing of numerical differentiation of $r(t)$ at discrete points t_i with known values of $r(t_i)$.

5. Numerical differentiation of $r(t_i)$ with discrete noisy data

We are interested in finding an approximation of $r'(t_i)$ from the given data $\{r_i^\varepsilon\}$ and substituting into $p(t) = \frac{r'(t)}{r(t)}$. We can write the numerical differentiation of a smooth function $r(t)$ as a Volterra integral equation

$$Ax(t) := 1 + \int_0^t x(\tau)d\tau = r(t),\ 0 \le t \le 1. \tag{26}$$

Below we assume that

$$r(0) = r_0^\varepsilon = 1, \tag{27}$$

i.e., the initial data are known exactly. Then it is clear that $x(t) = r'(t)$ is a unique solution of Eq. (26).

On the other hand, since only the noisy measurements $\{r_i^\varepsilon\}$ are available, one has a new equation

$$Ax = r,\ \|r - r_\delta\|_{L_2} \le \delta, \tag{28}$$

where r_δ and A are given and x and r are unknown.

The Tikhonov method for a noisy linear equation (28) consists of determining the regularized approximation x_α^δ as a unique solution of the following equation

$$\alpha x_\alpha^\delta + A^* A x_\alpha^\delta = A^* r_\delta, \tag{29}$$

where α is regularization parameter.

By applying the iterated Tikhonov method of order p, the regularized approximation $x_{\alpha,p}^\delta$ is determined by the recursion

$$\begin{aligned} &\alpha x_{\alpha,l}^\delta + A^* A x_{\alpha,l}^\delta = \alpha x_{\alpha,l-1}^\delta + A^* r_\delta,\ l = 1, 2, \cdots, p, \\ &x_{\alpha,0}^\delta = 0,\ x_{\alpha,1}^\delta = x_\alpha^\delta, \end{aligned} \tag{30}$$

i.e., the equation of the form (29) should be solve p times. The discrete form of the Eqs. (29), (30) can be constructed based on the Galerkin method. By considering the space of piecewise linear functions $V_{m+1} = span\{\varphi_i^m\}_{i=0}^m$, where

$$\varphi_i^m(t) = \varphi_i^{\sigma_m}(t),\ i = 1, 2, \cdots, m,\ \sigma_m = \{\frac{i}{m}\}_{i=0}^m,\ \varphi_0^m(t) = \varphi_m^m(1-t),$$

where piecewise linear interpolation functions φ_i^σ are defined as follows

$$\varphi_i^\sigma(t) = \begin{cases} \frac{t-t_{i-1}}{t_i - t_{i-1}}, & t \in [t_{i-1}, t_i], \\ \frac{t_{i+1}-t}{t_{i+1}-t_i}, & t \in [t_i, t_{i+1}], \\ 0, & t \notin [t_{i-1}, t_{i+1}], \end{cases} \qquad i = 1, 2, \cdots, m-1,$$

$$\varphi_m^\sigma(t) = \begin{cases} \frac{t-t_{m-1}}{t_m - t_{m-1}}, & t \in [t_{m-1}, t_m], \\ 0, & t \notin [t_{m-1}, t_m]. \end{cases}$$

Then the Galerkin approximation $x_{\alpha,l,m}^\delta$ of $x_{\alpha,l}^\delta$ has the form

$$x_{\alpha,l,m}^\delta(t) = \sum_{i=0}^m z_i^l \varphi_i^m(t), \tag{31}$$

and should solve the variational problem

$$\langle \nu, \alpha x_{\alpha,l,m}^\delta + A^* A x_{\alpha,l,m}^\delta - \alpha x_{\alpha,l-1,m}^\delta - A^* r_\delta \rangle = 0, \tag{32}$$

for all $\nu \in V_{m+1}$, where $\langle .,. \rangle$ is the standard inner product in Hilbert space $L_2(0,1)$. It is convenient to rewrite (32) as the following system of linear algebraic equations with respect to unknown coefficients z_i^l. From (31)

$$\alpha \sum_{i=0}^{m} z_i^l \langle \varphi_i^m, \varphi_j^m \rangle + \sum_{i=0}^{m} z_i^l \langle A\varphi_i^m, A\varphi_j^m \rangle = \alpha \sum_{i=0}^{m} z_i^{l-1} \langle \varphi_i^m, \varphi_j^m \rangle + \langle A\varphi_j^m, r_\delta \rangle, \quad (33)$$

$$j = 0, 1, \cdots, m, \; l = 1, 2, \cdots, p, \; z_i^0 = 0, \; i = 0, 1, \cdots, m,$$

The choice of the discretization parameter m and the regularization parameter α is crucial, we have the possibility to choose m in such a way that the error caused by the discretization will be dominated by the regularization error and for the selection of regularization parameter, we can refer to [11].

