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# MATHEMATICAL MODEL FOR THE DISPERSION OF TOXICITY THROUGH COMMUTERS

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Abstract. In the proposed paper, we consider three types of vehicle. One is those vehicles which are used for personal purposes called private vehicles. Second is for public transportation named as public vehicles. Third one is used for other activities known as cargo vehicles. These vehicles can be either polluted or non-polluted according to their fuel category. They produce toxic air pollutants which makes environment toxic. To compare the dispersion of toxicity through these vehicles, a mathematical model is formulated. In this model, dynamical system is developed with the help of non-linear differential equations. Threshold for each vehicle category is acquired and compared. Threshold and backward bifurcation govern the stability of the model. Simulation is prepared to brace the output.

Keywords: Dynamical system, Threshold, Backward bifurcation, Stability, Toxicity

AMS Subject Classification: 37Nxx

## 1. Introduction

In today's world, an individual has many ways to commute from one place to another. The form of transformation chosen by an individual is mainly depend upon the parameters of time and cost. The individuals who have a time constraint chose to travel by a private vehicle. In fact, for a long route or for individuals who have cost constraint, the public vehicle is used. But except human transportation, cargo transportation is also important for day-to-day life. Cargo vehicle is used for transporting packages or goods from the manufacturer to the buyer, picking up the waste, etc. Thus, private, public and cargo vehicles are very useful, every day. Depending upon their fuel category, not all vehicles have a good impact on the environment. Those which are not harmful to the environment, are considered as non-polluted vehicles. And vehicles that are harmful, are measured as polluted vehicles. Polluted vehicles directly have a negative impact on the environment while non-polluted vehicles have that indirectly. Vehicle pollution is the major

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source of an environmental disturbance. Vehicle pollution splits into two types: primary pollution and secondary pollution. Pollution which directly goes into the environment is called primary pollution. Secondary pollution occurs after the chemical reaction between the released pollutants into the atmosphere. This pollution contains Ozone ($O_3$), Particulate Matter (PM), Nitrogen Oxide ($NO_X$), Carbon Monoxide (CO), Sulphur dioxide ($SO_2$), Hazardous air pollutants (toxic) (https://www.ucsusa.org/clean-vehicles/vehicles-air-pollution-and-human-health/cars-trucks-air-pollution#.XF6NEjMzY2w). Those hazardous air pollutants alters into toxicity which leads to serious health issues.
Vehicular pollution is a worldwide problem. Therefore, research works regarding this have been done through many researchers: some review is done for air quality in relation to motor vehicle transportation using mathematical diffusion modeling techniques by Lamb *et al.* (1973). Beaton *et al.* (1972) have measured the impact of air quality on the highway for the factors of motor vehicle emission. Using the system dynamics, Sadovnikove *et al.* and Manohar *et al.* have studied this area. Sadovnikova *et al.* (2013) analysed the vehicle and landscaping impact on air quality where Manohar *et al.* (2014) evaluated the policies to reduce transportation pollution. Another approach through inventory is viewed for pollutants emitted through vehicles in Delhi by Goyal *et al.* (2013). Shah *et al.* and Bauver have worked under this real-valued problem using the area of mathematical modeling. Shah *et al.* (2018) have proposed the model which measures environmental pollution through vehicles where Bauver (1978) has prepared a model for highways consists of the spread of air pollutants. To reduce air pollution, the different idea was emerged through Shah *et al.* (2016). They have formed a model dynamic for the wrong parking habits leads to punishment. For environmental growth, graph theory was applied in mathematical modeling to support the idea of 'GO-CLEAN' by Shah *et al.* (2018).
In this paper, Mathematical models are designed for three different vehicle types: private, public and cargo vehicles in section 2 along with the existence of equilibrium points and their threshold. In section 3, the local and global stability about the equilibrium points is incorporated. Section 4 contains the existence of bifurcation. Numerical simulation is done in section 5 using validated data.

## 2. Mathematical model

We live in a society where transportation is very essential tool for day-to-day lifestyle. Some individuals opt for their personal transportation, some go for public transportation. Moreover, there are some activities such as factories, industries etc. those need some cargo materials to transport. Therefore, we have prepared three models viz. private vehicle model, public vehicle model and cargo vehicle model to study the spread of toxicity under some assumptions:
*Assumptions*:

(1) Toxicity free equilibrium point exists with non-polluted and polluted vehicles.
(2) Polluted vehicles turn into non-polluted because of advanced technology.
(3) Non-polluted vehicles create toxicity indirectly which is less than that by polluted vehicles.
(4) For the analytical purpose, escape rate from each compartment is taken as constant.

TABLE 1. Notation, description and its parametric values

| | | | | | | |
|---|---|---|---|---|---|---|
| $B_{P_R}$ | $B_{P_B}$ | $B_{C_V}$ | The growth rate of respective vehicles | 0.02 | 0.015 | 0.005 |
| $\beta_{P_R}$ | $\beta_{P_B}$ | $\beta_{C_V}$ | The respective vehicles which are non-polluted | 0.80 | 0.50 | 0.20 |
| $\delta_{P_R}$ | $\delta_{P_B}$ | $\delta_{C_V}$ | The respective vehicles which are non-polluted | 0.30 | 0.60 | 0.70 |
| $\eta_1$ | $\epsilon_1$ | $\gamma_1$ | The transfer rate of polluted vehicles into polluted for respective vehicle category | 0.40 | | |
| $\eta_2$ | $\epsilon_2$ | $\gamma_2$ | The transfer rate of non-polluted vehicles into polluted for respective vehicle category | 0.20 | | |
| $\eta_3$ | $\epsilon_3$ | $\gamma_3$ | The rate at which toxicity spread through non-polluted vehicles for respective vehicle category | 0.10 | | |
| $\eta_4$ | $\epsilon_4$ | $\gamma_4$ | The rate at which toxicity spread through polluted vehicles for respective vehicle category | 0.80 | | |
| $\mu_{P_R}$ | $\mu_{P_B}$ | $\mu_{C_V}$ | The respective vehicles which are non-polluted | 0.19 | 0.15 | 0.07 |

2.1. **Private vehicle model.** Figure 1 represents the transmission diagram of private vehicle model to analyse the spread of toxicity.

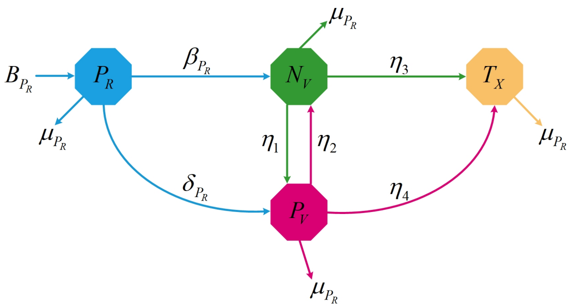

FIGURE 1. Transmission diagram of private vehicle model

Using figure 1 and accordingly parameter mentioned in the table 1, the system of non-linear differential equation is formulated:

$$\begin{aligned}
\frac{dP_R}{dt} &= B_{P_R} - \beta_{P_R}P_R - \delta_{P_R}P_R - \mu_{P_R}P_R \\
\frac{dN_V}{dt} &= \beta_{P_R}P_R - \eta_1 N_V + \eta_2 P_V - \eta_3 N_V T_X - \mu_{P_R}N_V \\
\frac{dP_V}{dt} &= \delta_{P_R}P_R + \eta_1 N_V - \eta_2 P_V - \eta_4 P_V T_X - \mu_{P_R}P_V \\
\frac{dT_X}{dt} &= \eta_3 N_V T_X + \eta_4 P_V T_X - \mu_{P_R}T_X
\end{aligned} \tag{1}$$

