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$K_n(\lambda)$ IS FULLY $\{P_7, S_4\}$-DECOMPOSABLE

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ABSTRACT. Let P_{k+1} denote a path of length k, S_m denote a star with m edges, and $K_n(\lambda)$ denote the complete multigraph on n vertices in which every pair of distinct vertices is joined by λ edges. In this paper, we have obtained the necessary conditions for a $\{P_{k+1}, S_m\}$-decomposition of $K_n(\lambda)$ and proved that the necessary conditions are also sufficient when $k = 6$ and $m = 4$.

Keywords: Decomposition, Complete multigraph, Path, Star.

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1. Introduction

All graphs considered here are finite and undirected with no loops. For the standard graph-theoretic terminology the reader is referred to [1]. A simple graph in which every pair of distinct vertices is joined by an edge is called a *complete graph*, denoted by K_n. If more than one edge joining two vertices are allowed, the resulting object is called a *multigraph*. Let $K_n(\lambda)$ denote the *complete multigraph* on n vertices in which every pair of distinct vertices is joined by λ edges. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = a$ and $|Y| = b$, such a graph is denoted by $K_{a,b}$. In $K_{a,b}(\lambda)$, we label the vertices in the partite set X as $\{x_1, x_2, \ldots, x_a\}$ and Y as $\{x_{a+1}, x_{a+2}, \ldots, x_{a+b}\}$. If $a = b$, the complete bipartite graph is referred to as *balanced*. A *path* is an open trail with no repeated vertex. A path with k edges is denoted by P_{k+1}. The complete bipartite graph $K_{1,m}$ is called a *star* and is denoted by S_m. For $m \geq 3$, the vertex of degree m in S_m is called the *center* and any vertex of degree 1 in S_m is called an *end vertex*.

Let G be a graph and G_1 be a subgraph of G. Then $G \backslash G_1$ is obtained from G by deleting the edges of G_1. Let G_1 and G_2 be subgraphs of G. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We say that G_1 and

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G_2 are *edge-disjoint* if they have no edge in common. If G_1 and G_2 are edge-disjoint, we denote their union by G_1+G_2. A *decomposition* of a graph G is a collection of edge-disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G is in exactly one G_i. Here it is said that G is *decomposed* or *decomposable* into $G_1, G_2, \ldots, G_n$. If G has a decomposition into p_1 copies of $G_1, \ldots, p_n$ copies of G_n, then we say that G has a $\{p_1G_1, \ldots, p_nG_n\}$-decomposition. If such a decomposition exists for all values of $p_1, \ldots, p_n$ satisfying trivial necessary conditions, then we say that G has a $\{G_1, \ldots, G_n\}_{\{p_1,\ldots,p_n\}}$-decomposition or G is fully $\{G_1, \ldots, G_n\}$-decomposable.

In [6], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of $\{pG_1, qG_2\}$-decomposition of $K_n(\lambda)$, when $(G_1, G_2) \in \{(P_n, S_{n-1}), (C_n, S_{n-1}), (P_n, C_n)\}$. In [7], Priyadharsini gave the necessary and sufficient conditions for the existence of $\{pP_n, qS_{n-1}\}$-decomposition of $K_{n+1}(\lambda)$. In [8], Shyu gave the necessary and sufficient conditions for a $\{P_4, S_3\}_{\{p,q\}}$-decomposition of K_n and also discussed the existence of $\{P_{k+1}, S_k\}_{\{p,q\}}$-decomposition of K_n, when $n \geq 4k$ such that either k is even and $p \geq \frac{k}{2}$ or k is odd and $p \geq k$. In [9], Shyu proved that the necessary conditions are also sufficient for the $\{P_{k+1}, S_k\}_{\{p,q\}}$-decomposition of K_n, when $n \geq 4k$. In [5], Ilayaraja and Muthusamy proved that K_n is fully $\{P_4, S_4\}$-decomposable. In [3], Lee and Chen showed the existence of $\{pP_{k+1}, qS_k\}$-decomposition of $K_n(\lambda)$ and $K_{b,b}(\lambda)$. In [2], Lee and Chen gave the necessary and sufficient conditions for a $\{F, S_3\}_{\{p,q\}}$-decomposition of K_n with $F \in \{P_n, C_n\}$. In [10], Shyu gave the necessary conditions for a $\{pC_k, qP_{k+1}, rS_k\}$-decomposition of K_n and proved that K_n is fully $\{C_4, P_5, S_4\}$-decomposable, when n is odd. In this paper we prove that $K_n(\lambda)$ is fully $\{P_7, S_4\}$-decomposable.

2. Preliminaries

For convenience we denote $V(K_n(\lambda)) = \{x_1, x_2, \ldots, x_n\}$. The notation $S(x_1; x_2 \cdots x_m)$ denotes an m-star with x_1 as center vertex and $x_2, \ldots, x_m$ as end vertices, and $[x_1x_2 \cdots x_{k+1}]$ is a $k+1$-path with vertices $x_1, x_2, \ldots, x_{k+1}$ and edges $x_1x_2, x_2x_3, \ldots, x_kx_{k+1}$.

We recall here some results on P_{k+1} and S_m-decompositions that are useful for our proofs.

