TWMS J. App. and Eng. Math. V.12, N.1, 2022, pp. 224-234

CYCLICAL NONLINEAR CONTRACTIVE MAPPINGS FIXED POINT THEOREMS WITH APPLICATION TO INTEGRAL EQUATIONS

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Abstract. In this paper, we present new nonlinear contractions based on altering distances and prove the existence and uniqueness of fixed points for cyclic operators. We prove here very interesting fixed point theorems in which we combine and extend the contractive conditions of Banach, Kannan, Chatterjea, and of many others. Our results shall serve as generalized versions of many fixed point results proved in the literature. Examples and application to integral equations that exploits Jensen inequality are given to illustrate the analysis and theory and validate our proved results.

Keywords: Cyclic operator, nonlinear contraction, fixed point theory, integral equation

AMS Subject Classification: 47H10, 46T99, 54H25.

1. Introduction and Preliminaries.

Contractions are the focus of fixed point theorists where the most common contraction on a metric space is the Banach's contraction [1] which is given by $d(Tx,Ty) \le \alpha d(x,y)$, for $0 < \alpha < 1$. Other common contractions are Kannan [2] and Chatterjea [3] which are given by $d(Tx,Ty) \le \beta [d(x,Tx) + d(y,Ty)]$ and $d(Tx,Ty) \le \gamma [d(x,Ty) + d(y,Tx)]$ for $0 \le \beta, \gamma < \frac{1}{2}$, respectively. In 1972 Zamfirescu [4] introduced a very nice fixed point theorem that combines the contractive conditions of Banach, Kannan, and Chatterjea and says that if T satisfies any of the above contractions, then T has a unique fixed point. Many fixed point results were proved using these types of contractions.

However, the cyclical extensions for these fixed point theorems were obtained at a later time by considering non-empty closed subsets $\{A_i\}_{i=1}^m$ of a complete metric space X and a cyclical operator T.

Definition 1.1. *Let $\{A_i\}_{i=1}^m$ be non-empty closed subsets of a complete metric space X. Then, $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ is said to be a cyclic operator if for all $i \in \{1, 2, \dots, m\}$ we have $T(A_i) \subseteq A_{i+1}$.*

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§ Manuscript received: February 01, 2020; accepted: April 13, 2020.

The cyclical extension for the Banach fixed point theorem was introduced by Kirk *et. al.* [5]. Later on, Rus [6] and Petric [7] proved the cyclical extensions for Kannan's theorem and for Chatterjea and Zamfirescu theorems respectively, using fixed point structure arguments.

Another enhancement in the fixed point theory field was the concept of a control function in terms of altering distances which was addressed by Khan *et. al.* [8]. These altering distance functions alter the metric distance between points and lead to a new category and relatively new classes of fixed point problems.

A substantial amount of work and studies have been carried out considering the cyclic contractive mappings as well as using the altering distances which have revealed many interesting results in fixed point theory, see for example [9]-[28] and references therein.

In this paper, we give extensions and generalized versions of many fixed point results proved in the literature. In particular, we present generalized versions of fixed point theorems of cyclic nonlinear contractions type using altering distance functions. At the end of the paper, the analysis and theory are illustrated and the proved results are validated by some examples and applications. In the application, besides our proved results, we shall use the well-known Jenson inequality [29] to prove the existence and uniqueness of solutions for integral equations under certain conditions. The following definitions and proposition shall be needed throughout the paper.

Definition 1.2. *The function $\phi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties are satisfied.*

(i) *ϕ is continuous and nondecreasing,*
(ii) *$\phi(t) = 0$ if and only if $t = 0$.*

Definition 1.3. *A function φ defined on an interval I is said to be convex if for each $x, y \in I$ and each λ, $0 \le \lambda \le 1$ we have*

$$\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

As consequence from Jensen inequality, we have the following proposition which needed in the application.

Proposition 1.1. *(Jensen Inequality, see* [29]*) Let φ be a convex, Borel measurable function on an interval I and let f be a real valued integrable function on* $[0,1]$*. Suppose that the range of f is a subset of I. Then*

$$\varphi\left(\int f(t)\,dt\right) \le \int \varphi(f(t))\,dt,$$

provided that $\varphi \circ f$ is integrable.

2. Fixed Points for Cyclic Operators.

We present in this section our main results in order to prove existence and uniqueness of fixed points for cyclic operators.