From above system, we have

$\frac{d}{dt}\left(P_R + N_V + P_V + T_X\right) = B_{P_R} - \mu_{P_R}\left(P_R + N_V + P_V + T_X\right)$

So that, $\lim_{t\to\infty} \sup\left(P_R + N_V + P_V + T_X\right) \leq \frac{B_{P_R}}{\mu_{P_R}}$

Hence, that feasible region of the model is

$\Lambda_{P_R} = \left\{(P_R, N_V, P_V, T_X) : P_R + N_V + P_V + T_X \leq \frac{B_{P_R}}{\mu_{P_R}}; P_R > 0, N_V > 0, P_V > 0, T_X \geq 0\right\}.$

Now, we take $\beta_{P_R} + \delta_{P_R} + \mu_{P_R} = x_{P_R}$ and get new modified system

$$\begin{aligned}
\frac{dP_R}{dt} &= B_{P_R} - x_{P_R}P_R \\
\frac{dN_V}{dt} &= \beta_{P_R}P_R - \eta_1 N_V + \eta_2 P_V - \eta_3 N_V T_X - \mu_{P_R}N_V \\
\frac{dP_V}{dt} &= \delta_{P_R}P_R + \eta_1 N_V - \eta_2 P_V - \eta_4 P_V T_X - \mu_{P_R}P_V \\
\frac{dT_X}{dt} &= \eta_3 N_V T_X + \eta_4 P_V T_X - \mu_{P_R}T_X
\end{aligned} \tag{2}$$

In this case, both the systems and their feasible regions are equivalent. So, we are considering $\Lambda_{P_R}$ as feasible region for the system (2).

2.2. **Public vehicle model.** Figure 2 denotes the transmission diagram of public vehicle model to study the toxicity spread.

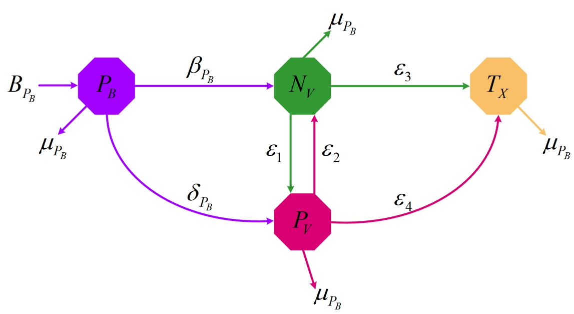


FIGURE 2. Transmission diagram of public vehicle model

From figure 1 and respective parameters displayed in the table 1, the system of non-linear differential equation is developed:

$$\begin{aligned}
\frac{dP_B}{dt} &= B_{P_B} - \beta_{P_B}P_B - \delta_{P_B}P_B - \mu_{P_B} \\
\frac{dN_V}{dt} &= \beta_{P_B}P_B - \epsilon_1 N_V + \epsilon_2 P_V - \epsilon_3 N_V T_X - \mu_{P_B}N_V \\
\frac{dP_V}{dt} &= \delta_{P_B}P_B + \epsilon_1 N_V - \epsilon_2 P_V - \epsilon_4 P_V T_X - \mu_{P_B}P_V \\
\frac{dT_X}{dt} &= \epsilon_3 N_V T_X + \epsilon_4 P_V T_X - \mu_{P_B}T_X
\end{aligned} \tag{3}$$

System 3 represents,
$\frac{d}{dt}(P_B + N_V + P_V + T_X) = B_{P_B} - \mu_{P_B}(P_B + N_V + P_V + T_X)$
So that, $\lim_{t\to\infty} \sup (P_B + N_V + P_V + T_X) \leq \frac{B_{P_B}}{\mu_{P_B}}$
Hence, that feasible region of the model is
$\Lambda_{P_B} = \left\{(P_B, N_V, P_V, T_X) : P_B + N_V + P_V + T_X \leq \frac{B_{P_B}}{\mu_{P_B}}; P_B > 0, N_V > 0, P_V > 0, T_X \geq 0\right\}$.
Now, substituting $\beta_{P_B} + \delta_{P_B} + \mu_{P_B} = x_{P_B}$, we get a new system shown below:

$$\begin{aligned}
\frac{dP_B}{dt} &= B_{P_B} - x_{P_B} P_B \\
\frac{dN_V}{dt} &= \beta_{P_B} P_B - \epsilon_1 N_V + \epsilon_2 P_V - \epsilon_3 N_V T_X - \mu_{P_B} N_V \\
\frac{dP_V}{dt} &= \delta_{P_B} P_B + \epsilon_1 N_V - \epsilon_2 P_V - \epsilon_4 P_V T_X - \mu_{P_B} P_V \\
\frac{dT_X}{dt} &= \epsilon_3 N_V T_X + \epsilon_4 P_V T_X - \mu_{P_B} T_X
\end{aligned} \tag{4}$$

System (3) and system (4) are equivalent. So, we are considering $\Lambda_{P_B}$ as feasible region for the system (4) in the invariant region.

2.3. **Cargo vehicle model.** Figure (3) shows the transmission diagram of toxicity spread through cargo vehicle.

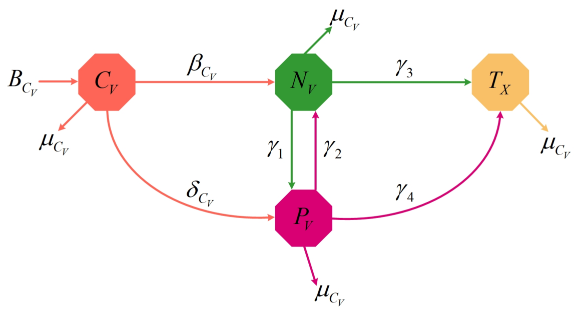


FIGURE 3. Transmission diagram of cargo vehicle model

Figure 3 and parameters assumed in the table 1 gives the system of non-linear differential equation.

$$\begin{aligned}
\frac{dC_V}{dt} &= B_{C_V} - \beta_{C_V} C_V - \delta_{C_V} C_V - \mu_{C_V} C_V \\
\frac{dN_V}{dt} &= \beta_{C_V} C_V - \gamma_1 N_V + \gamma_2 P_V - \gamma_3 N_V T_X - \mu_{C_V} N_V \\
\frac{dP_V}{dt} &= \delta_{C_V} C_V + \gamma_1 N_V - \gamma_2 P_V - \gamma_4 P_V T_X - \mu_{C_V} P_V \\
\frac{dT_X}{dt} &= \gamma_3 N_V T_X + \gamma_4 P_V T_X - \mu_{C_V} T_x
\end{aligned} \tag{5}$$

Adding system (5),
$\frac{d}{dt}(C_V + N_V + P_V + T_X) = B_{C_V} - \mu_{C_V}(C_V + N_V + P_V + T_X)$
So that, $\lim_{t\to\infty} \sup (C_V + N_V + P_V + T_X) \leq \frac{B_{C_V}}{\mu_{C_V}}$
Hence, that feasible region of the model is
$\Lambda_{C_V} = \left\{(C_V, N_V, P_V, T_X) : C_V + N_V + P_V + T_X \leq \frac{B_{C_V}}{\mu_{C_V}}; C_V > 0, N_V > 0, P_V > 0, T_X \geq 0\right\}.$
Now, substituting $\beta_{C_V} + \delta_{C_V} + \mu_{C_V} = x_{C_V}$ we get a new system shown below:

$$\begin{aligned}
\frac{dC_V}{dt} &= B_{C_V} - x_{C_V} C_V \\
\frac{dN_V}{dt} &= \beta_{C_V} C_V - \gamma_1 N_V + \gamma_2 P_V - \gamma_3 N_V T_X - \mu_{C_V} N_V \\
\frac{dP_V}{dt} &= \delta_{C_V} C_V + \gamma_1 N_V - \gamma_2 P_V - \gamma_4 P_V T_X - \mu_{C_V} P_V \\
\frac{dT_X}{dt} &= \gamma_3 N_V T_X + \gamma_4 P_V T_X - \mu_{C_V} T_x
\end{aligned} \tag{6}$$

System (5) and system (6) are equivalent and their dynamical behaviour. So, we are considering feasible region noted as $\Lambda_{C_V}$ for the system (6).

