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CERTAIN SUBCLASS OF UNIVALENT FUNCTIONS ASSOCIATED WITH M–SERIES BASED ON q–DERIVATIVE

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Abstract. By applying the q–derivative, M–series, convolution an subordination structures, we introduce a new subclass of univalent functions. For this subclass of functions, we obtain coefficient inequality, convexity and convolution preserving property. Some consequences of geometric properties are also considered.

Keywords: M–series, q–derivative, Convolution, Univalent function, Radii of starlikeness and convexity.

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1. Introduction

The M–series investigated by Sharma [10] and is denoted by:

$$\begin{aligned} {}_{x}^{\alpha}\mathcal{M}_{y}(b_1,\ldots,b_x;d_1,\ldots,d_y;z) &= {}_{x}^{\alpha}\mathcal{M}_{y}(z) \\ &= \sum_{k=0}^{\infty}\frac{(b_1)_k\cdots(b_x)_k}{(d_1)_k\cdots(d_y)_k}\frac{z^k}{\Gamma(\alpha k+1)}, \end{aligned} \tag{1}$$

where $\alpha, z \in \mathbb{C}$, $\mathrm{Re}\{a\alpha\} > 0$ and $(b_m)_k$, $(d_m)_k$ are the well-known Pochhammer symbols which are defined in terms of the Gamma function by:

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & , \quad k=0, \\ x(x+1)\ldots(x+k-1) & , \quad k\in\mathbb{N}=\{1,2,\ldots\}. \end{cases} \tag{2}$$

It is easy to see that by the ratio test, the series in (1) is convergence for all z if $x \leqslant y$. The extension of both Mittag-Leffler function and generalized hypergeometric function ${}_rF_s$ called generalized M–series was introduced in [12] and denoted by:

$${}_{x}^{\alpha}\mathcal{M}_{y}^{\beta}(z) = \sum_{k=0}^{\infty}\frac{(b_1)_k\cdots(b_x)_k}{(d_1)_k\cdots(d_y)_k}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (z,\alpha,\beta\in\mathbb{C}). \tag{3}$$

For more details see [6] and [11].

The series in (3) is convergence for all z if $x \leqslant y + \mathrm{Re}\{\alpha\}$. Also it is convergent for $|z| < \alpha^{\alpha}$, if $x = y + \mathrm{Re}\{\alpha\}$.

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The q–analogue of Pochhammer symbol is defined by:

$$(\gamma;q)_k=\prod_{n=0}^{k-1}(1-\gamma q^n),\qquad (k\in\mathbb{N}),\tag{4}$$

and for $k=0$ and $q\neq 1$, $(\gamma;q)_0=1$.

When $k\to\infty$, we shall assume that $|q|<1$, see [3]. Also q–derivative of a function $f(z)$ is defined by:

$$\big(\partial_q f\big)(z)=\frac{f(z)-f(qz)}{(1-q)z},\qquad (z\neq 0,\quad q\neq 0),\tag{5}$$

and

$$\lim_{q\to 1}\big(\partial_q f\big)(z)=f'(z).\tag{6}$$

By using (5), we conclude that:

$$\big(\partial^n_{\frac{1}{q}}f\big)(z)=q^n\big(\partial^n_q f\big)\big(\frac{z}{q^n}\big),\tag{7}$$

$$\partial^n_q z^\lambda=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-n+1)}z^{\lambda-n},\qquad (\mathrm{Re}\{\lambda\}+1\geqslant 0).\tag{8}$$

Indeed $\Gamma_q(z+1)=\frac{1-q^z}{1-q}\Gamma_q(z)$, see [3] and [5].

Further, the q–analogue of the Beta function is defined by:

$$\beta_q(x,y)=\int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}d_q(t)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)},\tag{9}$$

where $\mathrm{Re}\{x\}>0$, $\mathrm{Re}\{y\}>0$ and $\Gamma_q(\cdot)$ is the q–gamma function.