6. Numerical Experiment and Results

In order to assess the validity and accuracy of the numerical algorithm, we compare MFS solutions with the available exact solutions for various test examples. Numerical results are presented for different values of N, M and h, which were found to be sufficiently large to ensure that any further increasing in these numbers did not significantly improve the accuracy of the numerical solutions. The stability of the proposed method for test examples is investigated when some perturbed approximation of Eq. (7) is in hand. We examine the effect of this perturbed function on the solution of problem. The perturbed data at point t_i is produced by adding a noise ϵ_i to exact data $g(t)$ as follows

$$u_\epsilon(x^*, t_i) = g(t_i) + \epsilon_i, \; i = 0, 1, ..., N,$$

where ϵ_i are random variables which are generated from a Gaussian normal distribution with zero mean and standard deviation σ given by

$$\sigma = \delta \times Max_{t\in[0,1]} \mid g(t) \mid,$$

where δ represents the percentage of noise. The *normrand* command in MATLAB is used to generate the random variables $(\epsilon_i)_{i=0,1,...,N}$.

Also, we presents the root mean square error (RMSE) defined by

$$RMSE(p(t)) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [p(t_i) - p_{approx}(t_i)]^2},$$

to measure the values of error in our computations.

6.1. **Example 1.**

Let us consider the inverse problem (3)-(7) with input data

$$u_0(x) = x^2, \; f_0(t) = 2te^{(-t-\frac{t^2}{2})}, \; f_1(t) = (1+2t)e^{(-t-\frac{t^2}{2})},$$

$$g(t) = (\frac{1}{4} + 2t)e^{(-t-\frac{t^2}{2})}.$$

It is easy to check that the exact solutions of the problem (3)-(7) are $u(x,t) = (x^2 + 2t)e^{(-t-\frac{t^2}{2})}$ and $p(t) = 1 + t$. We take $x^* = 0.5$ and $T = 1$.

The value of $h > 0$ will be chosen appropriately. However, the accuracy of the approximation appears to decrease when $h < .5$ or $h > 1.5$.

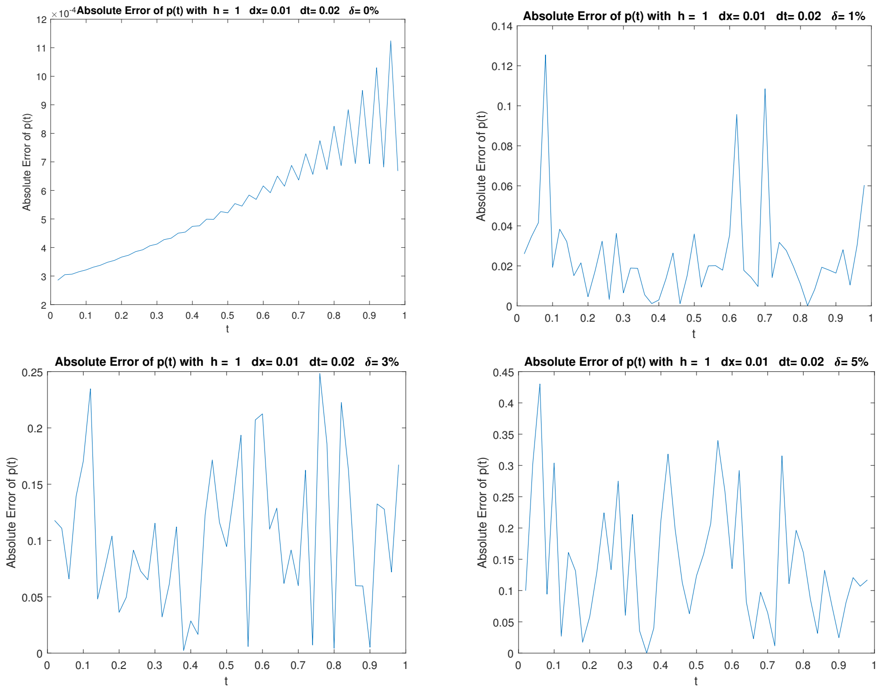

FIGURE 2. The absolute error of the numerical solution of $p(t)$ for different values of noise level δ with $M = 50, N = 100, h = 1$ for example 1.