2.4. **Existence equilibrium points.** In this section, equilibrium points of the models are carried out and also found their existence. Equilibrium points are obtained when each equation of the model is set to zero.

2.4.1. *Equilibrium points for private vehicle model.* For private vehicle model, we get two equilibrium points.

(1) Toxicity free equilibrium point for private vehicle model
$E_0\left(P_R\right) = \left(P_R^0, N_V^0, P_V^0, 0\right)$ where
$P_R^0 = \frac{B_{P_R}}{x_{P_R}}, N_V^0 = \frac{B_{P_R}\left(\beta_{P_R}\left(\eta_2+\mu_{P_R}\right)+\delta_{P_R}\eta_2\right)}{x_{P_R}\mu_{P_R}\left(\eta_1+\eta_2+\mu_{P_R}\right)}, P_V^0 = \frac{B_{P_R}\left(\beta_{P_R}\eta_1+\delta_{P_R}\left(\eta_1+\mu_{P_R}\right)\right)}{x_{P_R}\mu_{P_R}\left(\eta_1+\eta_2+\mu_{P_R}\right)}$
This equilibrium point exists trivially.

(2) Interior equilibrium point for private vehicle model
$E^*\left(P_R\right) = \left(P_R^*, N_V^*, P_V^*, T_X^*\right)$ where $P_R^* = \frac{B_{P_R}}{x_{P_R}}, N_V^* = r_1, P_V^* = \frac{\mu_{P_R}-\eta_3 r_1}{\eta_4}$,
$T_X^* = \frac{B_{P_R}\beta_{P_R}\eta_4+\left(\eta_2\mu_{P_R}-\left(\eta_2\eta_3+\eta_4\left(\eta_1+\mu_{P_R}\right)\right)r_1\right)x_{P_R}}{x_{P_R}\eta_3\eta_4 r_1}$ and

$$r_1 = rootof\{[(\eta_3 - \eta_4)x_{P_R}\eta_3\mu_{P_R}]Z^2 + [(\beta_{P_R} + \delta_{P_R})B_{P_R}\eta_3\eta_4 + ((\eta_2 - \mu_{P_R})\eta_3 + (\eta_1 + \mu_{P_R})\eta_4)x_{P_R}\mu_{P_R}]Z - \mu_{P_R}[B_{P_R}\beta_{P_R} + x_{P_R}\eta_4\mu_{P_R}]\}$$

The equilibrium point exists when
(a) $r_1$ is positive.
(b) $\frac{\eta_2}{r_1}\mu_{P_R} > \eta_2\eta_3 + \eta_4(\eta_1 + \mu_{P_R})$

2.4.2. *Equilibrium points for public vehicle model.* Solving the system for public vehicle model, we find two equilibrium points.

(1) Toxicity free equilibrium point for public vehicle model
$E_0(P_B) = \left(P_B^0, N_V^0, P_V^0, 0\right)$ where
$P_B^0 = \frac{B_{P_B}}{x_{P_B}}, N_V^0 = \frac{B_{P_B}\left(\beta_{P_B}\left(\epsilon_2+\mu_{P_B}\right)+\delta_{P_B}\epsilon_2\right)}{x_{P_B}\mu_{P_B}\left(\epsilon_1+\epsilon_2+\mu_{P_B}\right)}, P_V^0 = \frac{B_{P_B}\left(\beta_{P_B}\epsilon_1+\delta_{P_B}\left(\epsilon_1+\mu_{P_B}\right)\right)}{x_{P_B}\mu_{P_B}\left(\epsilon_1+\epsilon_2+\mu_{P_B}\right)}$
This point occurs trivially.

(2) Interior equilibrium point for public vehicle model
$E^*\left(P_B\right) = \left(P_B^*, N_V^*, P_V^*, T_X^*\right)$ where $P_B^* = \frac{B_{P_B}}{x_{P_B}}, N_V^* = r_2, P_V^* = \frac{\mu_{P_B}-\epsilon_3 r_2}{\epsilon_4}$,
$T_X^* = \frac{B_{P_B}\beta_{P_B}\epsilon_4+\left(\epsilon_2\mu_{P_B}-\left(\epsilon_2\epsilon_3+\epsilon_4\left(\epsilon_1+\mu_{P_B}\right)\right)r_2\right)x_{P_B}}{x_{P_B}\epsilon_3\epsilon_4 r_2}$ and

$$r_2 = rootof\{[(\epsilon_3 - \epsilon_4)x_{P_B}\epsilon_3\mu_{P_B}]Z^2 + [(\beta_{P_B} + \delta_{P_B})B_{P_B}\epsilon_3\epsilon_4 + ((\epsilon_2 - \mu_{P_B})\epsilon_3 + (\epsilon_1 + \mu_{P_B})\epsilon_4)x_{P_B}\mu_{P_B}]Z - \mu_{P_B}[B_{P_B}\beta_{P_B} + x_{P_B}\epsilon_4\mu_{P_B}]\}$$

The equilibrium point exists when

(a) $r_2$ is positive.
(b) $\frac{\epsilon_2}{r_2}\mu_{P_B} > \epsilon_2\epsilon_3 + \epsilon_4(\epsilon_1 + \mu_{P_B})$

2.4.3. *Equilibrium points for cargo vehicle model.* On solving system for cargo vehicle model, we get

(1) Toxicity free equilibrium point for cargo vehicle model
$E_0(C_V) = \left(C_V^0, N_V^0, P_V^0, 0\right)$ where
$C_V^0 = \frac{B_{C_V}}{x_{C_V}}, N_V^0 = \frac{B_{C_V}\left(\beta_{C_V}\left(\gamma_2+\mu_{C_V}\right)+\delta_{C_V}\gamma_2\right)}{x_{C_V}\mu_{C_V}\left(\gamma_1+\gamma_2+\mu_{C_V}\right)}, P_V^0 = \frac{B_{C_V}\left(\beta_{C_V}\gamma_1+\delta_{C_V}\left(\gamma_1+\mu_{C_V}\right)\right)}{x_{C_V}\mu_{C_V}\left(\gamma_1+\gamma_2+\mu_{C_V}\right)}$
The toxicity free equilibrium point exists unconditionally.
(2) Interior equilibrium point for cargo vehicle model
$E^*\left(C_V\right) = \left(C_V^*, N_V^*, P_V^*, T_X^*\right)$ where $C_V^* = \frac{B_{C_V}}{x_{C_V}}, N_V^* = r_3, P_V^* = \frac{\mu_{C_V}-\gamma_3 r_3}{\gamma_4}$,
$T_X^* = \frac{B_{C_V}\beta_{C_V}\gamma_4+\left(\gamma_2\mu_{C_V}-\left(\gamma_2\gamma_3+\gamma_4\left(\gamma_1+\mu_{C_V}\right)\right)r_3\right)x_{C_V}}{x_{C_V}\gamma_3\gamma_4 r_3}$ and

$$\begin{aligned} r_3 =& rootof\{[(\gamma_3-\gamma_4)x_{P_B}\gamma_3\mu_{C_V}]Z^2 + [(\beta_{C_V}+\delta_{C_V})B_{C_V}\gamma_3\gamma_4 + ((\gamma_2 \\ &-\mu_{C_V})\gamma_3 + (\gamma_1+\mu_{C_V})\gamma_4)x_{C_V}\mu_{C_V}]Z - \mu_{C_V}[B_{C_V}\beta_{C_V} + x_{C_V}\gamma_4\mu_{C_V}]\} \end{aligned}$$

2.5. **Computation of the threshold quantity.** Here, the threshold quantity for each vehicle model is computed. It is worked out using next generation matrix method (Diekmann et al., 2010). The threshold quantity is to study the density of commuters which creates concentration of toxicity into the environment. If threshold quantity is less than one then the model is stable otherwise the concentration is achieving epidemic state.

