TWMS J. App. and Eng. Math. V.11, N.4, 2021, pp. 1148-1157

DISJUNCTIVE TOTAL DOMINATION IN SOME GRAPHS DERIVED FROM THE SUBDIVISION GRAPH

CANAN ÇİFTÇİ[1], AYSUN AYTAÇ[2], §

ABSTRACT. For a set $S \subseteq V(G)$, if every vertex has a neighbor in S or has at least two vertices in S at distance two from it, then the set S is a disjunctive total dominating set of G. The minimum cardinality of such a set is equal to the disjunctive total domination number. In this study, we discuss disjunctive total domination number of some graphs derived from the subdivision graphs such as middle and central graphs.

Keywords: Domination, disjunctive total domination, central graphs, middle graphs

AMS Subject Classification: 05C12, 05C69, 05C76

1. INTRODUCTION

In network design, minimizing the exchange between resource allocation and redundancy is an important issue. However, key resources are expensive and cannot be allocated to every node of the network. Hence, a subset of nodes is selected depend on the closeness to the rest of the nodes. Thereby, the nodes in the subset share the resources to the other nodes. However, it can be a resource failure at any node. In this case the redundancy is important and needs extra resources to be allocated. This problem can be modelled by using graphs for a network and the subset of vertices of the graph forms a dominating set. Let the graph G be a model of a network. A subset of vertices S is a dominating set [7] of G if every vertex in $V(G) - S$ is adjacent to some vertex in S. In order to extend the domination problem including redundancy, a subset of vertices S is selected such that every vertex in $V(G)$ is adjacent to some vertex in S and this set is called as a total dominating set [3]. However, since determining dominating and total dominating sets are difficult, their implementations in modern networks are expensive. Although some restrictions are added to dominating and total dominating sets, the cost of implementation of these sets raises. Then, Goddard *et al.* [5] introduce *disjunctive domination* as a relaxation of the domination number. In a similar manner, Henning and Naicker [8] extend disjunctive domination as a relaxation of the total domination number and propose *disjunctive total domination*.

[1] Department of Mathematics, Faculty of Arts and Sciences, Ordu University, 52200, Ordu, Turkey.
e-mail: cananciftci@odu.edu.tr; ORCID: https://orcid.org/0000-0001-5397-0367
[2] Department of Mathematics, Faculty of Science, Ege University, 35100, Izmir, Turkey.
e-mail: aysun.aytac@ege.edu.tr; ORCID: https://orcid.org/0000-0003-2086-8969.
§ Manuscript received: December 19, 2019; accepted: March 23, 2020.

A set $S \subseteq V(G)$ is a *disjunctive total dominating set*, briefly DTD-set, if and only if every vertex $v \in V(G)$ has one of the following two properties.

P_1: v has a neighbor in S

P_2: v has at least two vertices in S at distance two from it.

If v has property P_1 or P_2, then v is *disjunctively totally dominated*, briefly DT-dominated, by vertices of S. Furthermore, if v has only property P_1, then v is *totally dominated* by vertices of S. If v has only property P_2, then v is *disjunctively dominated* by vertices of S. The minimum cardinality of a DTD-set is equal to the *disjunctive total domination number* $\gamma_t^d(G)$. A DTD-set which gives the value $\gamma_t^d(G)$ is called $\gamma_t^d(G)$-set.

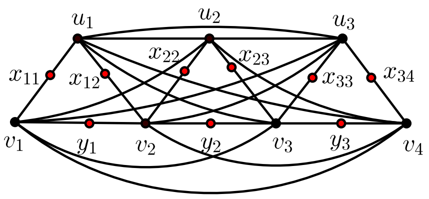
This parameter is studied on some graphs such as trees [8, 9], extremal graphs [14], claw-free graphs [8], grids [13], permutation graphs [19], and studied on some graphs operations such as corona, compositions of two graphs and subdivision [4, 10]. This paper is about the disjunctive total domination number of central and middle graphs.

2. Preliminaries

For a simple graph G, let $V(G)$ and $E(G)$ be vertex and edge sets of G, respectively. The cardinality of $V(G)$ is the *order* of G. Two vertices u and v in G are adjacent if there is an edge $e = uv$ joining them. The *distance* $d_G(u, v)$ between two vertices u and v in G is the length of a shortest path joining them in G. The degree $deg_G(v)$ of v in G is the number of edges of G incident to v. We follow [7] for graph theory terminology and notations which are not defined here for simplicity.

The *subdivision graph* of G is obtained by inserting a new vertex into each edge of G. This new vertex is called *subdivision vertex.* In Figure 1, the vertices x_{ij} and y_i are the subdivision vertices for $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$. The central and middle graph of a graph are introduced by Vernold [18] and Hamada *et al.* [6], respectively, and are derived from the subdivision graph by adding some edges with the following rules.

The *central graph* of a graph G, denoted by $C(G)$, is obtained from the subdivision graph of G by joining all the non-adjacent vertices of G. The *middle graph* of a graph G, denoted by $M(G)$, is obtained from the subdivision graph of G by joining the subdivision vertices by an edge if the two corresponding edges share the same vertex of G.