Now, we consider the q–analogue of generalized M–series as follow:

$${}_x^{\alpha}\mathcal{M}_y^{\beta}(z;q)=\sum_{k=0}^{\infty}\frac{(b_1;q)_k\cdots(b_x;q)_k}{(d_1;q)_k\cdots(d_y;q)_k(q;q)_k}\frac{z^k}{\Gamma_q(\alpha k+\beta)},\tag{10}$$

where $\alpha,\beta\in\mathbb{C}$, $\mathrm{Re}\{\alpha\}>0$, $|q|<1$, $(\gamma;q)_k$ is the q–analogue of Pochhammer symbol and $\Gamma_q(\cdot)$ is the q–gamma function, see [8]. We note that ${}_x^{\alpha}\mathcal{M}_y^{\beta}(z;q)$ is convergent, see [6].

Some special cases of ${}_x^{\alpha}\mathcal{M}_y^{\beta}(z;q)$ are:

(1) The q–Mittag-Leffler function [7].
(2) The generalized q–Mittag-Leffler function [12].
(3) The q–generalized M–series as a special case of the well-known q–Wright generalized hypergeometric function [9].

Let $\mathcal{A}$ denote the class of function $f(z)$ of the type:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k,\tag{11}$$

which are analytic in the open unit disk:

$$\mathbb{D}=\{z\in\mathbb{C}:|z|<1\},\tag{12}$$

and $\mathcal{N}$ be a subclass of $\mathcal{A}$ consisting of functions with negative coefficients of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k,\qquad (a_k\geqslant 0).\tag{13}$$

For $f(z)$ given by (11) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ the Hadamard product (convolution) of f and g denoted by $(f * g)$ is defined by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{14}$$

Further for f and g analytic in $\mathbb{D}$, we say that f is subordinate to g written $f \prec g$, if there exists a function w analytic in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Now we introduce a new subclass of $\mathcal{N}$ denoted by $\mathcal{S}(A, B, t)$ consisting of all functions in $\mathcal{N}$ for which:

$$\frac{zH'(z)}{f_t(z)} \prec \frac{1 + Az}{1 + Bz} \tag{15}$$

or equivalently

$$\left| \frac{\frac{zH'(z)}{f_t(z)} - 1}{A - Bz\frac{H'(z)}{f_t(z)}} \right| < 1, \tag{16}$$

where $0 \leqslant t \leqslant 1$, $-1 \leqslant B \leqslant 1$, $-1 \leqslant A \leqslant 1$,

$$H(z) = (f * F)(z), \qquad f_t(z) = (1 - t)z + tf(z), \tag{17}$$

$f(z) \in \mathcal{N}$ and

$$F(z) = \left(1 + \frac{(1 - b_1) \cdots (1 - b_x)}{(1 - d_1) \cdots (1 - d_y)\Gamma(\alpha k + \beta)}\right) z + \frac{1}{\Gamma_q(\beta)} - {}_x^{\alpha}\mathcal{M}_y^{\beta}(z; q). \tag{18}$$

From (14), (17) and (18) with a simple calculation, we get:

$$H(z) = z - \sum_{k=2}^{\infty} \frac{(b_1; q)_k \cdots (b_x; q)_k}{(d_1; q)_k \cdots (d_y; q)_k (q; q)_k \Gamma_q(\alpha k + \beta)} a_k z^k. \tag{19}$$

For more details about q–calculus, M–series and related areas, one may refer to the recent papers [1, 2] and [4] on the subject.

2. Main results

In this section, we shall obtain sharp coefficient estimates for functions in $\mathcal{S}(A, B, t)$. Also we will prove $\mathcal{S}(A, B, t)$ is a convex set.