2.5.1. *Threshold for private vehicle model.* The threshold for private vehicle model $\left(R_0^{P_R}\right)$ is the ratio of newly infected toxic air pollutants created by private vehicles affected by a single infectious toxic air pollutants created by private vehicles during its lifespan of spreading toxicity. It is calculated as follows:

$$F_{P_R} = \begin{bmatrix} \eta_3 N_V + \eta_4 P_V & 0 & \eta_3 T_X & \eta_4 T_X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $$V_{P_R} = \begin{bmatrix} \mu_{P_R} & 0 & 0 & 0 \\ 0 & x_{P_R} & 0 & 0 \\ \eta_3 N_V & -\beta_{P_R} & \eta_1 + \eta_3 T_X + \mu_{P_R} & -\eta_2 \\ \eta_4 P_V & -\delta_{P_R} & -\eta_1 & \eta_2 + \eta_4 T_X + \mu_{P_R} \end{bmatrix}$$

Here, $V_{P_R}$ is non-singular matrix.
The threshold of private vehicles is the spectral radius of matrix $F_{P_R}V_{P_R}^{-1}$.

$$R_0^{P_R} = \frac{B_{P_R}\left(\eta_3\left(\beta_{P_R}\left(\eta_2+\mu_{P_R}\right)+\delta_{P_R}\eta_2\right)+\eta_4\left(\beta_{P_R}\eta_1+\delta_{P_R}\left(\eta_1+\mu_{P_R}\right)\right)\right)}{\mu_{P_R}^2 x_{P_R}\left(\eta_1+\eta_2+\mu_{P_R}\right)} \quad (7)$$

2.5.2. *Threshold for public vehicle model.* The threshold for public vehicle model $\left(R_0^{P_B}\right)$ is the ratio of newly infected toxic air pollutants created by public vehicles affected by a single infectious toxic air pollutants created by public vehicles during its lifespan of spreading toxicity computed as in section 2.5.1.
The threshold of private vehicles is the spectral radius of matrix $F_{P_B}V_{P_B}^{-1}$.

$$R_0^{P_B} = \frac{B_{P_B}\left(\varepsilon_3\left(\beta_{P_B}\left(\varepsilon_2+\mu_{P_B}\right)+\delta_{P_B}\varepsilon_2\right)+\varepsilon_4\left(\beta_{P_B}\varepsilon_1+\delta_{P_B}\left(\varepsilon_1+\mu_{P_B}\right)\right)\right)}{\mu_{P_B}^2 x_{P_B}\left(\varepsilon_1+\varepsilon_2+\mu_{P_B}\right)} \tag{8}$$

2.5.3. *Threshold for cargo vehicle model.* The threshold for cargo vehicle model $\left(R_0^{C_V}\right)$ is the ratio of newly infected toxic air pollutants created by cargo vehicles affected by a single infectious toxic air pollutants created by cargo vehicles during its lifespan of spreading toxicity that is computed as 2.5.1.
The threshold of private vehicles is the spectral radius of matrix $F_{C_V}V_{C_V}^{-1}$.

$$R_0^{C_V} = \frac{B_{C_V}\left(\gamma_3\left(\beta_{C_V}\left(\gamma_2+\mu_{C_V}\right)+\delta_{C_V}\gamma_2\right)+\gamma_4\left(\beta_{C_V}\gamma_1+\delta_{C_V}\left(\gamma_1+\mu_{C_V}\right)\right)\right)}{\mu_{C_V}^2 x_{C_V}\left(\gamma_1+\gamma_2+\mu_{C_V}\right)} \tag{9}$$

## 3. Stability analysis of the equilibrium points

In this section, local and global stability are established to understand the behaviour of each equilibrium points.

3.1. **Local stability analysis.** Here, local stability analysis is worked for each model about their equilibrium points. It is established using Routh-Hurwitz criteria (Routh, 1877).

3.1.1. *Local stability of private vehicle model.* The Jacobian matrix for private vehicle model is derived from system (2), which has four distinct eigenvalues from that one is $-x_{P_R}$. Hence, the characteristics polynomial of Jacobian matrix is $C_{P_R}(\lambda) = (x_{P_R}+\lambda)\,Q_{P_R}(\lambda)$, where $Q_{P_R}(\lambda)$ is the characteristics polynomial of resultant matrix:

$$J(P_R) = \begin{bmatrix} -\eta_1-\eta_3T_X-\mu_{P_R} & \eta_2 & -\eta_3N_V \\ \eta_1 & \eta_2-\eta_4T_X-\mu_{P_R} & -\eta_4N_V \\ \eta_3T_X & \eta_4T_X & \eta_3N_V+\eta_4P_V-\mu_{P_R} \end{bmatrix} \tag{10}$$

Therefore, the local stability is calculated using above matrix.

**Theorem 3.1.** *The toxicity free equilibrium point for private vehicle $E_0(P_R)$ is locally asymptotically stable iff $R_0^{P_R} < 1$.*

*Proof.* From (10) at $E_0(P_R)$ and assuming $\eta_1+\mu_{P_R} = a_{11}, \eta_2+\mu_{P_R} = a_{22}, -\eta_3N_{V_0} - \eta_4P_{V_0}+\mu_{P_R}a_{33}$, then Jacobian has the characteristics polynomial
$\lambda_1^3+(a_{11}+a_{22}+a_{33})\lambda_1^2+\left(a_{33}\left(a_{22}+a_{11}\right)+\mu_{P_R}\left(a_{22}+a_{11}-\mu_{P_R}\right)\right)\lambda_1+a_{33}\mu_{P_R}(a_{22}+a_{11}-\mu_{P_R})$
Satisfying Routh-Hurwitz conditions, the toxicity free equilibrium point for private vehicle $E_0(P_R)$ is locally asymptotically stable if $a_{22}+a_{11}-\mu_{P_R} > 0$ (obvious) and $a_{33} > 0$ (if $R_0^{P_R} < 1$). Hence, the theorem. □

**Theorem 3.2.** *The endemic point for private vehicle model $E^*(P_R)$ is locally asymptotically stable.*

*Proof.* From (10) about $E^*(P_R)$ and taking $\eta_1+\eta_3T_X^*+\mu_{P_R} = b_{11}, \eta_2+\eta_4T_X^*+\mu_{P_R} = b_{22}, \eta_3N_V^*+\eta_4P_V^*-\mu_{P_R} = b_{33} = 0$, we have the characteristics polynomial
$\lambda_2^3+(b_{11}+b_{22})\lambda_2^2+(N_V^*T_X^*\eta_3^2+P_V^*T_X^*\eta_4^2+b_{22}b_{11}-\eta_1\eta_2)\lambda_2+(\eta_3N_V^*(b_{22}\eta_3+\eta_1\eta_4)+\eta_4P_V^*(b_{11}\eta_4+\eta_2\eta_3))T_X^*$
Here, from characteristics equation, $E^*(P_R)$ is locally asymptotically stable if $b_{22}b_{11} - \eta_1\eta_2 > 0$ which is obvious because it satisfies the conditions for Routh-Hurwitz criteria. □

3.1.2. *Local stability of public vehicle model.* From the system (4), the Jacobian matrix for public vehicle model is studied. The Jacobian matrix has four distinct eigenvalues. One of them is $-x_{P_B}$. Hence, the characteristics polynomial of Jacobian matrix is $C_{P_B}(\lambda) = (x_{P_B} + \lambda)\, Q_{P_B}(\lambda)$ where $Q_{P_B}$ is the characteristics polynomial of resultant matrix $J(P_B)$:

$$J(P_B) = \begin{bmatrix} -\epsilon_1 - \epsilon_3 T_X - \mu_{P_B} & \epsilon_2 & -\epsilon_3 N_V \\ \epsilon_1 & \epsilon_2 - \epsilon_4 T_X - \mu_{P_B} & -\epsilon_4 P_V \\ \epsilon_3 T_X & \epsilon_4 T_X & \epsilon_3 N_V + \epsilon_4 P_V - \mu_{P_B} \end{bmatrix} \quad (11)$$

Therefore, the local stability is calculated using above matrix.