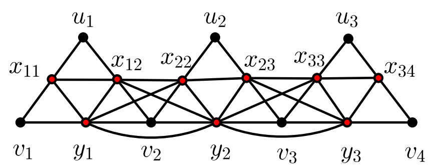


Figure 1. Central and middle graphs of triangular snake graph T_3

There have been many papers studying on properties of central and middle graphs and these graphs are studied on some graph parameters such as coloring and domination [1, 11, 12, 15, 17]. In this paper, we determine the disjunctive total domination number of central and middle graphs of some snake graphs such as triangular snake, double triangular snake and diamond snake graphs.

Triangular snake (or Δ_k-snake), which is defined by Rosa [16], is formed from a path $P = v_1, v_2, ..., v_{n+1}$ by joining v_i and v_{i+1} to a new vertex u_i for $i \in \{1, 2, ..., n\}$. It is obvious that its all blocks are triangles and k is the number of triangles in a Δ_k-snake. We use the notation T_n which is also given for this graph. A *double triangular snake* $D(T_n)$ [20] is formed from a path $P = v_1, v_2, ..., v_{n+1}$ by joining v_i and v_{i+1} to two new vertices u_i and w_i for each $i \in \{1, 2, ..., n\}$. The triangular snake is generalized by Barrientos [2] and the kC_n-snake is defined as a connected graph in which the k blocks are isomorphic to C_n. When $n = 4$, then the kC_4-snake is called as *diamond snake* graph which its each block is C_4. Equivalently, a diamond snake graph is formed by joining vertices v_i and v_{i+1} to two new vertices u_i and w_i for each $i \in \{1, 2, ..., n\}$. This graph is denoted by D_n.

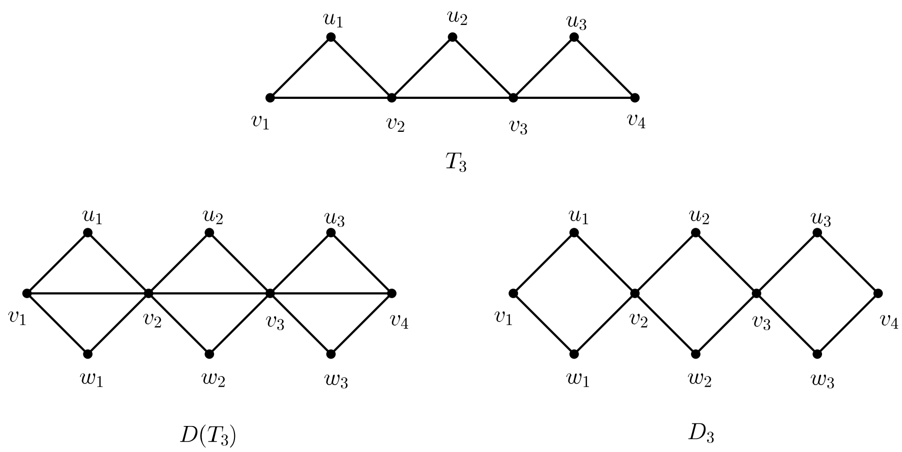


FIGURE 2. Triangular snake graph T_3, double triangular snake graph $D(T_3)$ and diamond snake D_3

3. Disjunctive Total Domination of Central Graphs

We first determine the disjunctive total domination number of triangular snake, double triangular snake and dimond snake graphs. Next, we obtain the disjunctive total domination number of central graph of them. The rest of the paper, we will represent vertices of T_n, $D(T_n)$ and D_n as they are mentioned in their definitions.

Theorem 3.1. *For a triangular snake graph T_n of order $2n + 1$ with $n \geq 2$,*

$$\gamma_t^d(T_n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \equiv 3 \ (\text{mod } 4) \\ \lceil \frac{n+2}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. In order to prove the formula we use induction on $n \geq 2$. As it is clear for $n \leq 7$, we may assume that the formula is true for all values less than $n \geq 8$ and we will now need to show that it is also true for n. For the upper bounds on $\gamma_t^d(T_n)$, let

$$X = \bigcup_{i=0}^{\lfloor \frac{n+1}{4} \rfloor - 1} \{v_{4i+2}, v_{4i+3}\}.$$

While if $n \equiv 0 \pmod 4$, let $Y = X \cup \{v_n\}$, if $n \equiv 1, 2 \pmod 4$, let $Y = X \cup \{v_n, v_{n+1}\}$. Further if $n \equiv 3 \pmod 4$, let $Y = X$. In all cases, the set Y is a DTD-set of T_n. Thus,

$$\gamma_t^d(T_n) \leq |Y| = \begin{cases} \frac{n+1}{2}, & \text{if } n \equiv 3 \pmod 4 \\ \lceil \frac{n+2}{2} \rceil, & \text{otherwise.} \end{cases}$$

Let S be a $\gamma_t^d(T_n)$-set in order to prove the opposite inequality. Since every vertex of Y has at least one adjacent vertex in Y, suppose that no two vertices in S are adjacent. Vertices of the first block of T_n are u_1, v_1 and v_2. Then there are two cases to DT-dominate these vertices: (1) $S \cap \{u_1, v_1, v_2\} \neq \emptyset$ (2) $|S \cap \{u_2, v_3\}| = 2$. For the first case, vertex v_2 must be in S for the minimality of S. Then vertices in the first and second blocks except v_2 are totally dominated by v_2. Since no two vertices in S are adjacent, vertices u_3 and v_4 must be in S to disjunctively dominate v_2. However, $u_3v_4 \in E(T_n)$ and this contradicts with our assumption. For the second case, let u_2 and v_3 be in S. Then vertices in the first and second blocks are DT-dominated. However, $u_2v_3 \in E(T_n)$ and this contradicts with our assumption. Therefore, the set S contains two adjacent vertices.