Theorem 2.1. [11] *A necessary and sufficient conditions for the existence of a P_{k+1}-decomposition of $K_n(\lambda)$ into edge-disjoint simple paths of length k is $\lambda\binom{n}{2} \equiv 0 \pmod{k}$ and $n \geq k+1$.*

Theorem 2.2. [12] *A necessary and sufficient conditions for the existence of a S_m-decomposition of $K_n(\lambda)$ is that: (i) $\lambda\binom{n}{2} \equiv 0 \pmod{m}$ (ii) $n \geq 2m$ for $\lambda = 1$ (iii) $n \geq m+1$ for even λ (iv) $n \geq m+1+\frac{m}{\lambda}$ for odd $\lambda \geq 3$.*

Theorem 2.3. [13] *Let k be a positive integer and let a and b be positive even integers such that $a \geq b$. $K_{a,b}(\lambda)$ has a P_{k+1}-decomposition if and only if $a \geq \lceil \frac{k+1}{2} \rceil, b \geq \lceil \frac{k}{2} \rceil$ and $\lambda ab \equiv 0 \pmod{k}$.*

Theorem 2.4. [4] *For positive integers a and b with $a \geq b$, the complete bipartite multigraph $K_{a,b}(\lambda)$ is S_m-decomposable if and only if $a \geq m$ and (i) $\lambda a \equiv 0 \pmod{m}$ if $b < m$ (ii) $\lambda ab \equiv 0 \pmod{m}$ if $b \geq m$.*

In the following Theorem, we discuss the necessary conditions for a $\{pP_{k+1}, qS_m\}$-decomposition of $K_n(\lambda)$, when $\lambda \geq 1$.

Theorem 2.5. *Let λ, n, k and m be positive integers. Let p and q be non-negative integers. The necessary condition for a $\{pP_{k+1}, qS_m\}$-decomposition of $K_n(\lambda)$ is $pk + qm = \lambda\binom{n}{2}$ and $n \geq \max\{k+1, m+1\}$.*

In this paper, we prove that the above necessary condition is sufficient for a $\{P_7, S_4\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$ in Theorem 3.1.

3. Main result

In this section, we discuss a $\{P_7, S_4\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$, when $\lambda \geq 1$. Since $K_n(\lambda)$ cannot be decomposed into P_7 and S_4 when $n \leq 6$, we discuss the decompositions for $n \geq 7$.

Remark 3.1. *The necessary conditions for the existence of a $\{P_7, S_4\}_{\{p,q\}}$-decomposition in $K_n(\lambda)$ is satisfied when $n \equiv 0, 1 \pmod 4$ if $\lambda \geq 1$ and $n \equiv 2, 3 \pmod 4$ if λ is even. i.e., there does not exist non-negative integers p and q satisfying $6p + 4q = \lambda\binom{n}{2}$ when $n \equiv 2, 3 \pmod 4$ if λ is odd.*

In the following two lemmas, we discuss $\{P_7, S_4\}_{\{p,q\}}$-decompositions of $K_{4,6}$ and $K_{3,6}(2)$ which we use further to decompose $K_n(\lambda)$ into $\{pP_7, qS_4\}$.

Lemma 3.1. *If p and q are non-negative integers such that $6p + 4q = 24$, then $K_{4,6}$ is fully $\{P_7, S_4\}$-decomposable.*

Proof. $(p, q) \in \{(4, 0), (2, 3), (0, 6)\}$. By Theorem 2.3, $K_{4,6}$ is $\{4P_7, 0S_4\}$-decomposable. $K_{4,6}$ can be decomposed into $2P_7 : [x_2x_8x_1x_9x_3x_{10}x_4], [x_3x_8x_4x_9x_2x_{10}x_1]$ and $3S_4 : S(x_5; x_1, x_2, x_3, x_4), S(x_6; x_1, x_2, x_3, x_4), S(x_7; x_1, x_2, x_3, x_4)$. By Theorem 2.4, $K_{4,6}$ is $\{0P_7, 6S_4\}$-decomposable. Therefore $K_{4,6}$ is fully $\{P_7, S_4\}$-decomposable. □

Lemma 3.2. *If p and q are non-negative integers such that $6p + 4q = 36$, then $K_{3,6}(2)$ is fully $\{P_7, S_4\}$-decomposable.*

Proof. $(p, q) \in \{(6, 0), (4, 3), (2, 6), (0, 9)\}$. $K_{3,6}(2)$ can be decomposed into $6P_7$: 2 copies of $[x_4x_1x_5x_2x_6x_3x_9], [x_5x_3x_8x_2x_7x_1x_6], [x_8x_1x_9x_2x_4x_3x_7]$. $K_{3,6}(2)$ can be decomposed into $4P_7 : [x_4x_3x_9x_2x_8x_1x_5], [x_4x_2x_6x_3x_7x_1x_9], [x_4x_1x_5x_2x_7x_3x_8], [x_4x_2x_5x_3x_6x_1x_8]$ and $3S_4 : S(x_1; x_4, x_6, x_7, x_9), S(x_2; x_6, x_7, x_8, x_9), S(x_3; x_4, x_5, x_8, x_9)$. $K_{3,6}(2)$ can be decomposed into $2P_7 : [x_4x_3x_9x_2x_8x_1x_5], [x_4x_2x_6x_3x_7x_1x_9]$ and $6S_4 : S(x_1; x_4, x_6, x_7, x_9), S(x_1; x_4, x_5, x_6, x_8), S(x_2; x_5, x_7, x_8, x_9), S(x_2; x_4, x_5, x_6, x_7), S(x_3; x_4, x_5, x_8, x_9), S(x_3; x_5, x_6, x_7, x_8)$. By Theorem 2.4, $K_{3,6}(2)$ is $\{0P_7, 9S_4\}$-decomposable. Therefore $K_{3,6}(2)$ is fully $\{P_7, S_4\}$-decomposable. □

We now prove our main result.