Theorem 2.1. *Let $\{A_i\}_{i=1}^{m}$ be non-empty closed subsets of a complete metric space (X,d) and let $T : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be a cyclic operator satisfying for any $x \in A_i, y \in A_{i+1}, i = 1,2,\dots,m$ the following condition*

$$\begin{aligned}\phi\left(d\left(Tx,Ty\right)\right) \le \phi\left(\max\left\{\left(\alpha d\left(x,Tx\right)+\beta d\left(y,Ty\right)\right),\gamma d(x,y)\right\}\right)\\ -\psi\left(d\left(x,Tx\right),d\left(y,Ty\right)\right),\end{aligned} \tag{1}$$

where $0 \leq \alpha$, $0 \leq \beta < 1$, $0 < \alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$, *and* ϕ *is an altering distance function defined in Definition 1.2 and* $\psi : [0, \infty)^2 \to [0, \infty)$ *is a continuous function such that* $\psi(x, y) = 0$ *if and only if* $x = y = 0$. *Then,* T *has a unique fixed point* $u^* \in \bigcap_{i=1}^{m} A_i$.

Proof. Take $x_0 \in X$ and consider the sequence given by $x_{n+1} = Tx_n, n \geq 0$. If there exists $n_0 \in N$ such that $x_{n_0+1} = x_{n_0}$, then the point of existence of the fixed point is proved. So, suppose that $x_{n+1} \neq x_n$ for any $n = 0, 1, \ldots$. Then there exists $i_n \in \{1, \ldots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Assume T satisfies (1). Then, we have

$$\begin{aligned} \phi(d(x_n, x_{n+1})) &= \phi(d(Tx_{n-1}, Tx_n)) \\ &\leq \phi(\max\{\alpha d(x_{n-1}, Tx_{n-1}) + \beta d(x_n, Tx_n), \gamma d(x_{n-1}, x_n)\}) \\ &\quad -\psi(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)) \\ &\leq \phi(\max\{\alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}, \gamma d(x_{n-1}, x_n))\}) \\ &\quad -\psi(d(x_{n-1}, x_n), d(x_n, x_{n+1})) \\ &\leq \phi(\max\{\alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}), \gamma d(x_{n-1}, x_n)\}). \end{aligned} \tag{2}$$

Let $L_n = \max\{\alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}), \gamma d(x_{n-1}, x_n)\}$. Then, (2) implies

$$\phi(d(x_n, x_{n+1})) \leq \phi(L_n).$$

Since ϕ is a nondecreasing function, we get

$$d(x_n, x_{n+1}) \leq L_n.$$

We have two cases to treat; either $L_n = \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1})$ or $L_n = \gamma d(x_{n-1}, x_n)$. Suppose first that $L_n = \gamma d(x_{n-1}, x_n)$. Then, we have

$$d(x_n, x_{n+1}) \leq \gamma d(x_{n-1}, x_n).$$

Since, $0 \leq \gamma \leq 1$, we get that $d(x_n, x_{n+1})$ is a nonincreasing sequence of nonnegative real numbers. Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Using the continuity of ϕ and ψ, we get

$$\begin{aligned} \phi(r) &\leq \phi(\gamma r) - \psi(r, r) \\ &\leq \phi(r) - \psi(r, r), \end{aligned}$$

which implies that $\psi(r, r) = 0$, and hence $r = 0$.

Similarly, if $L_n = \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1})$, then we get

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}),$$

which implies

$$d(x_n, x_{n+1}) \leq \frac{\alpha}{1-\beta} d(x_{n-1}, x_n). \tag{3}$$

Since $0 < \alpha + \beta \leq 1$, we get that $d(x_n, x_{n+1})$ is a nonincreasing sequence of nonnegative real numbers. Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Using the continuity of ϕ and ψ, we get

$$\begin{aligned} \phi(r) &\leq \phi((\alpha + \beta) r) - \psi(r, r) \\ &\leq \phi(r) - \psi(r, r), \end{aligned}$$

which implies that $\psi(r, r) = 0$, and hence $r = 0$.

In the sequel, we show that $\{x_n\}$ is a Cauchy sequence in (X, d). To do so, we need to prove first, the claim that for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ with $p - q \equiv 1\,(m)$, then $d\,(x_p, x_q) < \epsilon$. Suppose the contrary, i.e., there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $p_n > q_n \geq n$ with $p_n - q_n \equiv 1\,(m)$ satisfying $d\,(x_{p_n}, x_{q_n}) \geq \epsilon$. Now, we take $n > 2m$. Then corresponding to $q_n \geq n$, we can choose p_n in such a way that it is the smallest integer with $p_n > q_n$ satisfying $p_n - q_n \equiv 1\,(m)$ and $d\,(x_{p_n}, x_{q_n}) \geq \epsilon$. Therefore, $d\left(x_{q_n}, x_{p_{n-m}}\right) < \epsilon$. Using the triangular inequality,

$$\epsilon \leq d\,(x_{p_n}, x_{q_n}) \leq d\left(x_{q_n}, x_{p_{n-m}}\right) + \sum_{i=1}^{m} d\left(x_{p_{n-i}}, x_{p_{n-i+1}}\right) < \epsilon + \sum_{i=1}^{m} d\left(x_{p_{n-i}}, x_{p_{n-i+1}}\right).$$