To create S with the minimum cardinality, for the first case let $v_2 \in S$. Then either u_2 or v_3 is in S. However, when $\{v_2, v_3\} \subset S$, this is a better candidate than $\{v_2, u_2\} \subset S$ for the minimality. For the second case, u_2 and v_3 are in S as we say above. The cardinality of the set which is created by starting with these two vertices is equivalent to the cardinality of the set which is created by starting with v_2 and v_3. Thus, without loss of generality, let the set S contains two consecutive vertices on the path and then $\{v_2, v_3\} \subset S$. We will show that $S \cap \{v_4, v_5\} = \emptyset$.

Assume that $v_4 \in S$. If $v_5 \in S$, then replace v_4 with v_6. Let $v_5 \notin S$. If $v_6 \in S$, then replace v_4 with v_5. If $v_6 \notin S$ and $v_7 \in S$, then replace v_4 with v_6. Thus, we can select S such that $v_4 \notin S$.

Assume that $v_5 \in S$. If $v_6 \in S$, then change v_5 with v_7. Thus, we can select S such that $v_5 \notin S$. Therefore, $S \cap \{v_4, v_5\} = \emptyset$ which means that $\{v_6, v_7\} \subset S$.

Let $S' = S - \{v_2, v_3\}$ where $|S'| = |S| - 2$. Let $T_{n'}$ be a graph which is obtained from T_n by deleting the vertices u_i and v_i for $i \in \{1, 2, 3, 4\}$ in which $n' = n - 4 \geq 4$. Since S is a DTD-set of T_n, the set S' is a DTD-set of $T_{n'}$. Thus, $\gamma_e^*(T_{n'}) \leq |S'| = |S| - 2$. By the inductive hypothesis to $T_{n'}$, we have $\gamma_t^d(T_{n'}) = \frac{n'+1}{2} = \frac{n+1}{2} - 2$ when $n \equiv 3 \pmod 4$ and otherwise $\gamma_t^d(T_{n'}) = \lceil \frac{n'+2}{2} \rceil = \lceil \frac{n+2}{2} \rceil - 2$. This means that if $n \equiv 3 \pmod 4$, then $\gamma_t^d(T_n) = |S| \geq \frac{n+1}{2}$, otherwise $\gamma_t^d(T_n) = |S| \geq \lceil \frac{n+2}{2} \rceil$.

Consequently, the result follows from the lower and upper bounds. □

Theorem 3.2. *If $D(T_n)$ is a double triangular snake with $n \geq 2$, then $\gamma_t^d(D(T_n)) = \gamma_t^d(T_n)$.*

Proof. Since $D(T_n)$ consists of two triangular snakes on the common path, all vertices of $D(T_n)$ are DT-dominated by the set of Theorem 3.1. Thus, $\gamma_t^d(D(T_n)) = \gamma_t^d(T_n)$. □

Theorem 3.3. *For a diamond snake graph D_n with $n \geq 2$,*

$$\gamma_t^d(D_n) = \begin{cases} \frac{4n+10}{5}, & \text{if } n \equiv 0 \pmod 5 \\ \lceil \frac{4n+4}{5} \rceil, & \text{otherwise.} \end{cases}$$

Proof. In order to prove the formula we use induction on $n \geq 2$. As it is clear for $n \leq 6$, we may assume that the formula is true for all values less than $n \geq 7$ and we will now need to show that it is also true for n. For the upper bounds on $\gamma_t^d(D_n)$, let

$$X = \bigcup_{i=0}^{\lfloor \frac{n-1}{5} \rfloor} \{v_{5i+2}, w_{5i+1}\} \cup \bigcup_{i=0}^{\lfloor \frac{n-4}{5} \rfloor} \{v_{5i+4}, u_{5i+4}\}.$$

While if $n \equiv 0, 3 \pmod 5$, let $Y = X \cup \{u_n, v_n\}$, if $n \equiv 1, 4 \pmod 5$, let $Y = X$. Further if $n \equiv 2 \pmod 5$, let $Y = X \cup \{u_n\}$. In all cases, the set Y is a DTD-set of D_n. Moreover, $|Y| = \frac{4n+10}{5}$ for $n \equiv 0 \pmod 5$, while $|Y| = \lceil \frac{4n+4}{5} \rceil$ for the other cases. Therefore,

$$\gamma_t^d(D_n) \leq |Y| = \begin{cases} \frac{4n+10}{5}, & \text{if } n \equiv 0 \pmod 5 \\ \lceil \frac{4n+4}{5} \rceil, & \text{otherwise.} \end{cases}$$

