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COMPUTING μ-VALUES AND PSEUDO-SPECTRA FOR AIRY OPERATORS

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Abstract. Stability analysis play a vital role to design a linear feedback system in system theory. The stability of a feedback system is the direct measure of the roots of characteristic equation of transfer functions. The main objective of this article is to present numerical approximation of bounds of μ-values and computation of pseudo-spectrum for a class of Airy Operators. The comparison of the bounds of μ-values with the well-known MATLAB routine mussv is investigated which illustrate the behaviour of proposed methodology.

Keywords: Eigenvalues, Eigenvectors, Singular Values, Structured Singular Values, pseudo-spectrum, ordinary differential equations.

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1. Introduction

The Structured Singular Values "SSV" [7] is the most important and widely used tool in control to discuss the stability, instability, robustness and performance of linear systems. The structures of an admissible perturbation addressed by SSV undertakes almost all types of uncertainties. We refer to [1, 3, 4, 5, 6, 7, 8, 9] and the reference therein for applications and more discussions on SSV.

The robust stability and robust performance criterion of the linear feedback systems changes by imposing various assumptions on performance and uncertainties. The interconnected structure turns out to be very complicated whith dealing with the complex systems. In literature [9] there are several freely available software packages such as SIMULINK which generates the interconnecting structures. The uncertainty of the linear feedback systems can be modeled in terms of either by taking external input arguments or by taking the admissible perturbations to the nominal model. The outputs of the linear feedback systems and the error which occurs while measuring the performance and behavior of such systems. The designing of control systems demand various kinds of an elementary issues that cause to shift the boundaries of the particular application. These kind of issues are generic while designing objectives and procedure of the control. The main issue is then to provide the reliable performance while facing modeling errors, uncertainties and variations

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of the linear feedback systems. The feedback of the linear system is needed when system performance can not be achieved because of the presence of uncertainty.
The deterministic uncertainty and variation of an aeroelastic model was first developed by [10, 11] who made use of the μ-analysis [9] in order to perform the robustness of flutter analysis. The $\mu - k$ method [12, 13] based upon the frequency-domain flutter analysis discusses the robust analysis by means of an existing numerical algorithms. The $\mu - k$ analysis contributes very effectively while modeling the aerodynamics uncertainties which acts as a key factor with applications of aeroelastic, for more details see [14, 15, 16, 17].
In this article, we present the numerical approximation of lower bounds of SSV for a class of Airy operator by using low rank ordinary differential equations based methodology. The proposed methodology is based on two level algorithm, the inner algorithm and outer algorithm. In the inner algorithm, we compute the local extremizers and then construct and solve a gradient system of ordinary differential equations. This help us to derive the required admissible perturbation. In outer algorithm, we use fast Newton's iterations in order to vary the perturbation level. An Airy operator M is an operator which acts on the functions of $\widehat{x}$ defined on interval $[-1, 1]$ as

$$Mu = \epsilon\left(\frac{d^2u}{d\widehat{x}^2}\right) + i\widehat{x}u$$

where ϵ is a small parameter. The spectrum of such operator is unbounded discrete set which is contain in the half-strip, $Re\,(z) < 0$, $-1 < Img\,(z) < 1$. Furthermore, we also discuss the pseudo-spectrum of such an operator while making use of EigTool [18]. The pseudo-spectrum of an Airy operator was first studied by Reddy, Schmitd and Henningsen [19].

2. Preliminaries

Definition 2.1. *The spectrum of a square complex valued matrix $M \in \mathbb{C}^{n,n}$ is defined as*

$$\Lambda(M) = \{\lambda \in \mathbb{C} : \ |\,(\lambda I - M)\,| = 0\}.$$

Definition 2.2. *The pseudospectrum of a complex matrix $M \in \mathbb{C}^{n,n}$ with a small positive real parameter $\epsilon > 0$ is defined as*

$$\Lambda_\epsilon(M) = \{\lambda \in \mathbb{C} : \ |\,(\lambda I - M)^{-1}\,| \geq \frac{1}{\epsilon}\}.$$

Definition 2.3. *For a small positive parameter $\epsilon \geq 0$. A number λ belongs to epsilon-pseudo-spectrum of an operator A, denoted by $\Lambda_\epsilon(A)$ and satisfies the following equivalent conditions*
(i) $\lambda \in \Lambda(A + E)$ for some perturbation E having $\|E\| \leq \epsilon$;
(ii) $\exists\ u \in \mathbb{C}^{n,1}$ having $\|u\| = 1$ such that $\|Au - \lambda u\| \leq \epsilon$;
(iii) $\lambda \in \rho(A)$ and $\|\,(\lambda I - A)^{-1}\,\| \geq \frac{1}{\epsilon}$ or $\lambda \in \Lambda(A)$ where $\rho(A)$ denotes the spectral radius of the matrix A.