Letting $n \to \infty$ in the last inequality, and taking into account that $\lim_{n\to\infty} d\,(x_n, x_{n+1}) = 0$, we obtain $\lim_{n\to\infty} d\,(x_{p_n}, x_{q_n}) = \epsilon$. Again, by triangle inequality, we have

$$\begin{aligned}
\epsilon &\leq d(x_{q_n}, x_{p_n}) \\
&\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{p_{n+1}}) + d(x_{p_{n+1}}, x_{p_n}) \\
&\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{p_n}) + d(x_{p_n}, x_{p_{n+1}}) + d(x_{p_{n+1}}, x_{p_n}) \\
&\leq 2d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_n}, x_{p_n}) + 2d(x_{p_n}, x_{p_{n+1}}).
\end{aligned}$$

Taking the limit as $n \to \infty$, and taking into account that $\lim_{n\to\infty} d\,(x_n, x_{n+1}) = 0$, we get $\lim_{n\to\infty} d\left(x_{q_{n+1}}, x_{p_{n+1}}\right) = \epsilon$. Since x_{p_n} and x_{q_n} lie in different adjacently labelled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, assuming that T satisfies (1), we have

$$\begin{aligned}
\phi\left(d\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi\,(d\,(Tx_{q_n}, Tx_{p_n})) \\
&\leq \phi\,(\max\{\alpha d\,(x_{q_n}, Tx_{q_n}) + \beta d\,(x_{p_n}, Tx_{p_n})\,, \gamma d(x_{q_n}, x_{p_n})\}) \\
&\quad -\psi\,(d\,(x_{q_n}, Tx_{q_n})\,, d\,(x_{p_n}, Tx_{p_n}))\,.
\end{aligned} \tag{4}$$

Again, let $L_n = \max\{\alpha d\,(x_{q_n}, Tx_{q_n}) + \beta d\,(x_{p_n}, Tx_{p_n})\,, \gamma d(x_{q_n}, x_{p_n})\}$. If $L_n = \gamma d(x_{q_n}, x_{p_n})$, then by letting $n \to \infty$ in the last inequality, we obtain

$$\phi\,(\epsilon) \leq \phi\,(\gamma\epsilon) - \psi\,(0, 0) = \phi\,(\gamma\epsilon)\,.$$

Since, ϕ is a nondecreasing function and $0 \leq \gamma \leq 1$, we get $\epsilon = 0$ which is a contradiction. Now, if $L_n = \alpha d\,(x_{q_n}, Tx_{q_n}) + \beta d\,(x_{p_n}, Tx_{p_n})$, then by letting $n \to \infty$ in the last inequality, we obtain

$$\phi\,(\epsilon) \leq \phi\,(0) - \psi\,(0, 0) = 0.$$

Therefore, we get also $\epsilon = 0$ which is again a contradiction.

From the above proved claim, and for arbitrary $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that if $p, q > n_0$ with $p - q = 1(m)$, then $d\,(x_p, x_q) < \epsilon$. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we can find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\epsilon}{m}, \text{ for } n > n_1.$$

Now, for $r, s > \max\{n_0, n_1\}$ and $s > r$, there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r = k(m)$. Therefore, $s - r + j = 1(m)$ for $j = m - k + 1$. So, we have

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \cdots + d(x_{s+1}, x_s).$$

This implies

$$d(x_r, x_s) \leq \epsilon + \frac{\epsilon}{m}\sum_{j=1}^{m} 1 = 2\epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in $\bigcup_{i=1}^{m} A_i$. Consequently, $\{x_n\}$ converges to some $u^* \in \bigcup_{i=1}^{m} A_i$. However, in view of cyclical condition, the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for $i = 1, 2, \ldots, m$. Therefore, $u^* \in \bigcap_{i=1}^{m} A_i$.