Theorem 3.1. *For any non-negative integers p and q and any integer $n \geq 7$, there exists a $\{P_7, S_4\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$ if and only if $6p + 4q = \lambda\binom{n}{2}$.*

Proof. The necessary conditions are obvious. First we prove the result for $7 \leq n \leq 17$; then we use induction to settle the remaining cases. As we discuss $\{pP_7, qS_4\}$-decompositions of $K_n(\lambda)$ for all possible choices of p and q, we have the following cases:
Case 1: $n = 7$.
If $\lambda = 2$, then $(p, q) \in \{(7, 0), (5, 3), (3, 6), (1, 9)\}$. By Theorem 2.1, $K_7(2)$ is $\{7P_7, 0S_4\}$-decomposable. The graph $K_7(2)$ can be decomposed into $5P_7 : [x_1x_3x_2x_4x_7x_6x_5], [x_1x_4x_3x_5x_2x_6x_7], [x_2x_1x_6x_3x_7x_5x_4], [x_6x_1x_5x_4x_3x_2x_7], [x_1x_7x_3x_5x_2x_4x_6]$ and $3S_4 : S(x_1; x_2, x_3, x_4, x_5), S(x_6; x_2, x_3, x_4, x_5), S(x_7; x_1, x_2, x_4, x_5)$. $K_7(2)$ can be decomposed into $3P_7 : [x_7x_6x_1x_3x_2x_4x_5], [x_2x_1x_7x_5x_6x_3x_4], [x_1x_5x_3x_2x_6x_4x_7]$ and $6S_4 : S(x_1; x_2, x_3, x_4, x_6), S(x_2; x_4, x_5, x_6, x_7), S(x_3; x_4, x_5, x_6, x_7), S(x_4; x_1, x_5, x_6, x_7), S(x_5; x_1, x_2, x_6, x_7), S(x_7; x_1, x_2, x_3, x_6)$. $K_7(2)$ can be decomposed into a $P_7 : [x_1x_2x_3x_4x_5x_6x_7]$ and $9S_4 : S(x_1; x_3, x_4, x_5, x_6), S(x_1; x_3, x_5, x_6, x_7), S(x_2; x_1, x_4, x_6, x_7), S(x_2; x_3, x_4, x_5, x_7), S(x_3; x_4, x_5, x_6, x_7), S(x_4; x_1, x_5, x_6, x_7), S(x_5; x_2, x_3, x_6, x_7), S(x_6; x_2, x_3, x_4, x_7), S(x_7; x_1, x_3, x_4, x_5)$.

If $\lambda = 4$, then $(p, q) \in \{(14, 0), (12, 3), (10, 6), \dots, (0, 21)\}$ (we see that the values of p decreases by 2 and the values of q increases by 3). We write $K_7(4) = K_7(2) + K_7(2) = \{(7, 0), (5, 3), (3, 6), (1, 9)\}+\{(7, 0), (5, 3), (3, 6), (1, 9)\} = \{(14, 0), (12, 3), (10, 6), (8, 9), (6, 12), (4, 15), (2, 18)\}$. By Theorem 2.2, $K_7(4)$ is $\{0P_7, 21S_4\}$-decomposable.

If $\lambda \geq 6$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 4$: We write $K_7(\lambda) = \frac{\lambda}{4}K_7(4)$.

$\lambda \equiv 2 \pmod 4$: We write $K_7(\lambda) = K_7(\lambda - 2) + K_7(2) = \frac{\lambda-2}{4}K_7(4) + K_7(2)$. Therefore $K_7(\lambda)$ is fully $\{P_7, S_4\}$-decomposable.

Case 2: $n = 8$.

If $\lambda = 1$, then $(p, q) \in \{(4, 1), (2, 4), (0, 7)\}$. The graph K_8 can be decomposed into $4P_7 : [x_3x_1x_5x_8x_4x_2x_6], [x_8x_7x_6x_3x_2x_5x_4], [x_1x_8x_3x_7x_4x_6x_5], [x_6x_8x_2x_7x_5x_3x_4]$ and a $S_4 : S(x_1; x_2, x_4, x_6, x_7)$. K_8 can be decomposed into $2P_7 : [x_2x_1x_4x_3x_7x_8x_5], [x_2x_3x_8x_4x_7x_6x_5]$ and $4S_4 : S(x_1; x_3, x_5, x_7, x_8), S(x_2; x_4, x_6, x_7, x_8), S(x_5; x_2, x_3, x_4, x_7), S(x_6; x_1, x_3, x_4, x_8)$. By Theorem 2.2, K_8 is $\{0P_7, 7S_4\}$-decomposable.

If $\lambda = 2$, then $(p, q) \in \{(8, 2), (6, 5), (4, 8), \dots, (0, 14)\}$. By taking $K_8(2) = 2K_8$, we get all the above possible decompositions.

If $\lambda = 3$, then $(p, q) \in \{(14, 0), (12, 3), (10, 6), \dots, (0, 21)\}$. By Theorem 2.1, $K_8(3)$ is $\{14P_7, 0S_4\}$-decomposable. By taking $K_8(3) = K_8(2) + K_8$, we get all the above possible decompositions.

If $\lambda \geq 4$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 3$: We write $K_8(\lambda) = \frac{\lambda}{3}K_8(3)$.