Definition 2.4. *Unstructured uncertainty $\mathbb{B}$ is stable transfer matrix or structured stable transfer matrix having the form*

$$\mathbb{B} = \{diag(\delta_i I_i; \Delta_j) : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j,m_j}\}.$$

Definition 2.5. *For a given n-dimensional square matrix $M \in \mathbb{C}^{n,n}$ and underlying perturbation set $\mathbb{B}$, the μ-value is defined as*

$$\mu_{\mathbb{B}}(M) = \frac{1}{min\{\|\Delta\|_2 : \Delta \in \mathbb{B}, det(I - M\Delta) = 0\}}.$$

unless no such Δ *cause* $(I - M\Delta)$ *to be singular for which* $\mu_{\mathbb{B}}(M) = 0$.

Theorem 2.1. *Small Gain Theorem* [21]. *The feedback system is well-posed and stable for an admissible perturbation* Δ *with largest singular value bounded above by 1 if and only if*

$$\|M\|_{\infty} : Sup(\|M(jw)\|) < 1,$$

for some $w \in \mathbb{R}^{+}$*, the frequency.*

Theorem 2.2. [7]. *For two structured uncertainties* $\mathbb{B}_1 \subset \mathbb{B}_2$,

$$\mu_{\mathbb{B}_1}(\|M(jw)\|) < \mu_{\mathbb{B}_2}(\|M(jw)\|).$$

The feedback system is well-posed and internally stable for $\Delta \in \mathbb{B}$ *with* $\|\Delta\|_2 \leq 1$ *if and only if* $Sup(M(jw)) < 1$ *for some* $w \in \mathbb{R}^{+}$.

3. Reformulation of μ-values

In this section we reformulate the μ-values on the basis of structured spectral value sets. The key idea for the reformulation of the structured singular values is to shift the largest eigenvalue of the matrix valued function $I - M\Delta(t)$ such that $\lambda_{max} = 1$ and the new eigenvalue $\eta = 0$ as ($\eta = 1 - \lambda_{max}$) and it achieve the maximum value to be one when $\lambda_{max} = 0$. On the basis of this mathematical construction, the reformulation of structured singular values is given as below.

Definition 3.1. *For a given* $M \in \mathbb{C}^{n,n}$ *and perturbation level* $\epsilon > 0$*, the structured spectral value set is denoted by* $\Lambda_{\epsilon}^{\mathbb{B}}(M)$ *and is defined as*

$$\Lambda_{\epsilon}^{\mathbb{B}}(M) = \{\lambda \in \Lambda(\epsilon M\Delta), \Delta \in \mathbb{B}, \|\Delta\|_2 \leq 1\},$$

where $\Lambda(\epsilon M\Delta)$ *denotes the spectrum of the matrix valued function* $(\epsilon M\Delta)$*, which is simply a disk centered at origin 0.*

Definition 3.2. *The structured epsilon spectral value set for a given* $M \in \mathbb{C}^{n,n}$ *and* $\epsilon \geq 0$*, is defined as*

$$\Sigma_{\epsilon}^{\mathbb{B}}(M) = \{\eta : 1 - \lambda : \lambda \in \Lambda_{\epsilon}^{\mathbb{B}}(M)\}.$$

Definition 3.3. *For a given* $M \in \mathbb{C}^{n,n}$ *and an underlying perturbation set* $\mathbb{B}$ *the* μ*-value is defined as*

$$\mu_{\mathbb{B}}(M) = \frac{1}{arg\ min_{\epsilon > 0}\big\{max|\lambda| = 1, \lambda \in \Lambda_{\epsilon}^{\mathbb{B}}(M)\big\}}.$$

4. Pseudo-Spectrum

In this section we present the pseudospectra for a class of Airy operator. For this purpose we make use of the software package EigTool [18]. EigTool is routinely used for plotting unstructured pseudo-spectra of the matrices under consideration. In Figures 1-4, we show the computation of the pseudo-spectra for a class of Airy operators presented in section 6, that is, numerical experimentation. The spectrum of the eigenvalues corresponding to Airy operators in 3-dimensional space is shown in Figure 2 and Figure 2 by making use of Eigtool.