For the opposite inequality, let S be a $\gamma_t^d(D_n)$-set. Since every vertex of Y has at least one adjacent vertex in Y, suppose that no two vertices in S are adjacent. Then vertex u_1 or w_1 is in S to totally dominate v_1. Assume, without loss of generality, $u_1 \in S$. Then we have the set

$$S^* = \left\{ u_{3i+1} \mid 0 \leq i \leq \left\lfloor \frac{n-1}{3} \right\rfloor \right\} \cup \bigcup_{i=0}^{\lfloor \frac{n-2}{3} \rfloor} \{u_{3i+2}, w_{3i+2}\}$$

which no two vertices of it are adjacent. If $n \equiv 0 \pmod 3$, then let $S = S^* \cup \{u_n\}$; if $n \equiv 1 \pmod 3$, then let $S = S^* \cup \{u_{n-1}, w_{n-1}\}$; if $n \equiv 2 \pmod 3$, then let $S = S^*$. If $n \equiv 1 \pmod 3$, then $|S| = n + 2$ and for other cases $|S| = n + 1$. However, they contradict with our upper bounds. Therefore, the set S contains two adjacent vertices. Since vertex v_1 has exactly one vertex at distance two from it, vertex u_1 or w_1 must be in S to totally dominate v_1. Assume, without loss of generality, that $w_1 \in S$. In order to totally dominate w_1, vertex v_2 must be in S. Because, v_2 totally dominates also vertices u_2 and w_2. Thereby, only vertex v_3 among the vertices of the first and second blocks is not DT-dominated by w_1 and v_2. Thus, we may assume that $v_4 \in S$ and also that $u_4 \in S$ to totally dominate v_4. These means that we may assume that $\{v_2, w_1, v_4, u_4\} \subset S$ to DT-dominate vertices of the first four blocks. We will show that $\{v_5, w_4, v_6, w_5\} \cap S = \emptyset$.

Assume that $\{v_5, w_4\} \in S$. If $\{v_6, u_6\} \in S$, then replace v_5 with v_7 and w_4 with w_6. Let $\{v_6, u_6\} \notin S$. If $\{v_7, u_7\} \in S$, then replace v_5 with v_6 and w_4 with w_5. If $\{v_6, u_6\} \notin S$, $\{v_7, u_7\} \notin S$, $\{v_8, u_8\} \in S$, then replace v_5 with v_7 and w_4 with w_6. If $\{v_6, u_6\} \notin S$, $\{v_7, u_7\} \notin S$, $\{v_8, u_8\} \notin S$, $\{v_9, u_9\} \in S$, then replace v_5 with v_7 and w_4 with w_6. Thus, we can select S such that $\{v_5, w_4\} \notin S$.

Assume that $\{v_6, w_5\} \in S$. If $\{v_7, u_7\} \in S$, then replace v_6 with v_8 and w_5 with w_6. Let $\{v_7, u_7\} \notin S$. If $\{v_8, u_8\} \in S$, then replace v_6 with v_7 and w_5 with w_6. If $\{v_7, u_7\} \notin S$, $\{v_8, u_8\} \notin S$, $\{v_9, u_9\} \in S$, then replace v_6 with v_7 and w_5 with w_6. Thus, we can select S such that $\{v_6, w_5\} \notin S$.

As a consequence, $\{v_7, w_6, v_9, u_9\} \subset S$. Let $S' = S - \{v_2, w_1, v_4, u_4\}$ which yields $|S'| = |S| - 4$.

Let $D_{n'}$ be a graph which is obtained from D_n by deleting the vertices u_i, v_i and w_i for $1 \leq i \leq 5$ in which $n' = n - 5 \geq 2$. Since S is a DTD-set of D_n, the set S' is a DTD-set of $D_{n'}$. Thus, $\gamma_e^*(D_{n'}) \leq |S'| = |S| - 4$. By the inductive hypothesis to $D_{n'}$, we have if $n \equiv 0 \pmod 5$, then $\gamma_t^d(T_{n'}) = \frac{4n'+10}{5} = \frac{4n+10}{5} - 4$, otherwise $\gamma_t^d(D_{n'}) = \lceil \frac{4n'+4}{5} \rceil = \lceil \frac{4n+4}{5} \rceil - 4$. This yields that if $n \equiv 0 \pmod 5$, then $\gamma_t^d(D_n) = |S| \geq \frac{4n+10}{5}$, otherwise $\gamma_t^d(D_n) = |S| \geq \lceil \frac{4n+4}{5} \rceil$. □

Theorem 3.4. *If $C(T_n)$ is a central graph of a triangular snake with $n \geq 2$, then $\gamma_t^d(C(T_n)) = 3$.*

Proof. For a triangular snake T_n, let $C(T_n)$ be a central graph obtained from T_n by subdividing the edges u_iv_j and v_iv_{i+1} for each $i \in \{1, 2, ..., n\}$, $j \in \{i, i+1\}$ with subdivision vertices x_{ij} and y_i, respectively, and joining all the non-adjacent vertices of T_n.

We first prove the lower bound on $\gamma_t^d(C(T_n))$. Suppose that $\gamma_t^d(C(D(T_n))) = 2$ and that S be a γ_t^d-set of $C(D(T_n))$. Then vertices of S are adjacent and $S \cap \{u_i, v_j\} \neq \emptyset$ for any $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., n+1\}$. We may assume, without loss of generality, $u_1 \in S$. Since $d(y_1, u_1) = 3$, a vertex adjacent to y_1 must be in S in order to totally dominate y_1. This vertex is v_1 or v_2. However, none of two vertices are adjacent to u_1. This contradicts that vertices of S are adjacent. Thus, $\gamma_t^d(C(T_n)) \geq 3$.