TABLE 1. The comparison between the exact and numerical solutions of $p(t)$ for different values of M, N, h for example 1.

t	Exact solution	Numerical solution $M{=}50, N{=}100, h{=}.5$	Numerical solution $M{=}50, N{=}100, h{=}1$	Numerical solution $M{=}50, N{=}100, h{=}1.5$	Numerical solution $M{=}100, N{=}200, h{=}1$
0.0	1.00	1.020203	1.020272	1.020270	1.010067
0.1	1.10	1.103579	1.100322	1.100309	1.100077
0.2	1.20	1.204146	1.200366	1.200355	1.200089
0.3	1.30	1.303882	1.300412	1.300407	1.300102
0.4	1.40	1.403526	1.400474	1.400463	1.400116
0.5	1.50	1.503220	1.500522	1.500525	1.500131
0.6	1.60	1.602984	1.600616	1.600593	1.600148
0.7	1.70	1.702809	1.700636	1.700668	1.700167
0.8	1.80	1.802687	1.800825	1.800749	1.800187
0.9	1.90	1.902607	1.900694	1.900837	1.900209
1.0	2.00	1.952520	1.948481	1.950919	1.975231
$RMSE$	$--$	$6.2076e-05$	$6.0432e-05$	$5.5615e-05$	$7.0982e-06$

Fig. 2 displays the behavior of the absolute error for numerical solution of $p(t)$, when the overspecified condition (7) is contaminated by $\delta \in \{1, 3, 5\}\%$ noise. This figure shows that when the overdetermination condition (7) is perturbed by noise then the numerical approximation becomes unstable due to the noise.

The numerical results for comparison between the exact source term $p_{exact}(t)$ given in this example and the computed solution $p_{approx}(t)$ for different values of M, N and h along with their RMSE are shown in Table 1 and the comparison between the exact source term $p_{exact}(t)$ and the approximate solution $p_{approx}(t)$ for different values of noise level δ with their RMSE are presented in Table 2. From the figures and tables for this example, it can be been seen that when the noise level decrease to zero, the computed solution goes to the exact solution and the obtained numerical approximations are stable in the presence of noise in input data.

Table 2. The comparison between the exact and numerical solutions of $p(t)$ for different values of noise level with $M = 50, N = 100, h = 1$ for example 1.

t	Exact solution	Numerical solution $M{=}50, N{=}100, h{=}1, \delta{=}0\%$	Numerical solution $M{=}50, N{=}100, h{=}1, \delta{=}1\%$	Numerical solution $M{=}50, N{=}100, h{=}1, \delta{=}3\%$	Numerical solution $M{=}50, N{=}100, h{=}1, \delta{=}5\%$
0.0	1.00	1.020272	0.989119	0.863451	0.091973
0.1	1.10	1.100322	1.119308	0.929579	1.403863
0.2	1.20	1.200366	1.195399	1.163791	1.258247
0.3	1.30	1.300412	1.293478	1.184647	1.360364
0.4	1.40	1.400474	1.397027	1.428592	1.187839
0.5	1.50	1.500522	1.535932	1.594619	1.376355
0.6	1.60	1.600616	1.635583	1.812464	1.735068
0.7	1.70	1.700636	1.808493	1.759823	1.635478
0.8	1.80	1.800825	1.810818	1.795761	1.800749
0.9	1.90	1.900694	1.916411	1.894724	1.638915
1.0	2.00	1.948481	1.929861	2.364996	1.924565
$RMSE$	$--$	$6.0432e-05$	0.0013	0.0174	0.0456

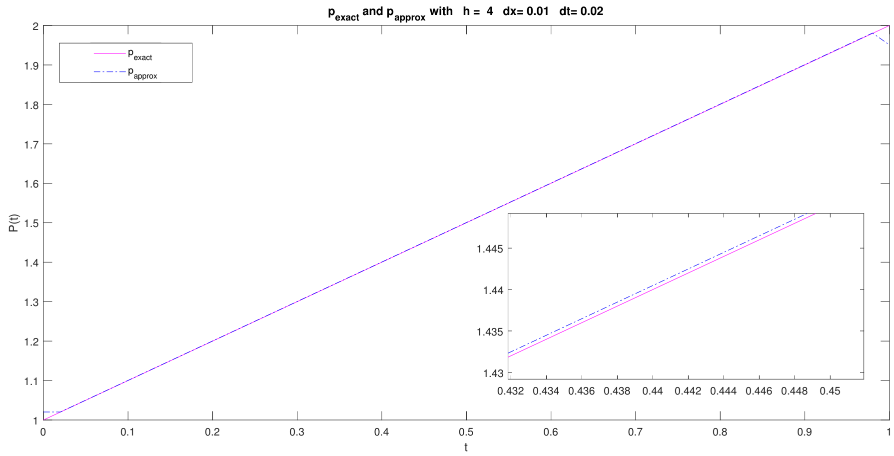

Figure 3. The comparison between the exact and numerical solutions of $p(t)$ with $M = 50, N = 100, h = 4$ and $\lambda = 5.9383 \times 10^{-14}$ for example 2.