$\lambda \equiv 1 \pmod 3$: We write $K_8(\lambda) = K_8(\lambda - 1) + K_8 = \frac{\lambda-1}{3}K_8(3) + K_8$.

$\lambda \equiv 2 \pmod 3$: We write $K_8(\lambda) = K_8(\lambda - 2) + K_8(2) = \frac{\lambda-2}{3}K_8(3) + K_8(2)$.

Case 3: $n = 9$.

If $\lambda = 1$, then $(p, q) \in \{(6, 0), (4, 3), (2, 6), (0, 9)\}$. By Theorem 2.1, K_9 is $\{6P_7, 0S_4\}$-decomposable. By Case 2, $K_8 = \{(4, 1), (2, 4), (0, 7)\}$. The graph $K_{1,8}$ is $\{0P_7, 2S_4\}$-decomposable. By taking $K_9 = K_8 + K_{1,8}$, we get all the above possible decompositions.

If $\lambda \geq 2$, $K_9(\lambda)$ can be decomposed into λ copies of K_9.

Case 4: $n = 10$.

If $\lambda = 2$, then $(p, q) \in \{(15, 0), (13, 3), (11, 6), \dots, (1, 21)\}$. By Theorem 2.1, $K_{10}(2)$ is $\{15P_7, 0S_4\}$-decomposable. We write $K_{10}(2) = (K_{10}(2)\backslash K_7(2)) + K_7(2)$. The graph $K_{10}(2)\backslash K_7(2)$ can be decomposed into $6P_7 : [x_1x_{10}x_2x_9x_3x_8x_4], [x_7x_{10}x_4x_9x_1x_8x_6], [x_3x_{10}x_4x_9x_7x_8x_5], [x_3x_{10}x_6x_9x_7x_8x_4], [x_1x_8x_2x_{10}x_6x_9x_3], [x_3x_8x_2x_9x_5x_{10}x_8]$ and $3S_4 : S(x_8; x_5, x_6, x_9, x_{10}), S(x_9; x_1, x_5, x_8, x_{10}), S(x_{10}; x_1, x_5, x_7, x_9)$. $K_{10}(2)\backslash K_7(2)$ can be decomposed into $12S_4$: 2 copies of $S(x_8; x_1, x_2, x_3, x_9), S(x_8; x_4, x_5, x_6, x_7), S(x_9; x_1, x_2, x_3, x_{10}), S(x_9; x_4, x_5, x_6, x_7), S(x_{10}; x_1, x_2, x_3, x_4), S(x_{10}; x_5, x_6, x_7, x_8)$. By Case 1, $K_7(2) = \{(7, 0), (5, 3), (3, 6), (1, 9)\}$. We have, $K_{10}(2) = (K_{10}(2)\backslash K_7(2))+K_7(2) = \{(6, 3), (0, 12)\}+\{(7, 0), (5, 3), (3, 6), (1, 9)\} = \{(13, 3), (11, 6), (9, 9), (7, 12), (5, 15), (3, 18), (1, 21)\}$.

If $\lambda = 4$, then $(p, q) \in \{(30, 0), (28, 3), (26, 6), \dots, (0, 45)\}$. By Theorem 2.2, $K_{10}(4)$ is $\{0P_7, 45S_4\}$-decomposable. By taking $K_{10}(4) = 2K_{10}(2)$, we get all the above possible decompositions.

If $\lambda \geq 6$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod 4$: We write $K_{10}(\lambda) = \frac{\lambda}{4}K_{10}(4)$.

$\lambda \equiv 2 \pmod 4$: We write $K_{10}(\lambda) = K_{10}(\lambda - 2) + K_{10}(2) = \frac{\lambda-2}{4}K_{10}(4) + K_{10}(2)$.

Case 5: $n = 11$.

If $\lambda = 2$, then $(p, q) \in \{(17, 2), (15, 5), (13, 8), \dots, (1, 26)\}$. We write $K_{11}(2) = (K_{11}(2)\backslash K_7(2))+K_7(2)$. The graph $K_{11}(2)\backslash K_7(2)$ can be decomposed into $10P_7 : [x_{11}x_5x_9x_6x_8x_7x_{10}], [x_6x_{10}x_4x_9x_7x_8x_1], [x_6x_{11}x_8x_2x_9x_3x_{10}], [x_1x_{10}x_{11}x_8x_3x_9x_4], [x_1x_9x_{10}x_8x_5x_{11}x_2], [x_3x_{11}x_1x_9$

$x_8x_4x_{10}], [x_{11}x_7x_9x_8x_5x_{10}x_2], [x_8x_1x_{10}x_5x_9x_{11}x_7], [x_6x_{11}x_9x_2x_8x_3x_{10}], [x_9x_6x_8x_4x_{11}x_{10}x_7]$ and $2S_4 : S(x_{10}; x_2, x_6, x_8, x_9), S(x_{11}; x_1, x_2, x_3, x_4)$. By Theorem 2.1, $K_7(2)$ is $\{7P_7, 0S_4\}$-decomposable. We have, $K_{11}(2) = (K_{11}(2)\backslash K_7(2)) + K_7(2) = \{(10, 2)\} + \{(7, 0)\} = \{(17, 2)\}$. The graph $K_{1,10}(2)$ is $\{0P_7, 5S_4\}$-decomposable. By taking $K_{11}(2) = K_{10}(2) + K_{1,10}(2)$, we get all the other possible decompositions.