Now, we will prove that u^* is a fixed point of T. Suppose $u^* \in A_i$, $Tu^* \in A_{i+1}$, and we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$. Then, assuming that T satisfies (1), we have

$$\begin{array}{rcl} \phi\left(d\left(x_{n_{k+1}}, Tu^*\right)\right) & = & \phi\left(d\left(Tx_{n_k}, Tu^*\right)\right) \\ & \leq & \phi\left(\max\{\alpha d\left(x_{n_k}, Tx_{n_k}\right) + \beta d\left(u^*, Tu^*\right), \gamma d(x_{n_k}, u^*)\}\right) \\ & & \quad -\psi\left(d\left(x_{n_k}, Tx_{n_k}\right), d\left(u^*, Tu^*\right)\right) \\ & \leq & \phi\left(\max\{\alpha d\left(x_{n_k}, Tx_{n_k}\right) + \beta d\left(u^*, Tu^*\right), \gamma d(x_{n_k}, u^*)\}\right). \end{array} \quad (5)$$

Letting $k \to \infty$, we have

$$\phi\left(d\left(u^*, Tu^*\right)\right) \leq \phi\left(\max\{\alpha d\left(u^*, u^*\right) + \beta d\left(u^*, Tu^*\right), \gamma d(u^*, u^*)\}\right),$$

and since ϕ is a nondecreasing function, we get

$$d\left(u^*, Tu^*\right) \leq \beta d\left(u^*, Tu^*\right).$$

Thus, since $0 \leq \beta < 1$, we have $d\left(u^*, Tu^*\right) = 0$, and hence $u^* = Tu^*$. □

Theorem 2.2. *Let $\{A_i\}_{i=1}^{m}$ be non-empty closed subsets of a complete metric space (X, d) and let $T : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be a cyclic operator satisfying for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m$ the following condition.*

$$\begin{array}{l} \phi\left(d\left(Tx, Ty\right)\right) \leq \phi(\max\{(\alpha d\left(x, Ty\right) + \beta d\left(y, Tx\right), \gamma d(x, y))\}) \\ \qquad -\psi\left(d\left(x, Ty\right), d\left(y, Tx\right)\right), \end{array} \quad (6)$$

where $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta$, $0 < \alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$, and ϕ is an altering distance function defined in Definition 1.2 and $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. Then, T has a unique fixed point $u^* \in \bigcap_{i=1}^{m} A_i$.

Proof. The proof follows exactly the same as the proof of Theorem 2.1. Therefore, for the seek of avoiding unnecessarily repetition, we shall mention here only the slight modifications. Assuming T satisfying (6), then (2) needs to be replaced by

$$\begin{array}{rcl} \phi\left(d\left(x_n, x_{n+1}\right)\right) & = & \phi\left(d\left(Tx_{n-1}, Tx_n\right)\right) \\ & \leq & \phi\left(\max\{\alpha d\left(x_{n-1}, Tx_n\right) + \beta d\left(x_n, Tx_{n-1}\right), \gamma d(x_{n-1}, x_n)\}\right) \\ & & \quad -\psi\left(d\left(x_{n-1}, Tx_n\right), d\left(x_n, Tx_{n-1}\right)\right) \\ & \leq & \phi\left(\max\{\alpha d\left(x_{n-1}, x_{n+1}\right) + \beta d\left(x_n, x_n\right), \gamma d(x_{n-1}, x_n)\}\right) \\ & & \quad -\psi\left(d\left(x_{n-1}, x_{n+1}\right), d\left(x_n, x_n\right)\right) \\ & \leq & \phi\left(\max\{\alpha d\left(x_{n-1}, x_{n+1}\right), \gamma d(x_{n-1}, x_n)\}\right). \end{array}$$

Since, ϕ is a nondecreasing function, we get

$$d\left(x_n, x_{n+1}\right) \leq \max\{\alpha d\left(x_{n-1}, x_{n+1}\right), \gamma d(x_{n-1}, x_n)\}. \quad (7)$$

Let $L_n = \max\{\alpha d\left(x_{n-1}, x_{n+1}\right), \gamma d(x_{n-1}, x_n)\}$. Then, we again have two cases. Assume first that $L_n = \gamma d(x_{n-1}, x_n)$, then as in the proof of Theorem 2.1, we get $\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) =$

0. Now, assume $L_n = \alpha d(x_{n-1}, x_{n+1})$, then by triangular inequality, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha d(x_{n-1}, x_{n+1}) \\ &\leq \alpha [d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n). \tag{8}$$

Since $0 \leq \alpha \leq \frac{1}{2}$, we get that $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence of nonnegative real numbers. Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Now, if $\alpha = 0$, then clearly, $r = 0$, and if $0 < \alpha < \frac{1}{2}$, then $\frac{\alpha}{1-\alpha} < 1$, and by induction, we have