6.2. **Example 2.** Consider the following inverse problem (3)-(7) with input data

$$u_0(x) = sin(\pi x), \ f_0(t) = 0, \ f_1(t) = 0,$$
$$g(t) = e^{(-t(\pi^2+1)-\frac{t^2}{2})}.$$

and $T = 1$. The exact solutions of the inverse problem (3)-(7) are as follow: $u(x,t) = sin(\pi x)e^{(-t(\pi^2+1)-\frac{t^2}{2})}$ and $p(t) = 1 + t$. We take $x^* = 0.5$ in our computations.

The accuracy of the approximation appears to decrease when $h < 1$ or $h > 7$. In Fig. 3 the exact solution and the MFS approximation $p(t)$ is plotted with $\lambda = 5.9383 \times 10^{-14}$ in the Tikhonov regularization for $M = 50, N = 100, h = 4$ when the input data given by Eq. (7) is exact.

Similar to previous example, the numerical results for comparison between the exact source term $p_{exact}(t)$ given in this example and the computed solution $p_{approx}(t)$ for different values of M, N and different values of noise level δ with their RMSE are shown in Table 3 and Table 4.

These results have significant implications and show that the regularization method plays an important role in order to obtain a stable solution of the ill-posed problem.

Table 3. The comparison between the exact and numerical solutions of $p(t)$ for different values of M, N, h for example 2.

t	Exact solution	Numerical solution $M{=}50, N{=}100, h{=}1.5$	Numerical solution $M{=}50, N{=}100, h{=}4$	Numerical solution $M{=}50, N{=}100, h{=}6.5$	Numerical solution $M{=}100, N{=}200, h{=}4$
0.0	1.00	1.020270	1.020270	1.020585	1.010067
0.1	1.10	1.100309	1.100309	1.100371	1.100077
0.2	1.20	1.200355	1.200355	1.200663	1.200089
0.3	1.30	1.300407	1.300407	1.300185	1.300102
0.4	1.40	1.400463	1.400463	1.398805	1.400116
0.5	1.50	1.500525	1.500525	1.496857	1.500131
0.6	1.60	1.600594	1.600593	1.594344	1.600148
0.7	1.70	1.700669	1.700668	1.692354	1.700167
0.8	1.80	1.800752	1.800749	1.791693	1.800188
0.9	1.90	1.900795	1.900836	1.881385	1.900208
1.0	2.00	1.945873	1.950918	1.924078	1.975256
$RMSE$	$--$	$6.5852e-05$	$5.5616e-05$	$1.9259e-04$	$7.0861e-06$

Table 4. The comparison between the exact and numerical solutions of $p(t)$ for different values of noise level with $M = 50, N = 100, h = 4$ for example 2.

t	Exact solution	Numerical solution $M{=}50, N{=}100, h{=}4, \delta{=}0\%$	Numerical solution $M{=}50, N{=}100, h{=}4, \delta{=}1\%$	Numerical solution $M{=}50, N{=}100, h{=}4, \delta{=}3\%$	Numerical solution $M{=}50, N{=}100, h{=}4, \delta{=}5\%$
0.0	1.00	1.020270	1.020270	1.020269	1.020267
0.1	1.10	1.100309	1.100308	1.100312	1.100307
0.2	1.20	1.200355	1.200358	1.200354	1.200370
0.3	1.30	1.300407	1.300413	1.300404	1.300487
0.4	1.40	1.400463	1.400429	1.400401	1.400416
0.5	1.50	1.500525	1.500550	1.500665	1.500020
0.6	1.60	1.600593	1.600978	1.600217	1.600754
0.7	1.70	1.700668	1.700736	1.701627	1.696306
0.8	1.80	1.800749	1.793154	1.816595	1.790316
0.9	1.90	1.900836	1.902233	1.898505	1.867805
1.0	2.00	1.950918	2.005793	1.843309	1.816337
$RMSE$	$--$	$5.5616e-05$	$3.1286e-05$	0.0012	0.0019

7. Conclusions

In this work a numerical approach based on the method of fundamental solutions combined with the Tikhonov regularization technique for solving an inverse time-dependent coefficient identification problem is presented. The key of the method is to employ the fundamental solutions as basis functions in the approximate solution of the inverse problem and so the original problem reduces to solve an ill-conditioned linear system. The stability of the proposed method is investigated by adding noise to input data. The proposed approach is applied for solving two test examples in one-dimensional case and the numerical results show that the method can be applied for parabolic inverse problem due to its efficiency. The obtained numerical approximations are accurate for noisy data and so the method is stable. The proposed method can be extended to solve such problem in higher dimensional problems.

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