Theorem 2.1. *Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{N}$. Then $f \in \mathcal{S}(A, B, t)$ if and only if*

$$\sum_{k=2}^{\infty} \left[\left(\frac{k(b_1; q)_k \cdots (b_x; q)_k}{(d_1; q)_k \cdots (d_y; q)_k (q; q)_k \Gamma(\alpha k + \beta)} - t \right) \frac{1 - B}{A - B} + t \right] a_k \leqslant 1, \tag{20}$$

where $-1 \leq A, B \leq 1$, $0 \leq t \leq 1$ and $B \leq A$.

Proof. Let $z \in \partial\mathbb{D} = \{z : |z| = 1\}$, so by (11) and (19) we have:

$$\begin{aligned}
X &= \left|zH'(z) - f_t(z)\right| - \left|Af_t(z) - BzH'(z)\right| \\
&= \left|z - \sum_{k=2}^{\infty} \frac{k(b_1;q)_k \cdots (b_x;q)_k a_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma_q(\alpha k + \beta)} z^k - (1-t)z - tf(z)\right| \\
&- \left|A(1-t)z + tf(z) - B\left(z - \sum_{k=2}^{\infty} \frac{k(b_1;q)_k \cdots (b_x;q)_k a_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma_q(\alpha k + \beta)} z^k\right)\right| \\
&= \left|-\sum_{k=2}^{\infty} \left[\frac{k(b_1;q)_k \cdots (b_x;q)_k a_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma_q(\alpha k + \beta)} - t\right] a_k z^k\right| \\
&- \left|(A-B)z - \sum_{k=2}^{\infty} \left[tA - Bk\frac{(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma_q(\alpha k + \beta)}\right] a_k z^k\right|.
\end{aligned}$$

By putting

$$\begin{aligned}
&tA - Bk\frac{(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)} = \\
&t(A-B) - \left[k\frac{(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)} - t\right] B,
\end{aligned}$$

the above expression reduces to

$$X \leqslant \left|\sum_{k=2}^{\infty} \left[\frac{k(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)}(1-B) + t(A-B)\right] a_k - (A-B)\right|,$$

and

$$\begin{aligned}
X &\leqslant \left|(A-B) - \sum_{k=2}^{\infty} \left[\left(\frac{k(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma_q(\alpha k + \beta)} - t\right)(1-B) + t(A-B)\right] a_k\right| \\
&\leqslant \left|1 - \sum_{k=2}^{\infty} \left[\left(\frac{k(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma_q(\alpha k + \beta)} - t\right)\frac{(1-B)}{(A-B)} + t\right] a_k\right| \\
&< 1.
\end{aligned}$$

By using (20), we get $X \leq 1$, so $f \in \mathcal{S}(A, B, t)$.

To prove the converse, let $f \in \mathcal{S}(A, B, t)$, thus:

$$\left|\frac{\frac{zH'(z)}{f_t(z)} - 1}{A - B\frac{zH'(z)}{f_t(z)}}\right| = \frac{\left|z - \sum_{k=2}^{\infty} \frac{k(b_1;q)_k \cdots (b_x;q)_k a_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)} z^k - (1-t)z - tf(z)\right|}{\left|A\big((1-t)z + tf(z)\big) - B\left(z - \sum_{k=2}^{\infty} \frac{k(b_1;q)_k \cdots (b_x;q)_k a_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)} z^k\right)\right|} < 1,$$

for all $z \in \mathbb{D}$. Since for all $z \in \mathbb{D}$, $\mathrm{Re}\{z\} \leqslant |z|$, we have:

$$\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty} \left[\frac{k(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)} - t\right] a_k z^k}{(A-B)z - \sum_{k=2}^{\infty} \left[tA - \frac{Bk(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)}\right] a_k z^k}\right\} < 1.$$