If $\lambda = 4$, then $(p, q) \in \{(36, 1), (34, 4), (32, 7), \dots, (0, 55)\}$. The graph $K_{11}(4)$ can be decomposed into $36P_7 : [x_6x_7x_9x_{11}x_8x_{10}x_5], [x_9x_5x_8x_1x_3x_4x_{10}], [x_5x_1x_6x_8x_7x_{11}x_{10}], [x_5x_3x_6 x_1x_9x_{10}x_7], [x_2x_5x_7x_3x_{11}x_1x_6]$, 4 copies of $[x_6x_{10}x_1x_2x_8x_4x_7], [x_{11}x_5x_6x_4x_2x_9x_8], [x_{11}x_4x_1 x_7x_2x_3x_8], [x_{11}x_2x_{10}x_3x_9x_4x_5]$, 3 copies of $[x_{11}x_6x_2x_5x_3x_7x_8], [x_{11}x_{10}x_5x_9x_7x_6x_3], [x_1x_9x_{10} x_8x_{11}x_7x_5], [x_{11}x_9x_6x_8x_5x_1x_3], [x_8x_1x_{11}x_3x_4x_{10}x_7]$ and a $S_4 : S(x_6; x_1, x_2, x_9, x_{11})$. By Theorem 2.2, $K_{11}(4)$ is $\{0P_7, 55S_4\}$-decomposable. By taking $K_{11}(4) = 2K_{11}(2)$, we get all the other possible decompositions.

If $\lambda = 6$, then $(p, q) \in \{(55, 0), (53, 3), (51, 6), \dots, (1, 81)\}$. By Theorem 2.1, $K_{11}(6)$ is $\{55P_7, 0S_4\}$-decomposable. By taking $K_{11}(6) = K_{11}(4) + K_{11}(2)$, we get all the above possible decompositions.

If $\lambda = 8$, then $(p, q) \in \{(72, 2), (70, 5), (68, 8), \dots, (0, 110)\}$. By taking $K_{11}(8) = 2K_{11}(4)$, we get all the above possible decompositions.

If $\lambda = 10$, then $(p, q) \in \{(91, 1), (89, 4), (87, 7), \dots, (1, 136)\}$. By taking $K_{11}(10) = K_{11}(6) + K_{11}(4)$, we get all the above possible decompositions.

If $\lambda = 12$, then $(p, q) \in \{(110, 0), (108, 3), (106, 6), \dots, (0, 165)\}$. By Theorem 2.2, $K_{11}(12)$ is $\{0P_7, 165S_4\}$-decomposable. By taking $K_{11}(12) = 2K_{11}(6)$, we get all the above possible decompositions.

If $\lambda \geq 14$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod{12}$: We write $K_{11}(\lambda) = \frac{\lambda}{12}K_{11}(12)$.

$\lambda \equiv 2 \pmod{12}$: We write $K_{11}(\lambda) = K_{11}(\lambda - 2) + K_{11}(2) = \frac{\lambda-2}{12}K_{11}(12) + K_{11}(2)$.

$\lambda \equiv 4 \pmod{12}$: We write $K_{11}(\lambda) = K_{11}(\lambda - 4) + K_{11}(4) = \frac{\lambda-4}{12}K_{11}(12) + K_{11}(4)$.

$\lambda \equiv 6 \pmod{12}$: We write $K_{11}(\lambda) = K_{11}(\lambda - 6) + K_{11}(6) = \frac{\lambda-6}{12}K_{11}(12) + K_{11}(6)$.

$\lambda \equiv 8 \pmod{12}$: We write $K_{11}(\lambda) = K_{11}(\lambda - 8) + K_{11}(8) = \frac{\lambda-8}{12}K_{11}(12) + K_{11}(8)$.

$\lambda \equiv 10 \pmod{12}$: We write $K_{11}(\lambda) = K_{11}(\lambda - 10) + K_{11}(10) = \frac{\lambda-10}{12}K_{11}(12) + K_{11}(10)$.

Case 6: $n = 12$.

If $\lambda = 1$, then $(p, q) \in \{(11, 0), (9, 3), (7, 6), \dots, (1, 15)\}$. By Theorem 2.1, K_{12} is $\{11P_7, 0S_4\}$-decomposable. We write $K_{12} = (K_{12}\backslash K_9) + K_9$. The graph $K_{12}\backslash K_9$ can be decomposed into $3P_7 : [x_1x_{12}x_3x_{10}x_8x_{11}x_7], [x_7x_{12}x_9x_{10}x_5x_{11}x_4], [x_1x_{11}x_2x_{10}x_6x_{12}x_4]$ and $3S_4 : S(x_{10}; x_1, x_4, x_7, x_{11}), S(x_{11}; x_3, x_6, x_9, x_{12}), S(x_{12}; x_2, x_5, x_8, x_{10})$. $K_{12}\backslash K_9$ can be decomposed into a $P_7 : [x_1x_{12}x_3x_{11}x_4x_{10}x_9]$ and $6S_4 : S(x_{10}; x_1, x_2, x_3, x_{11}), S(x_{10}; x_5, x_6, x_7 , x_8), S(x_{11}; x_1, x_2, x_5, x_{12}), S(x_{11}; x_6, x_7, x_8, x_9), S(x_{12}; x_2, x_4, x_5, x_9), S(x_{12}; x_6, x_7, x_8, x_{10})$. By Case 3, $K_9 = \{(6, 0), (4, 3), (2, 6), (0, 9)\}$. We have, $K_{12} = (K_{12}\backslash K_9) + K_9 = \{(3, 3), (1, 6)\} + \{(6, 0), (4, 3), (2, 6), (0, 9)\} = \{(9, 3), (7, 6), (5, 9), (3, 12), (1, 15)\}$.