**Theorem 3.3.** *The toxicity free equilibrium point for private vehicle $E_0(P_B)$ is locally asymptotically stable iff $R_0^{P_B} < 1$.*

*Proof.* Using (11) for $E_0(P_B)$ and assuming $\epsilon_1 + \mu_{P_B} = c_{11}, \epsilon_2 + \mu_{P_B} = c_{22}, -\epsilon_3 N_{V_0} - \epsilon_4 P_{V_0} + \mu_{P_B} = c_{33}$, the characteristics polynomial of the matrix is
$\lambda_3^3 + (c_{11} + c_{22} + c_{33})\lambda_3^2 + (c_{33}(c_{22} + c_{11}) + \mu_{P_B}(c_{22} + c_{11} - \mu_{P_B}))\lambda_3 + c_{33}\mu_{P_B}(c_{22} + c_{11} - \mu_{P_B})$
According to Routh-Hurwitz criteria and its conditions, the toxicity free equilibrium point for public vehicle $E_0(P_R)$ is locally asymptotically stable if $c_{22} + c_{11} - \mu_{P_B} > 0$ (obvious) and $c_{33} > 0$ if (if $R_0^{P_B}$). Hence, the theorem.

□

**Theorem 3.4.** *The endemic point for public vehicle model $E^*(P_B)$ is locally asymptotically stable.*

*Proof.* From (11) at $E^*(P_B)$ and assuming $\epsilon_1 + \epsilon_3 T_X^* + \mu_{P_B} = d_{11}, \epsilon_2 + \epsilon_4 T_X^* + \mu_{P_B} = d_{22}, \epsilon_3 N_V^* + \epsilon_4 P_V^* - \mu_{P_B} = d_{33} = 0$, we get the characteristics polynomial which is
$\lambda_4^3 + (d_{11} + d_{22})\lambda_4^2 + (N_V^* T_X^* \epsilon_3^2 + P_V^* T_X^* \epsilon_4^2 + d_{22}d_{11} - \epsilon_1\epsilon_2)\lambda_4 + (\epsilon_3 N_V^*(d_{22}\epsilon_3 + \epsilon_1\epsilon_4) + \epsilon_4 P_V^*(d_{11}\epsilon_4 + \epsilon_2\epsilon_4))T_X^*$
Here, from characteristics equation, $E^*(P_B)$ is locally asymptotically stable if $d_{22}d_{11} - \epsilon_1\epsilon_2 > 0$ which is obvious according to the conditions of Routh-Hurwitz criteria. □

3.1.3. *Local stability of cargo vehicle model.* The Jacobian matrix for cargo vehicle model is studied from the system (6). This has four distinct eigenvalues. One of them is $-x_{C_V}$. Hence, the characteristics polynomial of Jacobian matrix is $C_{C_V}(\lambda) = (x_{C_V} + \lambda)Q_{C_V}(\lambda)$ where $Q_{C_V}$ is the characteristics polynomial of resultant matrix $J(C_V)$

$$J(C_V) = \begin{bmatrix} -\gamma_1 - \gamma_3 T_X - \mu_{C_V} & \gamma_2 & -\gamma_3 N_V \\ \gamma_1 & -\gamma_2 - \gamma_4 T_X - \mu_{C_V} & -\gamma_4 P_V \\ \gamma_3 T_X & \gamma_4 T_X & \gamma_3 N_V + \gamma_4 P_V \mu_{C_V} \end{bmatrix} \quad (12)$$

Therefore, the local stability is calculated using above matrix.

**Theorem 3.5.** *The toxicity free equilibrium point for private vehicle $E_0(C_V)$ is locally asymptotically stable iff $R_0^{C_V} < 1$.*

*Proof.* Using (12) about $E_0(C_V)$ and assuming $\gamma_1 + \mu_{C_V} = e_{11}, \gamma_2 + \mu_{C_V} = e_{22}, -\gamma_3 N_{V_0} - \gamma_4 P_{V_0} + \mu_{C_V} = e_{33}$. The characteristics polynomial of that matrix is
$\lambda_5^3 + (e_{11} + e_{22} + e_{33})\lambda_5^2 + (e_{33}(e_{22} + e_{11}) + \mu_{C_V}(e_{22} + e_{11} - \mu_{C_V}))\lambda_5 + e_{33}\mu_{C_V}(e_{22} + e_{11} - \mu_{C_V})$
Accepting the Routh-Hurwitz criteria and its conditions, the toxicity free equilibrium point for cargo vehicle $E_0(C_V 0)$ is locally asymptotically stable if $e_{22} + e_{11} - \mu_{C_V}$ (obvious) and $e_{33} >$ (if $R_0^{C_V} < 1$). Hence, the theorem. □

**Theorem 3.6.** *The endemic point for private vehicle model $E^*(C_V)$ is locally asymptotically stable.*

*Proof.* From (12) about $E^*(C_V)$ and taking $\gamma_1 + \gamma_3 T_X^* + \mu_{C_V} = f_{11}, \gamma_2 + \gamma_4 T_X^* + \mu_{C_V} = f_{22}, \gamma_3 N_V^* + \gamma_4 P_V^* - \mu_{C_V} = f_{33} = 0$, we have the characteristics polynomial which is
$\lambda_6^3 + (f_{11} + f_{22})\lambda_6^2 + (N_V^* T_X^* \gamma_3^2 + P_V^* T_X^* \gamma_4^2 + f_{22} f_{11} - \gamma_1\gamma_2)\lambda_6 + (\gamma_3 N_V^*(f_{22}\gamma_3 + \gamma_1\gamma_4) + \gamma_4 P_V^*(f_{11}\gamma_4 + \gamma_2\gamma_4))T_X^*$
Here, Routh-Hurwitz conditions lead that from characteristics equation, $E^*(C_V)$ is locally asymptotically stable if $f_{22}f_{11} - \gamma_1\gamma_2 > 0$ which is obvious. □

3.2. **Global stability.** Global stability around each equilibrium points is calculated in this section. It is intended using Lyapunov function followed by Lasalle's Invariance Principle (La Salle, 1976).