For the upper bound on $\gamma_t^d(C(T_n))$, let $T = \{u_1, u_2, v_2\}$. Then the set T is a DTD-set of $C(T_n)$. Thus, $\gamma_t^d(C(T_n)) \leq 3$.

The proof is completed by combining the lower and upper bounds for $\gamma_t^d(C(T_n))$. □

Theorem 3.5. *Let $C(D(T_n))$ be a central graph of a double triangular snake with $n \geq 2$. Then $\gamma_t^d(C(D(T_n))) = 3$.*

Proof. Let $C(D(T_n))$ be a central graph obtained from $D(T_n)$ by subdividing the edges u_iv_j, v_iv_{i+1} and w_iv_j for $i \in \{1, 2, ..., n\}$, $j \in \{i, i+1\}$ with subdivision vertices x_{ij}, y_i and z_{ij}, respectively, and joining all non-adjacent vertices of $D(T_n)$.

We first prove the lower bound on $\gamma_t^d(C(T_n))$. Suppose that $\gamma_t^d(C(D(T_n))) = 2$ and that S be a γ_t^d-set of $C(D(T_n))$. Then vertices of S are adjacent and $S \cap \{u_i, w_i, v_j\} \neq \emptyset$ for any $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., n+1\}$. Furthermore, since $d(x_{ij}, x_{tk}) = d(z_{ij}, z_{tk}) = 3$ if $i \neq t$ or/and $j \neq t$ and $d(y_i, y_j) = 3$ if $j \neq i+1$, the set S does not contain subdivision vertices. This implies that S contains any two vertices of u_i, v_j or w_i depending on whether they are adjacent in $C(D(T_n))$. All vertices of the graph must be either adjacent to any vertex of S or at distance two both two vertices of S. However, if $S = \{v_i, v_j\}$ for $j \neq i-1$ or $j \neq i+1$, then $d(v_i, x_{i(i+1)}) = 3$, $d(v_j, x_{i(i+1)}) = 2$ and if $S = \{u_i, u_j\}$ or $S = \{w_i, w_j\}$ or $S = \{u_i, w_j\}$ or $S = \{u_i, v_j\}$ for $j \neq i$, $j \neq i+1$, then vertex y_i is at distance three from the first vertex of S and at distance two from the other vertex of S. Thereby there are some vertices that are not DT-dominated by vertices of S and then at least one vertex must be added to S. Therefore, $\gamma_t^d(C(D(T_n))) \geq 3$.

In order to prove the opposite inequality, let $T = \{u_1, u_2, v_2\}$. All vertices of $C(D(T_n))$ are DT-dominated by vertices of T. Thus, the set T is a DTD-set of $C(D(T_n))$, and then $\gamma_t^d(C(D(T_n))) \leq 3$.

Consequently, $\gamma_t^d(C(D(T_n))) = 3$ from the lower and upper bounds for $\gamma_t^d(C(D(T_n)))$. □

Theorem 3.6. *Let $C(D_n)$ be a central graph of a diamond snake with $n \geq 2$. Then $\gamma_t^d(C(D_n)) = 2$.*

Proof. The graph D_n is obtained from deleting the edge v_iv_{i+1} in $D(T_n)$ for all $i \in \{1, 2, ..., n\}$. Since $\gamma_t^d(C(D_n)) \geq 2$, it is sufficient to prove the upper bound. Let $S = \{u_1, u_2\}$. Then all vertices of $C(D_n)$ are DT-dominated by S. Hence, $\gamma_t^d(C(D_n)) = 2$. □

4. Disjunctive Total Domination of Middle Graphs

This section determines the disjunctive total domination number of middle graph of triangular snake, double triangular snake and dimond snake graphs.

Theorem 4.1. *If $M(T_n)$ is a middle graph of a triangular snake with $n \geq 2$, then $\gamma_t^d(M(T_n)) = \left\lceil \frac{2(n+1)}{3} \right\rceil$.*

Proof. Let $M(T_n)$ be a middle graph of T_n which is obtained from T_n by subdividing the edges u_iv_j and v_iv_{i+1} for $i \in \{1, 2, ..., n\}$, $j \in \{i, i+1\}$ with subdivision vertices x_{ij} and y_i, respectively, and joining the subdivision vertices by an edge if the two corresponding edges share the same vertex of T_n. It is clear that $deg(y_i) \geq deg(x_{ij})$ for all i and j.

In order to prove the formula we use induction on $n \geq 2$. As it is clear for $n \leq 5$, we may assume that the formula is true for all values less than $n \geq 6$ and we will now need to show that it is also true for n. We first establish the upper bound on $\gamma_t^d(M(T_n))$. Let

$$X = \bigcup_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} \{y_{3i+1}, y_{3i+2}\}.$$

If $n \equiv 0 \pmod 3$, then let $Y = X \cup \{y_n\}$, while if $n \equiv 1, 2 \pmod 3$, then let $Y = X \cup \{y_{n-1}, y_n\}$. In all cases, the set Y is a DTD-set of $M(T_n)$ and $|Y| = \left\lceil \frac{2(n+1)}{3} \right\rceil$. Thus, $\gamma_t^d(M(T_n)) \leq \left\lceil \frac{2(n+1)}{3} \right\rceil$.