If $\lambda = 2$, then $(p, q) \in \{(22, 0), (20, 3), (18, 6), \dots, (0, 33)\}$. By Theorem 2.2, $K_{12}(2)$ is $\{0P_7, 33S_4\}$-decomposable. By taking $K_{12}(2) = 2K_{12}$, we get all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.

$\lambda \equiv 0 \pmod{2}$: We write $K_{12}(\lambda) = \frac{\lambda}{2}K_{12}(2)$.

$\lambda \equiv 1 \pmod{2}$: We write $K_{12}(\lambda) = K_{12}(\lambda - 1) + K_{12} = \frac{\lambda-1}{2}K_{12}(2) + K_{12}$.

Case 7: $n = 13$.

If $\lambda = 1$, then $(p, q) \in \{(13, 0), (11, 3), (9, 6), \dots, (1, 18)\}$. By Theorem 2.1, K_{13} is $\{13P_7, 0S_4\}$-decomposable. The graph $K_{1,12}$ is $\{0P_7, 3S_4\}$-decomposable. By taking $K_{13} = K_{12} + K_{1,12}$, we get all the above possible decompositions.

If $\lambda = 2$, then $(p, q) \in \{(26, 0), (24, 3), (22, 6), \dots, (0, 39)\}$. By Theorem 2.2, $K_{13}(2)$ is $\{0P_7, 39S_4\}$-decomposable. By taking $K_{13}(2) = 2K_{13}$, we get all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.
$\lambda \equiv 0 \pmod 2$: We write $K_{13}(\lambda) = \frac{\lambda}{2}K_{13}(2)$.
$\lambda \equiv 1 \pmod 2$: We write $K_{13}(\lambda) = K_{13}(\lambda - 1) + K_{13} = \frac{\lambda-1}{2}K_{13}(2) + K_{13}$.

Case 8: $n = 14$.

By taking $K_{14}(\lambda) = K_8(\lambda) + K_7(\lambda) + \lambda K_{4,6} + \frac{\lambda}{2}K_{3,6}(2)$, we get all the possible decompositions.

Case 9: $n = 15$.

By taking $K_{15}(\lambda) = K_9(\lambda) + K_7(\lambda) + 2\lambda K_{4,6}$, we get all the possible decompositions.

Case 10: $n = 16$.

If $\lambda = 1$, then $(p, q) \in \{(20, 0), (18, 3), (16, 6), \dots, (0, 30)\}$. By Theorem 2.1, K_{16} is $\{20P_7, 0S_4\}$-decomposable. The graph $K_{1,8}$ is $\{0P_7, 2S_4\}$-decomposable. By taking $K_{16} = K_8 + K_9 + 2K_{6,4} + K_{1,8}$, we get all the above possible decompositions.

If $\lambda \geq 2$, $K_{16}(\lambda)$ can be decomposed into λ copies of K_{16}.

Case 11: $n = 17$.

By Theorems, 2.3 and 2.4, $K_{3,8}$ is $\{\{4P_7, 0S_4\}, \{0P_7, 6S_4\}\}$-decomposable. By taking $K_{17}(\lambda) = K_9(\lambda) + K_8(\lambda) + 2\lambda K_{6,4} + \lambda K_{3,8}$, we get all the possible decompositions.

Now we prove the result for $n > 17$. Let $n = 4r$, $n = 4r + 1$, $n = 4r + 2$, $n = 4r + 3$, where $r \geq 1$. We prove by mathematical induction on n, splitting the proof into four cases as follows:

$n \equiv 0 \pmod 4$. Let $n = 4r$, with $r \geq 5$. Assume that $K_{4t}(\lambda)$ is fully decomposable if $2 \leq t < r$. Write $K_{4r}(\lambda) = K_{4(r-3)}(\lambda) + K_{12}(\lambda) + K_{4(r-3),12}(\lambda) = K_{4(r-3)}(\lambda) + K_{12}(\lambda) + (r-3)K_{4,12}(\lambda) = K_{4(r-3)}(\lambda) + K_{12}(\lambda) + (2r-6)\lambda K_{4,6}$. Suppose the non-negative integers p and q satisfy the obvious necessary conditions for a $\{pP_7, qS_4\}$-decomposition in $K_{4r}(\lambda)$. Then we have $6p + 4q = \frac{\lambda(4r)\times(4r-1)}{2} = \frac{\lambda}{2}(16r^2 - 4r) = \lambda(8r^2 - 2r) = 8\lambda r^2 - 2\lambda r = 8\lambda r^2 - 2\lambda r + 144\lambda - 144\lambda = 8\lambda r^2 - 50\lambda r + 78\lambda + 66\lambda + 48\lambda r - 144\lambda = \lambda(8r^2 - 50r + 78) + 66\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(16r^2 - 100r + 156) + 66\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(16r^2 - 52r - 48r + 156) + 66\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(4r - 12)\times(4r - 13) + 66\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(4r - 12)\times(4r - 12 - 1) + 66\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(4(r-3)\times 4(r-3) - 1) + 66\lambda + 24\lambda(2r-6) = \frac{\lambda}{2}(4(r-3)\times 4(r-3) - 1) + \frac{132\lambda}{2} + 4\times 6\lambda(2r-6) = \frac{\lambda}{2}(4(r-3)\times 4(r-3) - 1) + \frac{\lambda}{2}(132) + (2r-6)\lambda 4\times 6 = \frac{\lambda}{2}(4(r-3)\times 4(r-3) - 1) + \frac{\lambda}{2}(12\times 11) + (2r-6)24\lambda = (6p_1 + 4q_1) + (6p_2 + 4q_2) + (6p_3 + 4q_3)$. By the induction hypothesis, there exists a $\{p_1P_7, q_1S_4\}$-decomposition of $K_{4(r-3)}(\lambda)$, by Case 6 there exists $\{p_2P_7, q_2S_4\}$-decomposition of $K_{12}(\lambda)$ and by Lemma 3.1 there exists $\{p_3P_7, q_3S_4\}$-decomposition of $K_{4,6}$. Therefore a $\{pP_7, qS_4\}$-decomposition of $K_{4r}(\lambda)$ exists. Hence by the method of induction, we have $K_{4r}(\lambda)$ is fully $\{P_7, S_4\}$-decomposable for any $r \geq 2$.