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, x_1),$$

and hence $r = 0$. Now, if $\alpha = \frac{1}{2}$, then from (7), we have

$$d(x_{n-1}, x_{n+1}) \geq 2d(x_n, x_{n+1}),$$

and hence

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \geq 2r,$$

but

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}),$$

and as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \leq 2r.$$

Therefore, $\lim\limits_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r$. Using the continuity of ϕ and ψ, we get

$$\begin{aligned} \phi(r) &\leq \phi\left(\frac{1}{2}2r\right) - \psi(2r, 0) \\ &= \phi(r) - \psi(2r, 0), \end{aligned}$$

which implies that $\psi(2r, 0) = 0$, and hence $r = 0$.
Another modification is (4) which needs to be replaced by

$$\begin{aligned} \phi\left(d\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi(d(Tx_{q_n}, Tx_{p_n})) \\ &\leq \phi(\max\{\alpha d(x_{q_n}, Tx_{p_n}) + \beta d(x_{p_n}, Tx_{q_n}), \gamma d(x_{q_n}, x_{p_n})\}) \\ &\quad -\psi(d(x_{q_n}, Tx_{p_n}), d(x_{p_n}, Tx_{q_n})). \end{aligned}$$

We again let $L_n = \max\{\alpha d(x_{q_n}, Tx_{p_n}) + \beta d(x_{p_n}, Tx_{q_n}), \gamma d(x_{q_n}, x_{p_n})\}$ and treat two cases. If $L_n = \gamma d(x_{q_n}, x_{p_n})$, then by letting $n \to \infty$ in the last inequality, we obtain

$$\phi(\epsilon) \leq \phi(\gamma\epsilon) - \psi(\epsilon, \epsilon).$$

Since ϕ is a nondecreasing function and $0 \leq \gamma \leq 1$, we have $\psi(\epsilon, \epsilon) = 0$, and hence $\epsilon = 0$, which is a contradiction. Now, if $L_n = \alpha d(x_{q_n}, Tx_{p_n}) + \beta d(x_{p_n}, Tx_{q_n})$, then by letting $n \to \infty$ in the last inequality, we obtain

$$\phi(\epsilon) \leq \phi((\alpha + \beta)\epsilon) - \psi(\epsilon, \epsilon).$$

Therefore, since $0 < \alpha + \beta \leq 1$, we again get $\psi(\epsilon, \epsilon) = 0$, and hence $\epsilon = 0$, which is again a contradiction.

Finally, (5) needs to be replaced by

$$\begin{aligned}\phi\left(d\left(x_{n_{k+1}},Tu^*\right)\right) &= \phi\left(d\left(Tx_{n_k},Tu^*\right)\right)\\ &\leq \phi\left(\max\{\alpha d\left(x_{n_k},Tu^*\right)+\beta d\left(u^*,Tx_{n_k}\right),\gamma d(x_{n_k},u^*)\}\right)\\ &\quad -\psi\left(d\left(x_{n_k},Tu^*\right),d\left(u^*,Tx_{n_k}\right)\right)\\ &\leq \phi\left(\max\{\alpha d\left(x_{n_k},Tu^*\right)+\beta d\left(u^*,Tx_{n_k}\right),\gamma d(x_{n_k},u^*)\}\right).\end{aligned}$$

Letting $k\to\infty$, we have

$$\phi\left(d\left(u^*,Tu^*\right)\right)\leq\phi\left(\max\{\alpha d\left(u^*,Tu^*\right)+\beta d\left(u^*,u^*\right),\gamma d(u^*,u^*)\}\right),$$

since ϕ is a nondecreasing function, we get

$$d\left(u^*,Tu^*\right)\leq\alpha d\left(u^*,Tu^*\right).$$

Thus, since $0\leq\alpha\leq\frac{1}{2}$, we have $d\left(u^*,Tu^*\right)=0$, and hence $u^*=Tu^*$. □

3. Examples and Applications.

In this section, as an application to our theory, we prove the existence and uniqueness of a non-negative solution for the integral equation given below in (9) under certain conditions. We also give two examples in order to validate the proved results.

Application 3.1. Let $X=C\left[0,1\right]$, the space of all continuous real valued functions on $[0,1]$ endowed with the max metric, $d(u,v)=\max\limits_{t\in[0,1]}|u(t)-v(t)|$. Consider the integral equation

$$v(s)=\int_0^1 K(s,t)h\left(t,v(t)\right)\,dt, \tag{9}$$

for all $s\in[0,1]$, where $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $K:[0,1]\times[0,1]\to[0,\infty)$ are continuous functions. Now, for $f,g\in X$, let $a,b\in\mathbb{R}$ be such that

$$a\leq f(s)\leq g(s)\leq b, \tag{10}$$

for all $s\in[0,1]$. Assume also that for all $s\in[0,1]$, we have

$$f(s)\leq\int_0^1 K(s,t)h\left(t,g(t)\right)\,dt \quad\text{and}\quad g(s)\geq\int_0^1 K(s,t)h\left(t,f(t)\right)\,dt. \tag{11}$$