For the opposite inequality, let S be a $\gamma_t^d(M(T_n))$-set. Suppose that no two vertices in S are adjacent. Hence, the set S contains vertex u_1 and all vertices of $\{v_1, v_2, ..., v_{n+1}\}$, and then $\gamma_t^d(M(T_n)) = |S| = n + 2$. However, this contradicts with our upper bound. Therefore, the set S contains two adjacent vertices. In order to DT-dominate vertex v_1, we have six cases: (1) $y_1 \in S$ (2) $x_{11} \in S$ (3) $x_{12} \in S$ and $y_2 \in S$ (4) $x_{12} \in S$ and $v_2 \in S$ (5) $x_{12} \in S$ and $u_1 \in S$ (6) $x_{12} \in S$ and $x_{22} \in S$. Since vertex y_i has the maximum degree in $M(T_n)$, we may assume $y_1 \in S$ to DT-dominate more vertices than the other cases. In order to totally dominate y_1, we may assume $y_2 \in S$. Because, all vertices of the first and second blocks are DT-dominated by y_1 and y_2 and also vertex y_2 is either adjacent to some vertices in the third block or at distance two from some vertices of the third and fourth blocks. Thus, $\{y_1, y_2\} \subset S$. Then we will show that we can choose S such that $S \cap \{y_3\} = \emptyset$.

Assume that $y_3 \in S$. If $y_4 \in S$, then we may replace y_3 with y_5. Thus, we can select S such that $y_3 \notin S$. Let $S' = S - \{y_1, y_2\}$ and this means $|S'| = |S| - 2$.

Let $M(T_{n'})$ be a graph which is obtained by deleting the vertices u_i, v_i, x_{ij} and y_i for each $i \in \{1, 2, 3\}$, $j \in \{i, i+1\}$ in which $n' = n - 3 \geq 3$. Since S is a DTD-set of $M(T_n)$, the set S' is a DTD-set of $M(T_{n'})$. Thus, $\gamma_t^d(M(T_{n'})) \leq |S'| = |S| - 2$. If we apply the inductive hypothesis to $M(T_{n'})$, we have $\gamma_t^d(M(T_{n'})) = \lceil \frac{2(n'+1)}{3} \rceil = \lceil \frac{2(n+1)}{3} \rceil - 2$. This means that $\gamma_t^d(M(T_n)) = |S| \geq \lceil \frac{2(n+1)}{3} \rceil$.

As a consequence, $\gamma_t^d(M(T_n)) = \lceil \frac{2(n+1)}{3} \rceil$ from the upper and lower bounds. □

Theorem 4.2. *If $M(D(T_n))$ is a middle graph of a double triangular snake with $n \geq 2$, then $\gamma_t^d(M(D(T_n))) = \gamma_t^d(M(T_n))$.*

Proof. Let $M(D(T_n))$ be a graph which is obtained by subdividing the edges u_iv_j, v_iv_{i+1} and v_iw_j for $i \in \{1, 2, ..., n\}$, $j \in \{i, i+1\}$ with subdivision vertices x_{ij}, y_i and z_{ij}, respectively, and joining the subdivision vertices by an edge if the two corresponding edges share the same vertex of $D(T_n)$.

We first establish the upper bound for $\gamma_t^d(M(D(T_n)))$. Let

$$X = \bigcup_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} \{y_{3i+1}, y_{3i+2}\}$$

as in the proof of Theorem 4.1. If $n \equiv 0 \pmod 3$, then let $Y = X \cup \{y_n\}$, while if $n \equiv 1, 2 \pmod 3$, then let $Y = X \cup \{y_{n-1}, y_n\}$. In all cases, the set Y is a DTD-set of $M(D(T_n))$ and $|Y| = \left\lceil \frac{2(n+1)}{3} \right\rceil$. Thus, $\gamma_t^d(M(D(T_n))) \leq \left\lceil \frac{2(n+1)}{3} \right\rceil$.

We now prove the reverse inequality. Assume $S = \{a_1, a_2, ..., a_s\}$ be a DTD-set of $M(D(T_n))$, where a_i is any vertex of $M(D(T_n))$. Let $f_x = d_{M(D(T_n))}(a_x, a_{x+2})$ for $x \in \{1, 2, ..., s-2\}$. We must prove $f_x \leq 3$ for $x \in \{1, 2, ..., s-2\}$.

Let us suppose that $f_x \geq 4$. We claim that $f_x = 4$ for $x \in \{1, 2, ..., s-2\}$. In accordance with this claim, we construct the set

$$\bigcup_{i=0}^{\lfloor \frac{n}{4} \rfloor - 1} \{y_{4i+1}, y_{4i+2}\}.$$

However some vertices, i.e. vertices u_3 and u_4, are not DT-dominated by this set. Thus, it is needed to add some new vertices to this set, which contradicts our claim. Therefore, $f_x \leq 3$ for $x \in \{1, 2, ..., s-2\}$ which implies that $\sum\limits_{x=1}^{s-2} f_x \leq 3(s-2)$.