$n \equiv 1 \pmod 4$. Let $n = 4r+1$, with $r \geq 5$. Assume that $K_{4t+1}(\lambda)$ is fully decomposable if $2 \leq t < r$. Write $K_{4r+1}(\lambda) = K_{4(r-3)+1}(\lambda) + K_{13}(\lambda) + K_{4(r-3),12}(\lambda) = K_{4(r-3)+1}(\lambda) + K_{13}(\lambda) + (r-3)K_{4,12}(\lambda) = K_{4(r-3)+1}(\lambda) + K_{13}(\lambda) + (2r-6)\lambda K_{4,6}$. Suppose the non-negative integers p and q satisfy the obvious necessary conditions for a $\{pP_7, qS_4\}$-decomposition in $K_{4r+1}(\lambda)$. Then we have $6p + 4q = \frac{\lambda(4r+1)\times(4r+1)-1}{2} = \frac{\lambda}{2}(16r^2 + 4r) = \lambda(8r^2 + 2r) = 8\lambda r^2 + 2\lambda r = 8\lambda r^2 + 2\lambda r + 144\lambda - 144\lambda = 8\lambda r^2 - 46\lambda r + 66\lambda + 78\lambda + 48\lambda r - 144\lambda = \lambda(8r^2 - 46r + 66) + 78\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(16r^2 - 92r + 132) + 78\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(16r^2 - 48r - 44r + 132) + 78\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(4r - 11)\times(4r - 12) + 78\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(4r - 12 + 1)\times(4r - 12) + 78\lambda + 48\lambda r - 144\lambda = \frac{\lambda}{2}(4(r-3) + 1\times 4(r-3)) + 78\lambda + 24\lambda(2r-6) = \frac{\lambda}{2}(4(r-3) + 1\times 4(r-3) + 1 - 1) + \frac{156\lambda}{2} + 4\times 6\lambda(2r-6) = \frac{\lambda}{2}(4(r-3) + 1\times 4(r-3) +$

$1-1)+\frac{\lambda}{2}(156)+(2r-6)\lambda 4\times 6=\frac{\lambda}{2}(4(r-3)+1\times 4(r-3)+1-1)+\frac{\lambda}{2}(13\times 12)+(2r-6)24\lambda=(6p_1+4q_1)+(6p_2+4q_2)+(6p_3+4q_3)$. By the induction hypothesis, there exists a $\{p_1P_7, q_1S_4\}$-decomposition of $K_{4(r-3)+1}(\lambda)$, by Case 7 there exists $\{p_2P_7, q_2S_4\}$-decomposition of $K_{13}(\lambda)$ and by Lemma 3.1 there exists $\{p_3P_7, q_3S_4\}$-decomposition of $K_{4,6}$. Therefore a $\{pP_7, qS_4\}$-decomposition of $K_{4r+1}(\lambda)$ exists. Hence by the method of induction, we have $K_{4r+1}(\lambda)$ is fully $\{P_7, S_4\}$-decomposable for any $r \geq 2$.