Let A be an n-dimensional matrix and let $\Lambda(A)$ denotes the set of all eigenvalues of matrix A. Let $\|A\|$ denotes the matrix-norm of the matrix A induced by an inner product space $\langle \cdot, \cdot \rangle$. The computation of the pseudo-spectra of an operator is very straightforward but at the same time is very costly. The boundaries associated with the pseudo-spectrum are nothing but just the level curves of the resolvent corresponding to operator A, that is, $\| (\lambda I - A)^{-1} \|$. The computation of the level curves involve the computation of the

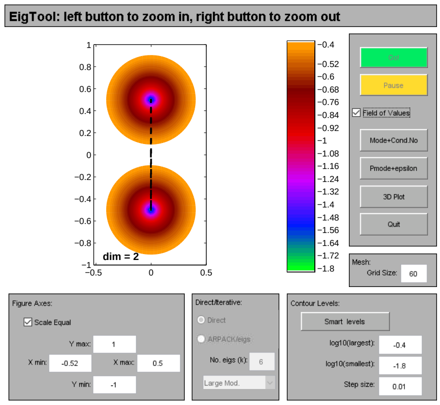

FIGURE 1. Pseudospectrum of 2-dimensional Airy Operator

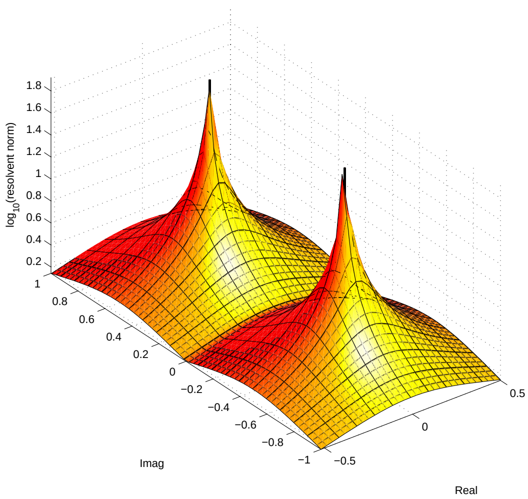

FIGURE 2. The spectrum of 2-dimensional Airy Operator

numerical values of λ at the grid point in the complex plane and then to compute the desired contour plots as shown in Figure 1 and Figure 3.
For the computation of the ϵ-pseudo-spectrum, the computation of an admissible perturbation E such that $\|E\| = \epsilon$ for the perturbed matrix $A + E$ is to essential to compute. For the computation of the ϵ-pseudo-spectrum the determination of the sets L_ϵ and U_ϵ are essential such that $L_\epsilon \leq \Lambda_\epsilon \leq U_\epsilon$.
Here, $L_\epsilon(A) = \{\lambda \in \rho(A) : b(\lambda) \geq \frac{1}{\epsilon}\} \cup \Lambda(A)$ acts as a lower bound of the ϵ-pseudo-spectrum with $\epsilon \geq 0$. For an upper bounds of the pseudo-spectrum, $U_\epsilon(A) = \{\lambda \in \rho(A) : B(\lambda) \geq \| (\lambda I - A)^{-1} \|\}$ for all $\lambda \in \rho(A)$. For a complete detail we refer [19] and the reference therein.

5. Proposed Methodology

In order to solve the maximization problem discussed in Definition 3.3, we make use of numerical method based upon low-rank ordinary differential equations technique. The numerical method is mainly composed of two-level algorithm, that is, inner-algorithm and outer-algorithm. In the inner-algorithm the main objective is to first construct then solve