If $n \equiv 0 \pmod 3$, then $S \subseteq X \cup \{y_n\}$. Hence, we have

$$3\Big(\left\lceil \frac{2n+1}{3} \right\rceil - 3\Big) + 2 = \sum_{x=1}^{s-3} f_x + f_{s-2} \leq 3(s-2),$$

in which $f_{s-2} = 2$. Since $\lceil \frac{2n+1}{3} \rceil = \frac{2n+3}{3}$ for $n \equiv 0 \pmod 3$, we have $|S| = s \geq \lceil \frac{2n+2}{3} \rceil$.

If $n \equiv 1 \pmod 3$, then $S \subseteq X \cup \{y_{n-1}, y_n\}$. Hence, we have

$$3\Big(\left\lceil \frac{2n+1}{3} \right\rceil - 3\Big) + 4 = \sum_{x=1}^{s-4} f_x + f_{s-3} + f_{s-2} \leq 3(s-2),$$

in which $f_{s-3} = f_{s-2} = 2$. Since $\lceil \frac{2n+1}{3} \rceil = \frac{2n+1}{3}$ for $n \equiv 1 \pmod 3$, we have $|S| = s \geq \lceil \frac{2n+2}{3} \rceil$.

If $n \equiv 2 \pmod 3$, then $S \subseteq X \cup \{y_{n-1}, y_n\}$. Hence, we have

$$3\Big(\left\lceil \frac{2n+1}{3} \right\rceil - 2\Big) = \sum_{x=1}^{s-2} f_x \leq 3(s-2).$$

Since $\lceil \frac{2n+1}{3} \rceil = \frac{2n+2}{3}$ for $n \equiv 2 \pmod 3$, we have $|S| = s \geq \frac{2n+2}{3}$.

The proof is completed by combining the lower and upper bounds for $\gamma_t^d(M(D(T_n)))$ and Theorem 4.1. □

Theorem 4.3. *If $M(D_n)$ is a middle graph of a diamond snake with $n \geq 2$, then $\gamma_t^d(M(D_n)) = n + 1$.*

Proof. For a diamond snake graph D_n, the graph $M(D_n)$ is obtained from $M(D(T_n))$ by deleting the vertex y_i for all $i \in \{1, 2, ..., n\}$. Vertices of D_n and subdivision vertices of $M(D_n)$ are labeled as we use for $M(D_n)$. However, for brevity we use x_k and z_k rather than x_{ij} and z_{ij} for $i \in \{1, 2, ..., n\}$, $j \in \{i, i+1\}$, where k is equal to $i + j - 1$, that is $k \in \{1, 2, ..., 2n\}$.

In order to prove the formula we use induction on $n \geq 2$. As it is clear for $n \leq 3$, we may assume that the formula is true for all values less than $n \geq 4$ and we will now need to show that it is also true for n. Firstly we establish the upper bound on $\gamma_t^d(M(D_n))$. Let

$$X = \bigcup_{i=0}^{\lfloor \frac{2n-1}{4} \rfloor} \{x_{4i+2}, z_{4i+2}\}.$$

If $n \equiv 0, 2 \pmod 4$, then let $Y = X \cup \{x_{2n}\}$ while if $n \equiv 1, 3 \pmod 4$, then let $Y = X$. In all cases, $|Y| = n+1$ and the set Y is a DTD-set of $M(D_n)$. Therefore, $\gamma_t^d(M(D_n)) \leq n+1$.

Let S be a $\gamma_t^d(M(D_n))$-set to prove the opposite inequality and suppose that no two vertices in S are adjacent. Hence, $S = \{x_{2i+1} \mid 0 \leq i \leq n-1\} \cup \{w_1, w_n, v_{n+1}\}$ or $S = \{z_{2i+1} \mid 0 \leq i \leq n-1\} \cup \{u_1, u_n, v_{n+1}\}$ and then $\gamma_t^d(M(D_n)) = |S| = n+3$. However, this contradicts with our upper bound. Therefore, the set S must contain two adjacent vertices. In $M(D_n)$, the vertices which have maximum degree are x_t and y_t for every $t \in \{2, 3, ..., n-1\}$. Thus, we may assume $x_2 \in S$. The vertices in the first block except v_1, z_1 and w_1 are totally dominated by x_2. To totally dominate these vertices and x_2, we may assume $z_2 \in S$. Hence, $\{x_2, z_2\} \subset S$. We will show that $S \cap \{x_3, x_4, x_5, z_3, z_4, z_5\} = \emptyset$.

Assume that $x_3 \in S$. If $x_4 \in S$ (respectively $z_4 \in S$), then replace x_3 with z_4 (respectively with x_4). Let $x_4 \notin S$ (respectively $z_4 \notin S$). If $x_5 \in S$ (respectively $z_5 \in S$), then replace x_3 with z_5 (respectively with x_5). If $x_5 \notin S$, $x_6 \in S$ (respectively $z_5 \notin S$, $z_6 \in S$), then replace x_3 with z_6 (respectively with x_6). Therefore, we can select S such that $x_3 \notin S$.