$n \equiv 2 \pmod 4$. Let $n = 4r+2$, with $r \geq 4$. Assume that $K_{4t+2}(\lambda)$ is fully decomposable if $2 \leq t < r$. Write $K_{4r+2}(\lambda) = K_{4(r-1)}(\lambda) + K_7(\lambda) + K_{4(r-1)-1,6}(\lambda) = K_{4(r-1)}(\lambda) + K_7(\lambda) + (r-2)\lambda K_{4,6} + \frac{\lambda}{2}K_{3,6}(2)$. Suppose the non-negative integers p and q satisfy the obvious necessary conditions for a $\{pP_7, qS_4\}$-decomposition in $K_{4r+2}(\lambda)$. Then we have $6p+4q=\frac{\lambda(4r+2)\times(4r+2)-1}{2}=\frac{\lambda}{2}(16r^2+12r+2)=\lambda(8r^2+6r+1)=8\lambda r^2+6\lambda r+\lambda=8\lambda r^2+6\lambda r+49\lambda-48\lambda=8\lambda r^2-18\lambda r+10\lambda+21\lambda+24\lambda r-48\lambda+18\lambda=\lambda(8r^2-18r+10)+21\lambda+24\lambda r-48\lambda+18\lambda=\frac{\lambda}{2}(16r^2-36r+20)+21\lambda+24\lambda r-48\lambda+18\lambda=\frac{\lambda}{2}(16r^2-20r-16r+20)+21\lambda+24\lambda r-48\lambda+18\lambda=\frac{\lambda}{2}(4r-4)\times(4r-5)+21\lambda+24\lambda r-48\lambda+18\lambda=\frac{\lambda}{2}((4r-4)\times(4r-4)-1)+21\lambda+24\lambda r-48\lambda+18\lambda=\frac{\lambda}{2}(4(r-1)\times 4(r-1)-1)+21\lambda+24\lambda(r-2)+18\lambda=\frac{\lambda}{2}(4(r-1)\times 4(r-1)-1)+\frac{42\lambda}{2}+4\times 6\lambda(r-2)+\frac{\lambda}{2}(36)=\frac{\lambda}{2}(4(r-1)\times 4(r-1)-1)+\frac{\lambda}{2}(42)+(r-2)\lambda 4\times 6+\frac{\lambda}{2}(2\times 3\times 6)=\frac{\lambda}{2}(4(r-1)\times 4(r-1)-1)+\frac{\lambda}{2}(7\times 6)+(r-2)24\lambda+\frac{\lambda}{2}(36)=\frac{\lambda}{2}(4(r-1)\times 4(r-1)-1)+\frac{\lambda}{2}(7\times 6)+(r-2)24\lambda+18\lambda=(6p_1+4q_1)+(6p_2+4q_2)+(6p_3+4q_3)+(6p_4+4q_4)$. By the induction hypothesis, there exists a $\{p_1P_7, q_1S_4\}$-decomposition of $K_{4(r-1)}(\lambda)$, by Case 1 there exists $\{p_2P_7, q_2S_4\}$-decomposition of $K_7(\lambda)$, by Lemma 3.1 there exists $\{p_3P_7, q_3S_4\}$-decomposition of $K_{4,6}$ and by Lemma 3.2 there exists $\{p_4P_7, q_4S_4\}$-decomposition of $K_{3,6}(2)$. Therefore a $\{pP_7, qS_4\}$-decomposition of $K_{4r+2}(\lambda)$ exists. Hence by the method of induction, we have $K_{4r+2}(\lambda)$ is fully $\{P_7, S_4\}$-decomposable for any $r \geq 2$.

$n \equiv 3 \pmod 4$. Let $n = 4r+3$, with $r \geq 4$. Assume that $K_{4t+3}(\lambda)$ is fully decomposable if $1 \leq t < r$. Write $K_{4r+3}(\lambda) = K_{4(r-1)+1}(\lambda) + K_7(\lambda) + K_{4(r-1),6}(\lambda) = K_{4(r-1)+1}(\lambda) + K_7(\lambda) + (r-1)\lambda K_{4,6}$. Suppose the non-negative integers p and q satisfy the obvious necessary conditions for a $\{pP_7, qS_4\}$-decomposition in $K_{4r+3}(\lambda)$. Then we have $6p+4q=\frac{\lambda(4r+3)\times(4r+3)-1}{2}=\frac{\lambda}{2}(16r^2+20r+6)=\lambda(8r^2+10r+3)=8\lambda r^2+10\lambda r+3\lambda=8\lambda r^2+10\lambda r+27\lambda-24\lambda=8\lambda r^2-14\lambda r+6\lambda+21\lambda+24\lambda r-24\lambda=\lambda(8r^2-14r+6)+21\lambda+24\lambda r-24\lambda=\frac{\lambda}{2}(16r^2-28r+12)+21\lambda+24\lambda r-24\lambda=\frac{\lambda}{2}(16r^2-16r-12r+12)+21\lambda+24\lambda r-24\lambda=\frac{\lambda}{2}(4r-3)\times(4r-4)+21\lambda+24\lambda r-24\lambda=\frac{\lambda}{2}((4r-4+1)\times(4r-4)+1-1)+21\lambda+24\lambda r-24\lambda=\frac{\lambda}{2}((4(r-1)+1)\times 4(r-1)+1-1)+21\lambda+24\lambda(r-1)=\frac{\lambda}{2}(4(r-1)+1)\times(4(r-1)+1-1)+\frac{42\lambda}{2}+4\times 6\lambda(r-1)=\frac{\lambda}{2}(4(r-1)+1)\times(4(r-1)+1-1)+\frac{\lambda}{2}(42)+(r-1)\lambda 4\times 6=\frac{\lambda}{2}(4(r-1)+1)\times 4(r-1)+1-1)+\frac{\lambda}{2}(7\times 6)+(r-1)24\lambda=(6p_1+4q_1)+(6p_2+4q_2)+(6p_3+4q_3)$. By the induction hypothesis, there exists a $\{p_1P_7, q_1S_4\}$-decomposition of $K_{4(r-1)+1}(\lambda)$, by Case 1 there exists $\{p_2P_7, q_2S_4\}$-decomposition of $K_7(\lambda)$ and by Lemma 3.1 there exists $\{p_3P_7, q_3S_4\}$-decomposition of $K_{4,6}$. Therefore a $\{pP_7, qS_4\}$-decomposition of $K_{4r+3}(\lambda)$ exists. Hence by the method of induction, we have $K_{4r+3}(\lambda)$ is fully $\{P_7, S_4\}$-decomposable for any $r \geq 1$. □

4. Conclusions

In this paper, we have obtained the necessary conditions for a $\{P_{k+1}, S_m\}$-decomposition of $K_n(\lambda)$ and proved that the necessary conditions are also sufficient when $k = 6$ and $m = 4$.

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