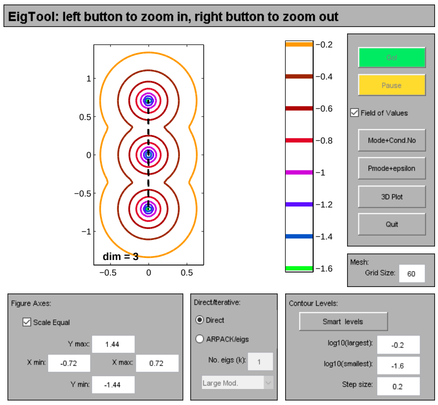

FIGURE 3. Pseudospectrum of 3-dimensional Airy Operator

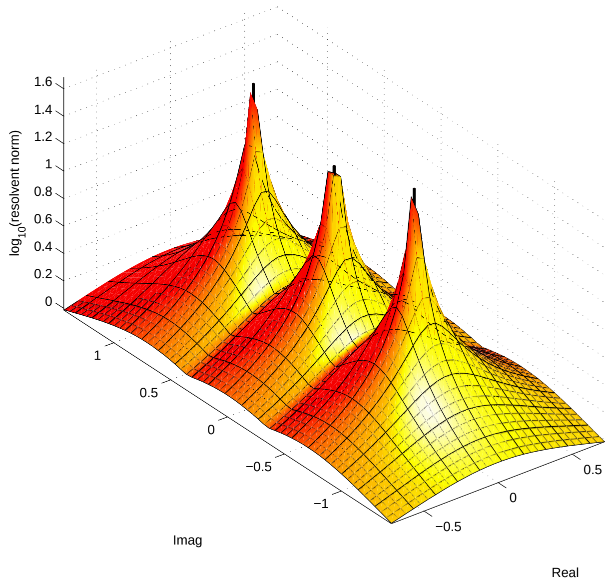

FIGURE 4. The spectrum of 3-dimensional Airy Operator

a gradient system of ordinary differential equations. On the other hand in the Outer-algorithm we vary the perturbation level $\epsilon > 0$ by means of fast Newton iteration. The outer-algorithm computes an exact derivative of an extremizer say $\Delta(\epsilon)$ for $\Delta \in \mathbb{B}$ and $\epsilon > 0$. A complete detail of numerical method under consideration is given in [20, 22, 24].

Next, we discuss the computation of an extremizer. For this purpose, we first approximate the derivative of an eigenvalue matrix $\Lambda(p)$ of a smooth matrix family say $A(p)$ for some fixed parameter p.

5.1. **Approximation of an Extremizers.** A matrix valued function $\Delta \in \mathbb{B}$ having the largest singular value bounded above by 1 and the matrix valued function $(I - \epsilon M\Delta)$ having a smallest eigenvalue which minimizes the modulus of structured spectral value set $\sum_{\epsilon}^{\mathbb{B}}(M)$ is known as an extremizer. The following theorem computes extremizer for a chosen smallest complex number belonging to the set $\sum_{\epsilon}^{\mathbb{B}}(M)$.

Theorem 5.1. For a perturbation $\Delta \in \mathbb{B}$ having the block diagonal structure

$$\Delta = \{diag(\delta_1 I_1, \ldots \delta_{s'} I_{s'}, \delta_{s'+1} I_{s'+1}, \ldots \delta_S I_S; \Delta_1, \ldots, \Delta_F\},$$

with $\|\Delta\|_2 = 1$, acts as a local extremizer of structured spectral value set. For a simple smallest eigenvalue $\lambda = |\lambda| e^{\iota\theta}, \theta \in \mathbb{R}$ of matrix valued function $(I - \epsilon M\Delta)$ having right and left eigenvectors x and y scaled as $S = e^{\iota\theta} y^* x$ and let $z = M^* y$. The non-degeneracy conditions

$$\begin{aligned} z_k^* x_k &\neq 0, \ \forall k = 1 : S' \\ Re(z_k^* x_k) &\neq 0, \ \forall k = 1 : S' + 1 : S \\ \text{and} \quad ||z_{s+h}||.||x_{s+h}|| &\neq 0, \ \forall h = 1 : F, \end{aligned}$$

holds. Then magnitude of each complex scalar $\delta_i \ \forall i = 1 : s$ appears to be exactly equal to 1 while each full block possesses a unit 2-norm.
Proof. For proof we refer to [20].