Further, assume that for all $t\in[0,1]$, $h(t,.)$ be a nonincreasing function on $\mathbb{R}$, that is,

$$\text{for } x,y\in\mathbb{R},\ x\geq y\Rightarrow h(t,x)\leq h(t,y), \tag{12}$$

and for all $s\in[0,1]$, for all $x,y\in\mathbb{R}$ with ($x\leq b$ and $y\geq a$) or ($x\geq a$ and $y\leq b$), we have

$$|h(s,x)-h(s,y)|\leq\rho\left(|x-y|\right), \tag{13}$$

where ρ is a real valued continuous function satisfying

$$\rho\left(t\right)\leq\delta\,t, 0<\delta\leq1. \tag{14}$$

Theorem 3.1. *Let φ be a convex continuous altering distance function satisfying $\varphi(xy)\leq\varphi(x)\varphi(y)$. Then if the conditions (10)-(14) are satisfied, the integral equation (9) has a unique solution $v^*\in\{v\in X: f(t)\leq v(t)\leq g(t), t\in[0,1]\}$, where $K:[0,1]\times[0,1]\to[0,\infty)$ is a continuous function that satisfies*

$$\sup_{t\in[0,1]}\int_0^1 K(t,s)\,ds\leq1.$$

Proof. In order to prove the existence of a unique non-negative solution of (9), we define the map $T : X \to X$ as

$$Tv(s) = \int_0^1 K(s,t)h\left(t, v(t)\right)\, dt,$$

and the two closed subsets A_1, A_2 of X as

$$A_1 = \{u \in X : u \leq g\} \text{ and } A_2 = \{u \in X : u \geq f\}.$$

First, we will show that $T : A_1 \cup A_2 \to A_1 \cup A_2$ is a cyclic map. Let $u \in A_1$. Then for all $t \in [0,1]$, we have $u(t) \leq g(t)$. Now, since $h\left(s, .\right)$ is a nonincreasing function on $\mathbb{R}$ and $K(s,t) \geq 0$ for all $t, s \in [0,1]$, we get

$$K(s,t)h\left(t, u(t)\right) \geq K(s,t)h\left(t, g(t)\right),$$

for all $t, s \in [0,1]$. Consequently, we have

$$\int_0^1 K(s,t)h\left(t, u(t)\right)\, dt \geq \int_0^1 K(s,t)h\left(t, g(t)\right)\, dt \geq f(s),$$

for all $s \in [0,1]$. Hence, $Tu \in A_2$. Similarly, if $u \in A_2$, then

$$\int_0^1 K(s,t)h\left(t, u(t)\right) dt \leq \int_0^1 K(s,t)h\left(t, f(t)\right) dt \leq g(s)$$

for all $s \in [0,1]$ and hence $Tu \in A_1$. Thus, T is a cyclic map from $A_1 \cup A_2$ into $A_1 \cup A_2$. Now, for a convex continuous altering distance function φ and $(u,v) \in A_1 \times A_2$, $u(s) \leq b, v(s) \geq a$, for all $s \in [0,1]$, we have

$$\begin{aligned} \varphi(|Tu - Tv|) &= \varphi\left(\left|\int_0^1 K(s,t)h\left(t, u(t)\right) dt - \int_0^1 K(s,t)h\left(t, v(t)\right) dt\right|\right) \\ &\leq \varphi\left(\int_0^1 K(s,t)\left|h\left(t, u(t)\right) - h\left(t, v(t)\right)\right|\, dt\right). \end{aligned}$$

Using Jenson Inequality 1.1, and properties of the functions φ, h and ρ, we have

$$\begin{aligned} \varphi\left(|Tu - Tv|\right) &\leq \int_0^1 \varphi\left(K(s,t)\left|h\left(t, u(t)\right) - h\left(t, v(t)\right)\right|\right)\, dt \\ &\leq \int_0^1 \varphi\left(K(s,t)\right)\varphi\left(\left|h\left(t, u(t)\right) - h\left(t, v(t)\right)\right|\right)\, dt \\ &\leq \int_0^1 \varphi\left(K(s,t)\right)\varphi\left(\rho\left(|u(t) - v(t)|\right)\right)\, dt \\ &\leq \varphi\left(\rho\left(d\left(u,v\right)\right)\right)\int_0^1 \varphi\left(K(s,t)\right)\, ds \\ &\leq \varphi\left(\gamma d\left(u,v\right)\right) \\ &\leq \varphi\left(\max\left\{\alpha d\left(u, Tu\right) + \beta d\left(v, Tv\right), \gamma d\left(u,v\right)\right\}\right) \\ &\quad - \left(\varphi\left(\max\left\{\alpha d\left(u, Tu\right) + \beta d\left(v, Tv\right), \gamma d\left(u,v\right)\right\}\right) - \varphi\left(\gamma d\left(u,v\right)\right)\right). \end{aligned}$$