3.2.1. *Global stability of private vehicle model.*

**Theorem 3.7.** *The toxicity free equilibrium point of private vehicle model is globally asymptotically stable.*

*Proof.* Let us consider a Lyapunov function $L_0^{P_R}(t) = N_V(t) + P_V(t) + T_X(t)$, then

$$L_0'^{P_R} = N_V'(t) + P_V'(t) + T_X'(t) = B_{P_R} - \mu_{P_R}(P_R + N_V + P_V) - \mu_{P_R} T_X$$

Now, $P_R \leq \frac{B_{P_R}}{x_{P_R}}, N_V \leq \frac{B_{P_R}(\beta_{P_R}(\eta_2+\mu_{P_R})+\delta_{P_R}\eta_2)}{x_{P_R}\mu_{P_R}(\eta_1+\eta_2+\mu_{P_R})}, P_V \leq \frac{B_{P_R}(\beta_{P_R}\eta_1+\delta_{P_R}(\eta_1+\mu_{P_R}))}{x_{P_R}\mu_{P_R}(\eta_1+\eta_2+\mu_{P_R})}$
Threfore, $P_R + N_V + P_V \leq \frac{B_{P_R}}{\mu_{P_R}}$
Hence, $L_0'^{P_R} \leq -\mu_{P_R} T_X \leq 0$
We get $\frac{dL_0^{P_R}}{dt} \leq 0$ whereas $\frac{dL_0^{P_R}}{dt} = 0$ only if $T_X = 0$.
Hence, using LaSalle's Invariance Principle, we get $E_0^{P_R}$ is globally asymptotically stable. □

**Theorem 3.8.** *The interior equilibrium point $E^*(P_R)$ is globally asymptotically stable.*

*Proof.* Consider $L_{P_R}^*(t) = \frac{1}{2}\left[(P_R - P_R^*) + (N_V - N_V^*) + (P_V - P_V^*) + (T_X - T_X^*)\right]^2$ and then

$$\begin{aligned} L_{P_R}'^*(t) =& \left[(P_R - P_R^*) + (N_V - N_V^*) + (P_V - P_V^*) + (T_X - T_X^*)\right]\left[B_{P_R} - \mu_{P_R}P_R - \mu_{P_R}N_V \right. \\ & \left. - \mu_{P_R}P_V - \mu_{P_R}T_X\right] \\ =& - \mu_{P_R}\left[(P_R - P_R^*) + (N_V - N_V^*) + (P_V - P_V^*) + (T_X - T_X^*)\right]^2 \leq 0 \end{aligned}$$

where $B_{P_R} = \mu_{P_R}P_R^* + \mu_{P_R}N_V^* + \mu_{P_R}P_V^* + \mu_{P_R}T_X^*$
Therefore, the interior point for private vehicle model $E^*(P_R)$ is globally asymptotically stable. □

3.2.2. *Global stability of public vehicle model.*

**Theorem 3.9.** *The toxicity free equilibrium point of public vehicle model is globally asymptotically stable.*

*Proof.* Let us consider a Lyapunov function $L_0^{P_B}(t) = N_V(t) + P_V(t) + T_X(t)$, then

$$L_0'^{P_B} = N_V'(t) + P_V'(t) + T_X'(t) = B_{P_B} - \mu_{P_B}(P_R + N_V + P_V) - \mu_{P_B} T_X$$

Now, $P_B \leq \frac{B_{P_B}}{x_{P_B}}, N_V \leq \frac{B_{P_B}(\beta_{P_B}(\epsilon_2+\mu_{P_B})+\delta_{P_B}\epsilon_2)}{x_{P_B}\mu_{P_B}(\epsilon_1+\epsilon_2+\mu_{P_B})}, P_V \leq \frac{B_{P_B}(\beta_{P_B}\epsilon_1+\delta_{P_B}(\epsilon_1+\mu_{P_B}))}{x_{P_B}\mu_{P_B}(\epsilon_1+\epsilon_2+\mu_{P_B})}$
Threfore, $P_R + N_V + P_V \leq \frac{B_{P_B}}{\mu_{P_B}}$
Hence, $L_0'^{P_B} \leq -\mu_{P_B} T_X \leq 0$
We get $\frac{dL_0^{P_B}}{dt} \leq 0$ whereas $\frac{dL_0^{P_B}}{dt} = 0$ only if $T_X = 0$.

Hence, using LaSalle's Invariance Principle, we get $E_0^{P_B}$ is globally asymptotically stable. □

**Theorem 3.10.** *The interior equilibrium point $E^*(P_B)$ is globally asymptotically stable.*

*Proof.* Consider $L^*_{P_B}(t) = \frac{1}{2}\left[(P_B - P^*_B) + (N_V - N^*_V) + (P_V - P^*_V) + (T_X - T^*_X)\right]^2$ and then

$$\begin{aligned} L'^*_{P_B}(t) =& \left[(P_B - P^*_B) + (N_V - N^*_V) + (P_V - P^*_V) + (T_X - T^*_X)\right]\left[B_{P_B} - \mu_{P_B} P_B - \mu_{P_B} N_V \right. \\ & \left. - \mu_{P_B} P_V - \mu_{P_B} T_X\right] \\ =& - \mu_{P_B}\left[(P_B - P^*_B) + (N_V - N^*_V) + (P_V - P^*_V) + (T_X - T^*_X)\right]^2 \leq 0 \end{aligned}$$

where $B_{P_B} = \mu_{P_B} P^*_B + \mu_{P_B} N^*_V + \mu_{P_B} P^*_V + \mu_{P_B} T^*_X$
Therefore, the interior point for private vehicle model $E^*(P_B)$ is globally asymptotically stable. □

3.2.3. *Global stability of cargo vehicle model.*

**Theorem 3.11.** *The toxicity free equilibrium point of cargo vehicle model is globally asymptotically stable.*

*Proof.* Let us consider a Lyapunov function $L_0^{C_V}(t) = N_V(t) + P_V(t) + T_X(t)$, then

$$L_0'^{C_V} = N'_V(t) + P'_V(t) + T'_X(t) = B_{C_V} - \mu_{C_V}(P_R + N_V + P_V) - \mu_{C_V} T_X$$

Now, $C_V \leq \frac{B_{C_V}}{x_{C_V}}, N_V \leq \frac{B_{C_V}(\beta_{C_V}(\gamma_2+\mu_{C_V})+\delta_{C_V}\gamma_2)}{x_{C_V}\mu_{C_V}(\gamma_1+\gamma_2+\mu_{C_V})}, P_V \leq \frac{B_{C_V}(\beta_{C_V}\gamma_1+\delta_{C_V}(\gamma_1+\mu_{C_V}))}{x_{C_V}\mu_{C_V}(\gamma_1+\gamma_2+\mu_{C_V})}$
Threfore, $C_V + N_V + P_V \leq \frac{B_{C_V}}{\mu_{C_V}}$
Hence, $L_0'^{C_V} \leq -\mu_{C_V} T_X \leq 0$
We get $\frac{dL_0^{C_V}}{dt} \leq 0$ whereas $\frac{dL_0^{C_V}}{dt} = 0$ only if $T_X = 0$.
Hence, using LaSalle's Invariance Principle, we get $E_0^{C_V}$ is globally asymptotically stable. □

**Theorem 3.12.** *The interior equilibrium point $E^*(C_V)$ is globally asymptotically stable.*

*Proof.* Consider $L^*_{C_V}(t) = \frac{1}{2}\left[(C_V - C^*_V) + (N_V - N^*_V) + (P_V - P^*_V) + (T_X - T^*_X)\right]^2$ and then

$$\begin{aligned} L'^*_{C_V}(t) =& \left[(C_V - C^*_V) + (N_V - N^*_V) + (P_V - P^*_V) + (T_X - T^*_X)\right]\left[B_{C_V} - \mu_{C_V} P_B - \mu_{C_V} N_V \right. \\ & \left. - \mu_{C_V} P_V - \mu_{C_V} T_X\right] \\ =& - \mu_{C_V}\left[(C_V - C^*_V) + (N_V - N^*_V) + (P_V - P^*_V) + (T_X - T^*_X)\right]^2 \leq 0 \end{aligned}$$

where $B_{C_V} = \mu_{C_V} C^*_V + \mu_{C_V} N^*_V + \mu_{C_V} P^*_V + \mu_{C_V} T^*_X$
Therefore, the interior point for private vehicle model $E^*(C_V)$ is globally asymptotically stable. □

## 4. Bifurcation analysis

In this section, backward bifurcation is analysed for non-polluted vehicles (Khan et al. (2014), Wangari et al. (2016)).