By letting $z \to 1$, through positive real values and choose the values of z such that $\frac{zH'(z)}{f_t(z)}$ is real, we get:

$$\sum_{k=2}^{\infty}\left(\frac{k(b_1;q)_k\cdots(b_x;q)_k}{(d_1;q)_k\cdots(d_y;q)_k(q;q)_k\Gamma(\alpha k+\beta)}-t\right)a_k$$
$$\leqslant (A-B)-\sum_{k=2}^{\infty}\left(tA-\frac{Bk(b_1;q)_k\cdots(b_x;q)_k}{(d_1;q)_k\cdots(d_y;q)_k(q;q)_k\Gamma(\alpha k+\beta)}\right)a_k,$$

and this gives the required results. □

Remark 2.1. *We note that the function*

$$G(z)=z-\frac{A-B}{\left(\frac{2(b_1;q)_2\cdots(b_x;q)_2}{(d_1;q)_2\cdots(d_y;q)_2(q;q)_2\Gamma(2\alpha+\beta)}-t\right)(1-B)+t(A-B)}z^2, \tag{21}$$

shows that the inequality (20) *is sharp. Also for all* $k\geqslant 2$*, we have:*

$$a_k\leqslant\frac{(A-B)}{\left(\frac{k(b_1;q)_k\cdots(b_x;q)_k}{(d_1;q)_k\cdots(d_y;q)_k(q;q)_k\Gamma(\alpha k+\beta)}-t\right)(1-B)+t(A-B)}. \tag{22}$$

Theorem 2.2. *$\mathcal{S}(A,B,t)$ is a convex set, where $-1\leq A\leq 1$, $-1\leq B\leq 1$ and $0\leq t\leq 1$.*

Proof. To establish the required result, it is sufficient to prove that if the functions $f_j(z)$, $(j=1,2,\dots,m)$ be in the class $\mathcal{S}(A,B,t)$, then the function $h(z)=\sum_{j=1}^{m}\lambda_j f_j(z)$, ($\lambda_j\geqslant 0$, $\sum_{j=1}^{m}\lambda_j=1$) is also in $\mathcal{S}(A,B,t)$. But by definition of $h(z)$, we obtain:

$$h(z)=\sum_{j=1}^{m}\lambda_j\left(z-\sum_{k=2}^{\infty}a_{k,j}z^k\right)$$
$$=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}\lambda_j a_{k,j}\right)z^k.$$

But by Theorem 2.1, we have:

$$\sum_{k=2}^{\infty}\left[\left(\frac{k(b_1;q)_k\cdots(b_x;q)_k}{(d_1;q)_k\cdots(d_y;q)_k(q;q)_k\Gamma(\alpha k+\beta)}-t\right)(1-B)+t(A-B)\right]\left(\sum_{j=1}^{m}\lambda_j a_{k,j}\right)$$
$$=\sum_{j=1}^{m}\lambda_j\left\{\sum_{k=2}^{\infty}\left[\left(\frac{k(b_1;q)_k\cdots(b_x;q)_k}{(d_1;q)_k\cdots(d_y;q)_k(q;q)_k\Gamma(\alpha k+\beta)}-t\right)(1-B)+t(A-B)\right]a_{k,j}\right\}$$
$$\leqslant\sum_{j=1}^{m}\lambda_j(A-B)=A-B,$$

which completes the proof. □

3. Geometric properties of $\mathcal{S}(A,B,t)$

In the last section, we show that the class $\mathcal{S}(A,B,t)$ is closed under convolution. Also radii of starlikeness convexity are introduced.

Theorem 3.1. *Let the function f and g defined by:*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = z - \sum_{k=2}^{\infty} b_k z^k,$$

be in the class $\mathcal{S}(A,B,t)$, then $f * g$ given by (14) *belongs to $\mathcal{S}(A,B,t)$, where $B_0 \leqslant \frac{AY-1}{Y-1}$,*

$$Y = \frac{t + \Big[\big(Q(\alpha,\beta) - t\big)\big(\frac{1-B}{A-B}\big) + t\Big]^2}{Q(\alpha,\beta) - t}, \tag{23}$$

$$Q(a,b) = \frac{k(b_1;q)_k \cdots (b_x;q)_k}{(d_1;q)_k \cdots (d_y;q)_k (q;q)_k \Gamma(\alpha k + \beta)}. \tag{24}$$

Proof. It is sufficient to show that:

$$\sum_{k=2}^{\infty} \Big[\big(Q(\alpha,\beta) - t\big)\big(\frac{1-B}{A-B}\big) + t\Big] a_k b_k \leqslant 1,$$

where $Q(a,b)$ is defined by (24).