Therefore, the map T is a cyclic contractive satisfying the conditions of Theorem 2.1. Hence, T has a unique fixed point v^* in $A_1 \cap A_2 = \{u \in X = C\left[0,1\right] : f\left(t\right) \leq u\left(t\right) \leq g\left(t\right)$ for all $t \in [0,1]\}$. Hence, v^* is a solution of the integral equation (9). □

Example 3.1. Let X be a complete metric space, m positive integer, $A_1, \ldots, A_m$ non-empty closed subsets of X, and $X = \bigcup\limits_{i=1}^{m} A_i$. Let $T : X \to X$ be an operator such that

a) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T.

b) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping satisfies $\int_0^t \rho(s)\, ds > 0$ for $t > 0$, we have one of the following:

$$\int_0^{d(Tx,Ty)} \rho(t)\, dt \leq \int_0^{\max\{(\alpha d(x,Tx)+\beta d(y,Ty)),\gamma d(x,y)\}} \rho(t)\, dt,$$

or

$$\int_0^{d(Tx,Ty)} \rho(t)\, dt \leq \int_0^{\max\{(\alpha d(y,Tx)+\beta d(x,Ty)),\gamma d(x,y)\}} \rho(t)\, dt.$$

Then T has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

In order to see this, one shall let $\phi : [0, \infty) \to [0, \infty)$ be defined as $\phi(t) = \int_0^t \rho(s)\, ds > 0$. Then ϕ is altering distance function, and by taking $\psi(t) = 0$, we get the result.

Example 3.2. Let $X = [-1, 1] \subseteq \mathbb{R}$ with $d(x, y) = |x - y|$. Let $T : [-1, 1] \to [-1, 1]$ be given by

$$T(x) = \begin{cases} -\frac{1}{2} x e^{-\frac{1}{|x|}}, & x \in (0, 1], \\ 0, & x = 0, \\ -\frac{1}{3} x e^{-\frac{1}{|x|}}, & x \in [-1, 0). \end{cases}$$

By taking $\psi(t) = 0$, $\phi(t) = t$, and $x \in [0, 1]$, $y \in [-1, 0]$, we have

$$\begin{aligned} |Tx - Ty| &= |-\tfrac{1}{2} x e^{-\frac{1}{|x|}} + \tfrac{1}{3} y e^{-\frac{1}{|y|}}| \\ &\leq \tfrac{1}{2}|x| + \tfrac{1}{3}|y| \\ &\leq \tfrac{1}{2}|x + \tfrac{1}{2} x e^{-\frac{1}{|x|}}| + \tfrac{1}{3}|y + \tfrac{1}{3} y e^{-\frac{1}{|y|}}| \\ &= \tfrac{1}{2}|Tx - x| + \tfrac{1}{3}|Ty - y|, \end{aligned}$$

which implies that T has a unique fixed point in $[-1, 0] \cap [0, 1]$ which is $z = 0$.

4. Conclusions.

Using nonlinear contractions based on altering distances, we prove new fixed point theorems that generalize and extend many previous theorems in the literature in the sense that those previous results are special cases of our new proved results. Furthermore, we prove the existence and uniqueness of solutions for integral equations under certain conditions using Jenson inequality and our presented fixed point results.

References

[1] Banach, B., (1922), Surles operations dans les ensembles et leur application aux equation sitegrales, Fund. Math., 3, pp. 133-181.

[2] Kannan, R., (1968), Some results on fixed points, Bull. Calcutta Math. Soc., 10, pp. 71-76.

[3] Chatterjea, S., (1972), Fixed point theorems, C. R. Acad. Bulgare Sci., 25, pp. 727-730.

[4] Zamfirescu, T., (1972), Fixed point theorems in metric spaces, Arch. Math., (Basel) 23, 292–298.

[5] Kirk, W. A., Srinivasan, P. S. and Veeramani, P., (2003), Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory, 4(1), pp. 79-89.

[6] Rus, I., (2005), Cyclic representations and fixed points, Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, vol.3, pp. 171-178.

[7] Petric, M., (2010), Some results concerning cyclical contractive mappings, General Mathematics, 18(4), pp. 213-226.

[8] Khan, M., Swaleh, M. and Sessa, S., (1984), Fixed point theorem by altering distances between points, Bull. Austral. Math. soc., 30(1), pp. 1-9.

[9] Sastry, K. and Babu, G., (1999), Some fixed point theorems by altering distances between the points, Indian Journal of Pure and Applied Mathematics, 30(6), pp. 641-647.