5.2. **Gradiant System of ODE's.** The gradient system of odes for an admissible perturbation $\Delta \in \mathbb{B}$ to approximate a local extremizer of smallest eigenvalue $\lambda = |\lambda| e^{i\theta}$, is obtained as,

$$\begin{aligned} \dot{\delta_i} &= \nu_i (x_i^* z_i - Re(x_i^* z_i \bar{\delta_i}) \delta_i); \quad i = 1 : s' \\ \dot{\delta_l} &= sign(Re(z_l^* x_l) \Psi_{(-1,1)}(\delta_l); \quad l = s' + 1 : s \\ \dot{\Delta_j} &= \nu_j (z_{s+j} x_{s+j}^* - Re\langle \Delta_j; z_{s+j} x_{s+j}^* \rangle); \quad j = 1 : F, \end{aligned}$$

where $\delta_i \in \mathbb{C}, \forall \ i = 1 : s'$, $\delta_l \in \mathbb{R}$ for $l = s' + 1$ and $\Psi_{(-1,1)}$, the characteristic function. For more discussion on the construction of gradient system of odes in above equations, we refer to [20].

5.3. **Outer-Algorithm.** In outer-algorithm the main aim is to vary $\epsilon > 0$, the perturbation level by means of fast Newton's itaration. In turn $\frac{1}{\epsilon}$ will provide us the approximation of lower bound of μ-values.

We make use of fast newton's iteration in order to solve

$$|\lambda(\epsilon)| = 1. \tag{1}$$

In Equ. 1, $\epsilon > 0$. In order to solve Equ. 1, we need to compute

$$\frac{d}{d\epsilon} \left(|\lambda(\epsilon)| \right),$$

the derivative.

The following theorem 5.2 help us to compute $\frac{d}{d\epsilon}(|\lambda(\epsilon)|)$, when $|\lambda(\epsilon)|$ are simple and $\Delta(0)$, $\lambda(0)$ are assumed to remains smooth in the neighboring region of perturbation level $\epsilon > 0$

Theorem 5.2 Consider matrix valued function $\Delta \in \mathbb{B}$. Let x and y as a function of perturbation level $\epsilon > 0$ acts as right and left eigenvectors of matrix valued function $(\epsilon M\Delta)$. Consider the scaling of these vector according to theorem 5.1. Let $z = M^* y$ and assume that non-degenracy conditions as discussed in theorem 5.1. hold, then,

$$\frac{d}{d\epsilon}(|\lambda(\epsilon)|) = \frac{1}{|y(\epsilon^*) x(\epsilon)|} \sum_{i=1}^{s} |z_i(\epsilon)^* x_i(\epsilon)|$$

$$+ \sum_{j=1}^{F} ||z_{s+j}(\epsilon)||.||y_{s+j}(\epsilon)|| > 0.$$

Proof. For proof we refer to [20].

5.4. **Choice of suitable initial value matrix and initial perturbation level.** For a suitable choice of initial value matrix Δ_0 and an initial perturbation level ϵ_0, we refer to [20].

6. Numerical Experimentation

Example 1.
Case-I Consider a two dimensional complex valued matrix M

$$M = \begin{bmatrix} -0.001 + 0.5i & 0.0008 \\ 0.0008 & -0.001 - 0.5i \end{bmatrix}.$$

We take the underlying perturbation as

$$\Delta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{2,2}\}.$$

The well-known MATLAB routine mussv approximates the bounds of SSV along with the required perturbation $\widehat{\Delta}$ as

$$\widehat{\Delta} = \begin{bmatrix} -0.003 - 0.998i & 0.998 + i \\ 0.998 - i & -0.003 + 0.998i \end{bmatrix}.$$

The matrix 2-norm of $\widehat{\Delta}$, that is, $\|\widehat{\Delta}\|_2 = 1.9968$. The mussv routine computes an upper bound $\mu_u^{mussv} = 0.5008$ and a same lower bound $\mu_l^{mussv} = 0.5008$. Algorithm [20] computes the lower bound of SSV as follows while the admissible perturbation $\epsilon^*\Delta^*$ is obtained as

$$\Delta^* = \begin{bmatrix} -i & 0.0008 + i \\ 0.0008 + i & -0.003 + i \end{bmatrix},$$

and the perturbation level is computed as $\epsilon^* = 2$. The lower bound of SSV is obtained as $\mu_l^{New} = 0.5000$.
Case-II Again consider a two dimensional complex valued matrix M.

$$M = \begin{bmatrix} -0.001 + 0.5i & 0.0008 \\ 0.0008 & -0.001 - 0.5i \end{bmatrix}$$

We take the underlying perturbation as

$$\Delta_{\mathbb{B}} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{C}\}.$$