4.1. **Bifurcation analysis for private vehicle.** The backward bifurcation exists when non-polluted vehicles are at least non-zero. On solving system (2) for $N_V^*$, we have

$$f_1(N_V^*) = A_1 N_V^{*2} + B_1 N_V^* + C_1 = 0 \tag{13}$$

where $A_1 = (\eta_3 - \eta_4)\eta_3 \mu_{P_R} x_{P_R}$
$B_1 = (\beta_{P_R} + \delta_{P_R}) B_{P_R} \eta_3 \eta_4 + ((\eta_1 + \mu_{P_R})\eta_4 \mu_{P_R} + (\eta_2 - \mu_{P_R})\eta_3 \mu_{P_R}) x_{P_R}$
$C_1 = \left( \frac{\mu_{P_R}^3 (\eta_2 \mu_{P_R} x_{P_R} + \eta_4 B_{P_R} \beta_{P_R})(\eta_1 + \eta_2 + \mu_{P_R}) x_{P_R}}{(((\eta_2 + \mu_{P_R})\eta_3 + \eta_1 \eta_4)\beta_{P_R}(\eta_2\eta_3 + (\eta_1 + \mu_{P_R})\eta_4)\delta_{P_R}) B_{P_R} - \mu_{P_R}^2 (\eta_1 + \eta_2 + \mu_{P_R}) x_{P_R}} \right) \left(1 - R_0^{P_R}\right)$

The coefficient $A_1$ must be always positive and $C_1$ should depend upon the value of $R_0^{P_R}$, if $R_0^{P_R} < 1$ then $C_1$ is positive and if $R_0^{P_R} < 1$ then $C_1$ is negative. For $A_1 > 0$, the positive result depends upon the sign of $B_1$ and $C_1$. The equation (13) have two roots; from that one is positive and other is negative for $R_0^{P_R} > 1$. Now, if $R_0^{P_R} = 1$ then $C_1 = 0$ and we obtain a non-zero solution of equation (13) as $\frac{-B_1}{A_1}$ which is positive if and only if $B_1 < 1$. For $B_1 < 1$, there exists a positive interior equilibrium point for $R_0^{P_R} = 1$ that means the equilibria continuously depends upon $R_0^{P_R}$, indicating that there exists an interval for $R_0^{P_R}$ which have two positive equilibria $I_1^{P_R} = \frac{-B_1 - \sqrt{B_1^2 - 4A_1C_1}}{2A_1}$, $I_2^{P_R} = \frac{-B_1 + \sqrt{B_1^2 - 4A_1C_1}}{2A_1}$. For, backward bifurcation putting the discriminant $B_1^2 - 4A_1C_1 = 0$ and then solving for the critical points of $R_0^{P_R}$ gives $R_C^{P_R} = 1 - \frac{B_1^2((((\eta_2 + \mu_{P_R})\eta_3 + \eta_1\eta_4)\beta_{P_R} + (\eta_2\eta_3 + (\eta_1 + \mu_{P_R})\eta_4)\delta_{P_R}) B_{P_R}) - \mu_{P_R}^2 (\eta_1 + \eta_2 + \mu_{P_R}) x_{P_R}}{4A_1 \mu_{P_R}^3 x_{P_R} (\eta_2 \mu_{P_R} x_{P_R} + \eta_4 B_{P_R} \beta_{P_R})(\eta_1 + \eta_2 + \mu_{P_R})}$. If $R_C^{P_R} < R_0^{P_R}$ then $B_1^2 - 4A_1C_1 > 0$ and for the point of $R_0^{P_R}$ backward bifurcation exists such that $R_C^{P_R} < R_0^{P_R} < 1$.

4.2. **Bifurcation analysis for public vehicle.** To find backward bifurcation, we take non-polluted vehicles non-zero. Let us solve system (4) for $N_V^*$ then we get

$$f_2(N_V^*) = A_2 N_V^{*2} + B_2 N_V^* + C_2 = 0 \tag{14}$$

where $A_2 = (\epsilon_3 - \epsilon_4)\epsilon_3 \mu_{P_B} x_{P_B}$
$B_2 = (\beta_{P_B} + \delta_{P_B}) B_{P_B} \epsilon_3 \epsilon_4 + ((\epsilon_1 + \mu_{P_B})\epsilon_4 \mu_{P_B} + (\epsilon_2 - \mu_{P_B})\epsilon_3 \mu_{P_B}) x_{P_B}$
$C_2 = \left( \frac{\mu_{P_B}^3 (\epsilon_2 \mu_{P_B} x_{P_B} + \epsilon_4 B_{P_B} \beta_{P_B})(\epsilon_1 + \epsilon_2 + \mu_{P_B}) x_{P_B}}{(((\epsilon_2 + \mu_{P_B})\epsilon_3 + \epsilon_1 \epsilon_4)\beta_{P_B}(\epsilon_2\epsilon_3 + (\epsilon_1 + \mu_{P_B})\epsilon_4)\delta_{P_B}) B_{P_B} - \mu_{P_B}^2 (\epsilon_1 + \epsilon_2 + \mu_{P_B}) x_{P_B}} \right) \left(1 - R_0^{P_B}\right)$

Using the cocept of section 4.1, we have
$R_C^{P_B} = 1 - \frac{B_2^2((((\epsilon_2 + \mu_{P_B})\epsilon_3 + \epsilon_1\epsilon_4)\beta_{P_B} + (\epsilon_2\epsilon_3 + (\epsilon_1 + \mu_{P_B})\epsilon_4)\delta_{P_B}) B_{P_B}) - \mu_{P_B}^2 (\epsilon_1 + \epsilon_2 + \mu_{P_B}) x_{P_B}}{4A_2 \mu_{P_B}^3 x_{P_B} (\epsilon_2 \mu_{P_B} x_{P_B} + \epsilon_4 B_{P_B} \beta_{P_B})(\epsilon_1 + \epsilon_2 + \mu_{P_B})}$. If $R_C^{P_B} < R_0^{P_B}$ then $B_2^2 - 4A_2C_2 > 0$ and for the point of $R_0^{P_B}$ backward bifurcation exists such that $R_C^{P_B} < R_0^{P_B} < 1$.

4.3. **Bifurcation analysis for cargo vehicle.** For cargo vehicle model backward bifurcation exists when compartment of non-polluted vehicles is non-zero. Let us solve system (6) for $N_V^*$ then we get

$$f_3(N_V^*) = A_3 N_V^{*2} + B_3 N_V^* + C_3 = 0 \tag{15}$$

where $A_3 = (\gamma_3 - \gamma_4)\gamma_3 \mu_{C_V} x_{C_V}$
$B_3 = (\beta_{C_V} + \delta_{C_V}) B_{C_V} \gamma_3 \gamma_4 + ((\gamma_1 + \mu_{C_V})\gamma_4 \mu_{C_V} + (\gamma_2 - \mu_{C_V})\gamma_3 \mu_{C_V}) x_{C_V}$
$C_3 = \left( \frac{\mu_{C_V}^3 (\gamma_2 \mu_{C_V} x_{C_V} + \gamma_4 B_{C_V} \beta_{C_V})(\gamma_1 + \gamma_2 + \mu_{C_V}) x_{C_V}}{(((\gamma_2 + \mu_{C_V})\gamma_3 + \gamma_1 \gamma_4)\beta_{C_V}(\gamma_2\gamma_3 + (\gamma_1 + \mu_{C_V})\gamma_4)\delta_{C_V}) B_{C_V} - \mu_{C_V}^2 (\gamma_1 + \gamma_2 + \mu_{C_V}) x_{C_V}} \right) \left(1 - R_0^{C_V}\right)$

Here, the cocept of section 4.1 is applied which results

$R_C^{C_V} = 1 - \frac{B_3^2((((\gamma_2+\mu_{C_V})\gamma_3+\gamma_1\gamma_4)\beta_{C_V}+(\gamma_2\gamma_3+(\gamma_1+\mu_{C_V})\gamma_4)\delta_{C_V})B_{C_V})-\mu_{C_V}^2(\gamma_1+\gamma_2+\mu_{C_V})x_{C_V}}{4A_3\mu_{C_V}^3 x_{C_V}(\gamma_2\mu_{C_V}x_{C_V}+\gamma_4 B_{C_V}\beta_{C_V})(\gamma_1+\gamma_2+\mu_{C_V})}$. If $R_C^{C_V} < R_0^{C_V}$ then $B_3^2 - 4A_3C_3 > 0$ and for the point of $R_0^{C_V}$ backward bifurcation exists such that $R_C^{C_V} < R_0^{C_V} < 1$.