In similar manner as above, it can be shown that $\{x_4, x_5, z_3, z_4, z_5\} \notin S$, separately. This implies that $\{x_6, z_6\} \subset S$. Let $S' = S - \{x_2, z_2\}$ and then this means $|S'| = |S| - 2$.

Let $M(D_{n'})$ be a graph which is obtained by deleting the vertices u_i, v_i, w_i, x_j and z_j for $i \in \{1, 2\}$, $j \in \{1, 2, 3, 4\}$ in which $n' = n - 2 \geq 2$. Since S is a DTD-set of $M(D_n)$, the set S' is a DTD-set of $M(D_{n'})$. Thus, $\gamma_t^d(M(D_{n'})) \leq |S'| = |S| - 2$. If we apply the inductive hypothesis to $M(D_{n'})$, we have $\gamma_t^d(M(D_{n'})) = n' + 1 = n + 1 - 2$. This means that $\gamma_t^d(M(D_n)) = |S| \geq n + 1$.

As a consequence, $\gamma_t^d(M(D_n)) = n + 1$. □

References

[1] Akiyama, J., Hamada, T. and Yoshimura, I., (1975), On characterizations of the middle graphs, TRU Math, 11, pp. 35-39.

[2] Barrientos, C., (2001), Graceful labelings of cyclic snakes, Ars Combinatoria, 60, pp. 85-96.

[3] Cockayne, E. J., Dawes, R. M. and Hedetniemi, S. T., (1980), Total domination in graphs, Networks, 10(3), pp. 211-219.

[4] Çiftçi C. and Aytaç, V., (2020), Disjunctive total domination subdivision number of graphs, Fundamenta Informaticae, volume 174.

[5] Goddard, W., Henning, M. A., and McPillan, C. A., (2014), The disjunctive domination number of a graph, Quaestiones Mathematicae, 37(4), pp. 547-561.

[6] Hamada, T. and Yoshimura, I., (1976), Traversability and connectivity of the middle graph of a graph, Discrete Mathematics, 14(3), pp. 247-255.

[7] Haynes T.W., Hedetniemi S.T. and Slater P.J., (1998), Fundamentals of Domination in Graphs, Marcel Dekker Inc., New York.

[8] Henning M.A. and Naicker V., (2016), Disjunctive total domination in graphs, Journal of Combinatorial Optimization, 31(3), pp. 1090-1110.

[9] Henning M.A. and Naicker V., (2016), Bounds on the disjunctive total domination number of a tree, Discussiones Mathematicae Graph Theory, 36(1), pp. 153-171.

[10] Jamil, F. P. and Malalay, R. P., (2019), On disjunctive domination in graphs, Quaestiones Mathematicae, pp. 1-20, DOI: 10.2989/16073606.2018.1547797.

[11] Kalpana, M. and Vijayalakshmi, D., (2019), On b-coloring of central graph of some graphs, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(1), pp. 1229-1239.

[12] Kazemnejad F. and Moradı S., (2019), Total domination number of central graphs, Bull. Korean Math. Soc. 56 (4), pp. 1059-1075.

[13] Lin C.F, Peng S.L. and Yang H.D., (2016), Disjunctive total domination numbers of grid graphs, International Computer Symposium (ICS), IEEE, pp. 80-83.

[14] Naicker, V., and Henning, M. A., (2015), Graphs with large disjunctive total domination number. Discrete Mathematics & Theoretical Computer Science, 17(1), pp. 255-281.

[15] Natarajan, C., Ayyaswamy, S. K., and Sathiamoorthy, G., (2018), A note on hop domination number of some special families of graphs, International Journal of Pure and Applied Mathematics, 119(12), pp. 14165-14171.

[16] Rosa, A., (1967), Cyclic Steiner triple systems and labeling of triangular cacti, Scientia, 5, pp. 87-95.

[17] Thilagavathi, K. and Roopesh, N., (2009), Generalization of achromatic colouring of central graphs, Electronic Notes in Discrete Mathematics, 33, pp. 147-152.

[18] Vivin, J.V., (2007), Harmonious coloring of total graphs, n-leaf, central graphs and circumdetic graphs, Ph.D. Thesis, Bharathiar University, Coimbatore, India.

[19] Yi E., (2017), Disjunctive total domination in permutation graphs, Discrete Mathematics, Algorithms and Applications, 9 (1), 1750009 .

[20] Yue, X., Yuansheng, Y. and Liping, W., (2008), On harmonious labeling of the double triangular snake, Indian Journal of Pure & Applied Mathematics, 39(2), pp. 177-184.

Canan Çiftçi received her MSc degree from Karadeniz Technical University, Turkey, and received her PhD degree in mathematics from Ege University, Turkey in 2012 and 2017, respectively. She has been a member of Ordu University, Turkey, in the department of mathematics since 2010. Her research interests include graph theory, especially vulnerability parameters in graphs, domination and coloring in graphs, combinatorics and discrete mathematics.

Aysun Aytaç graduated from Department of Mathematics, Ege University, Izmir, Turkey in 1993. She received her MSc and PhD degrees in mathematics and computer science from Ege University in 1996 and 2001, respectively. She has been a member of Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey since 1996. Her research interests focus mainly on graph theory, especially vulnerability parameters in graphs, domination in graphs, optimization, combinatorics and discrete mathematics.