The well-known MATLAB routine mussv approximates the bounds of SSV along with the required perturbation $\widehat{\Delta}$ as

$$\widehat{\Delta} = \begin{bmatrix} -0.006 - 1.996i & 0 \\ 0 & -0.006 + 1.996i \end{bmatrix}.$$

The matrix 2-norm of $\widehat{\Delta}$, that is, $\|\widehat{\Delta}\|_2 = 1.9968$. The mussv routine computes an upper bound $\mu_u^{mussv} = 0.5008$ and lower bound is computed as $\mu_l^{mussv} = 0.5008$. Algorithm [20] computes the lower bound of SSV while the admissible perturbation $\epsilon^*\Delta^*$ is obtained as

$$\Delta^* = \begin{bmatrix} 0.003 - i & 0 \\ 0 & -0.002 + i \end{bmatrix}.$$

The perturbation level is computed as $\epsilon^* = 4$. The lower bound of SSV is obtained as $\mu_l^{New} = 0.2500$.

Example 2. Consider a three dimensional complex valued matrix M.

$$M = \begin{bmatrix} -0.004 + 0.707i & 0.001 & -0.0006 \\ 0.001 & -0.001 & 0.001 \\ -0.000 & 0.001 & -0.004 - 0.707i \end{bmatrix}$$

n	$\Delta_{\mathbb{B}}$	μ_l^{mussv}	μ_l^{New}
03	$\{diag(\delta_i I_3) : \delta_i \in \mathbb{C}, \forall\ i = 1:3\}$	0.70	0.35
03	$\{diag(\delta_i I_3) : \delta_i \in \mathbb{R}, \forall\ i = 1:3\}$	0	0.35
03	$\{diag(\Delta_1, \Delta_2) : \Delta_1 \in \mathbb{C}^{2,2}, \Delta_2 \in \mathbb{C}^{1,1}\}$	0.70	0.70
03	$\{diag(\delta_1 I_2, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{R}\}$	0	0.35
03	$\{diag(\delta_1 I_2, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{C}\}$	0.70	0.35

TABLE 1. Computation of bounds of μ-values

We take the underlying perturbation as

$$\Delta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{3,3}\}.$$

The well-known MATLAB routine mussv approximates the bounds of SSV as along with the required perturbation $\widehat{\Delta}$ as

$$\widehat{\Delta} = \begin{bmatrix} -0.004 - 0.706i & 0.001 + 0.001i & -0.706 - 0.001i \\ 0.001 + 0.001i & i & 0.001 - 0.001i \\ -0.706 + 0.001i & 0.001 - 0.001i & -0.004 + 0.706i \end{bmatrix}.$$

The matrix 2-norm of $\widehat{\Delta}$, that is, $\|\widehat{\Delta}\|_2 = 1.4130$. The mussv routine computes an upper bound $\mu_u^{mussv} = 0.7077$ and same lower bound is computed $\mu_l^{mussv} = 0.7077$. Algorithm [20] computes the lower bound of SSV as follows while the admissible perturbation $\epsilon^* \Delta^*$ is obtained as

$$\Delta^* = \begin{bmatrix} i & i & -0.0004 - i \\ i & -i & 0.002 + i \\ -0.0004 - i & 0.001 + i & -0.005 + i \end{bmatrix}.$$

The perturbation level is computed as $\epsilon^* = 1.4142$. The lower bound of SSV is obtained as $\mu_l^{New} = 0.7071$.

Table 1 show the numerical comparison of bounds of SSV, that is, both lower and upper bounds computed with MATLAB routine mussv and algorithm [20] for the matrix M. In very first column, n denotes the size of the matrix M. The second column highlights the structure of set of block diagonal matrices denoted with $\Delta_{\mathbb{B}}$. The third and fourth columns present the bunds of SSV approximated by mussv routine and with the algorithm [20].

7. CONCLUSION.

In this article, we have presented numerical approximations and a comparison of lower and upper bounds of SSV for a family of Airy Operators. The lower bounds of SSV provides sufficient conditions about the instability of linear feedabck systems. The numerical approximation of upper bounds of SSV discuss the stability of linear feedback systems. It is investigated that in most of experiments, the approximated bounds by making use of MATLAB routine mussv and the algorithm presented in [20] are tightly related while the gaps between lower bounds of SSV is minimum.

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