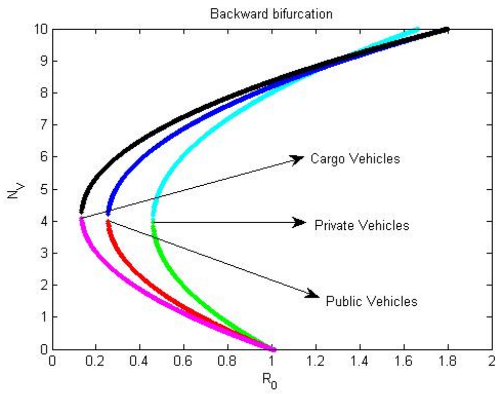


FIGURE 4. Backward bifurcation

The backward bifurcation interprets that cargo vehicles are at low risk.

## 5. Numerical simulation

In this section, numerical analysis is carried out to support the model results and to compare the spread of toxicity through private vehicles, public vehicles and cargo vehicles.

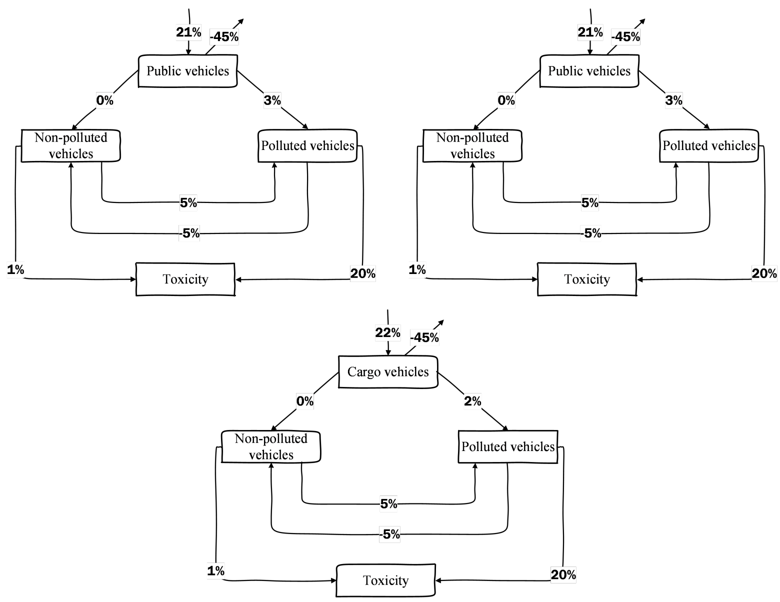


FIGURE 5. Simulation results from the model

Some observations are made from the simulations as shown below:

(1) Need for public and cargo vehicle is more in comparison to private vehicles.

(2) Private vehicle can be non-polluted but public and cargo vehicles cannot be non-polluted because of their fuel category.
(3) Private and cargo vehicles are less polluted as compared to public vehicles. Because cargo vehicles move on highways and its entry to the city are prohibited in day time.
(4) Statistics shows that there is an excessive possibility to become polluted private vehicle from non-polluted ones. This is a result of individuals' tendency to live a comfortable and luxurious life.
(5) There are more chances of becoming non-polluted private vehicles than that of public and cargo vehicles. The transfer rate of non-polluted into polluted vehicles is equivalent in the case of public and cargo vehicles. This happens because both these types of vehicles run on diesel.
(6) Toxicity is spreading with the uniform rate for public and cargo vehicles and it is higher than that of public vehicles.
(7) The scrapped rate of public and cargo vehicle is low/negligible which creates high toxicity compared to private vehicles.

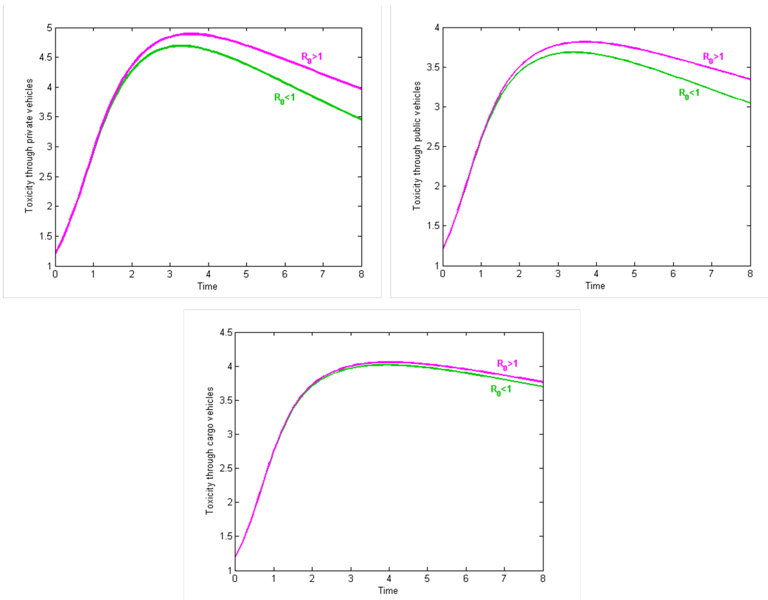


FIGURE 6. Concentration of toxicity with respect to growth rate

Figures propose that when the growth rate is increased by 10% then the significant change is observed in toxicity. For private, public and cargo vehicles, toxicity is increased by 14.94%, 9.82% and 1.89%, respectively. This suggests private vehicles are responsible for more toxicity because of its density. Also, this analysis indicates that there is a dire need to curb the growth rate of each vehicle category.

## 6. CONCLUSIONS

In this paper, three consequent models are compared for toxic commuters. Here, private, public and cargo vehicles are assumed as toxic commuters. The models are developed to compare the dispersion of toxicity through these vehicles. The threshold quantity is computed to calculate the density of vehicles affecting toxicity. The observed threshold for private, public and cargo vehicle is 0.2364, 0.3255 and 0.5443, respectively. This says that only 23.64% non-polluted private vehicles exist which is les than among the density of public and cargo vehicles. On the threshold quantity, backward bifurcation is dependent. The critical threshold is observed for three of the models. The critical threshold is 0.46, 0.255 and 0.134 for private, public and cargo vehicle. This suggest that cargo vehicles are at low risk for polluting environment. The stability of the models suggest that the ratio of newly infected toxic air pollutants created by vehicular pollution should be less than 1. This research advocates that the growth rate of each types of vehicles should be under controlled, especially for private vehicles. Vehicles should be run on that fuel or with the technology which is less harmful to the environment. To support this idea, advanced technology should be adopted. With the advanced technology, electric public transport and trucks are made just like electric private transport.

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