By using Cauchy-Schwartz inequality, from (20), we obtain:

$$\sum_{k=2}^{\infty} \Big[\big(Q(\alpha,\beta) - t\big)\big(\frac{1-B}{A-B}\big) + t\Big] \sqrt{a_k b_k} \leqslant 1.$$

Hence, we find the largest B_0 such that:

$$\sum_{k=2}^{\infty} \Big[\big(Q(\alpha,\beta) - t\big)\big(\frac{1-B_0}{A-B_0}\big) + t\Big] a_k b_{\leqslant} \sum_{k=2}^{\infty} \Big[\big(Q(\alpha,\beta) - t\big)\big(\frac{1-B}{A-B}\big) + t\Big] \sqrt{a_k b_k} \leqslant 1,$$

or equivalently

$$\sqrt{a_k b_k} \leqslant \frac{\big(Q(\alpha,\beta) - t\big)\big(\frac{1-B}{A-B}\big) + t}{\big(Q(\alpha,\beta) - t\big)\big(\frac{1-B_0}{A-B_0}\big) + t}, \qquad (k \geqslant 2).$$

This inequality holds if

$$\frac{A-B}{\big(Q(\alpha,\beta) - t\big)(1-B) + t(A-B)} \leqslant \frac{\Big[\big(Q(\alpha,\beta) - t\big)(1-B) + t(A-B)\Big](A-B_0)}{\Big[\big(Q(\alpha,\beta) - t\big)(1-B_0) + t(A-B_0)\Big](A-B)},$$

or equivalently

$$B_0 \leqslant \frac{AY-1}{Y-1},$$

where Y is given by (23), and this completes the proof. □

Theorem 3.2. *If $f(z) \in \mathcal{S}(A,B,t)$, then:*

(1) *f is univalently starlike of order δ $(0 \leqslant \delta < 1)$ in $|z| < R_1$, where:*

$$R_1 = \inf_{k \geqslant 2} \left[\frac{1-\delta}{k-\delta} \left(\big(kQ(\alpha,\beta) - t\big)\big(\frac{1-B}{A-B}\big) + t\right)\right]^{\frac{1}{k-1}},$$

and $Q(\alpha,\beta)$ is given by (14).

(2) *f is univalently convex of order δ $(0 \leqslant \delta < 1)$ in $|z| < R_2$, where:*

$$R_2 = \inf_{k \geqslant 2} \left[\frac{1-\delta}{k(k-\delta)} \left(\big(kQ(\alpha,\beta) - t\big)\big(\frac{1-B}{A-B}\big) + t\right)\right]^{\frac{1}{k-1}}.$$

Proof.

(1) It is sufficient to show that $\left|\frac{zf'}{f}-1\right| \leqslant 1-\delta$ for $|z|<R_1$. But

$$\left|\frac{zf'}{f}-1\right| = \left|\frac{\sum\limits_{k=2}^{\infty}(k-1)a_k z^k}{z-\sum\limits_{k=2}^{\infty}a_k z^k}\right| \leqslant \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k|z|^{k-1}} \leqslant 1-\delta,$$

or

$$\sum_{k=2}^{\infty}\left(\frac{k-\delta}{1-\delta}\right)a_k|z|^{k-1} \leqslant 1.$$

By applying (22), we conclude the result.

(2) Since f is convex "if and only if zf' is starlike", we get the required result, so the proof is complete.

□

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