[10] Sastry, K., Naidu, S., Babu, G. R. and Naidu, G. A., (2000), Generalization of common fixed point theorems for weakly commuting map by altering distances, Tamkang Journal of Mathematics, 31(3), pp.243–250.

[11] Naidu, S., (2003), Some fixed point theorems in metric spaces by altering distances, Czechoslovak Mathematical Journal, 53(1), pp. 205-212.

[12] Abdeljawad, T., Alzabut, J., Mukheimer, A. and Zaidan, Y., (2012), Banach contraction principle for cyclical mappings on partial metric spaces, Fixed Point Theory Applications, Article number: 154(2012).

[13] Karapinar, E., Yildiz-Ulus, A. and Erhan, I. M., (2012), Cyclic Contractions on G-Metric Spaces, Abstract and Applied Analysis, Volume 2012, Article ID 182947, 15 pages.

[14] Khandaqji, M., Al-Sharif, Sh. and Al-Khaleel, M., (2012), Property P and some fixed point results on (ψ, ϕ)-weakly contractive G-metric spaces, International Journal of Mathematics and Mathematical Sciences, Volume 2012, Article ID 675094, 11 pages, DOI:10.1155/2012/675094.

[15] Khandaqji, M., Al-Sharif, Sh. and Al-Khaleel, M., (2012), Coupled fixed point theorems for nonlinear contractions in partial metric spaces, International Journal of Mathematics and Mathematical Sciences, Volume 2012, Article ID 763952, 13 pages, DOI:10.1155/2012/763952.

[16] Khandaqji, M., Al-Sharif, Sh. and Al-Khaleel, M., (2013), Results in coupled fixed point in nonlinear contractive conditions, Int. Journal of Physics: Conference Series, 435(2013) 012019, 12 pages, DOI:10.1088/1742-6596/435/1/012019.

[17] Al-Khaleel, M., Awad, A. and AlShareef, Sh., (2013), Some results for cyclic nonlinear contractive mappings in metric spaces, Acta Math. Acad. Paed. Ny., 29, pp. 9-18.

[18] Sang, Y. and Meng, Q., (2015), Fixed point theorems with generalized altering distance functions in partially ordered metric spaces via w-distances and applications, Fixed Point Theory and Applications, ArticleID 168(2015), 25 pages, DOI 10.1186/s13663-015-0408-0.

[19] Radenović, S., (2015), A note on fixed point theory for cyclic φ-contractions, Fixed Point Theory and Applications, Article ID. 189(2015).

[20] Radenović, S., Došenović, T., Aleksić-Lampert, T. and Golubović, Z., (2016), A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations, Applied Mathematics and Computation, 273 (2016), pp. 155-164.

[21] Klineam, Ch. and Kaskasem, P., (2016), Fixed Point Theorems for Cyclic Contractions in C^*-Algebra-Valued b-Metric Spaces, Journal of Function Spaces, Volume 2016, Article ID 7827040, 16 pages.

[22] Aydi, H., Felhi, A. and Sahmim, S., (2017), Related fixed point results for cyclic contractions on G-metric spaces and application, Filomat, 31:3(2017), pp. 853–869.

[23] Ješić, S., Ćirović, N. and O'Regan, D., (2017), Altering distances and a common fixed point theorem in menger probability metric spaces, Filomat, 31(2), pp. 175-181.

[24] Al-Sharif, Sh., Alahmari, A., Al-Khaleel, M. and Salim, A. , (2017), New results on fixed points for an infinite sequence of mappings in G-metric space, Italian Journal of Pure and Applied Mathematics, 2017(37), pp. 517-540.

[25] Kumari, P. S., Nantadilok, J. and Sarwar, M., (2018), Fixed point theorems for a class of generalized weak cyclic compatible contractions, Fixed Point Theory and Applications, Article number: 13 (2018).

[26] Karapinar, E., Erhan, I. M. and Yildiz-Ulus, A., (2018), Fixed Point Theorem for Cyclic Maps on Partial Metric spaces, Applied Mathematics and Information Sciences, 6(2), pp. 239-244.

[27] Nurwahyu, B., (2019), Fixed Point Theorems for Cyclic Weakly Contraction Mappings in Dislocated Quasi Extended b-Metric Space, Journal of Function Spaces, Volume 2019, Article ID 1367879, 10 pages.

[28] Al-Khaleel, M. and Al-Sharif, Sh., (2019), On cyclic (ϕ - ψ)-Kannan and (ϕ - ψ)-Chatterjea contractions in metric spaces, Annals of the University of Craiova - Mathematics and Computer Science Series, 46(2), pp.320-327.

[29] Royden, H. L., (1988), Real analysis, Third edition, Prentice Hall, England Cliffs, NJ 07632.

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