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THE BOUNDS FOR THE LARGEST EIGENVALUES OF FIBONACCI-SUM AND LUCAS-SUM GRAPHS

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Abstract. In this paper, we first get the degree of each point in Lucas-sum graph based on Lucas numbers. After that, we obtain lower and upper bounds for the largest eigenvalues λ and μ of the adjacency matrices of Fibonacci-sum and Lucas-sum graphs, respectively.

Keywords: Fibonacci-sum graph, Lucas-sum graph, bounds, vertex degree.

AMS Subject Classification: 05C07, 05C50, 11B39

1. Introduction

For each $n \geq 1$, a graph G is a pair of sets (V, E), where $V = \{1, 2, \ldots, n\}$ is the vertex set and $E = \{ij : i, j \in V\}$ is the edge set . In any graph G for the vertices $i, j \in V$, if ij is an edge then, i and j vertices are called as adjacent vertices and indicated by $i \sim j$. In graph theory, the number of edges that are incident to i th vertex is called the degree of i th vertex and denoted by $d(i)$.

For any simple graph G with n vertices, the adjacency matrix of G is $A(G) = (a_{ij})_{n\times n}$ and the elements of this matrix are defined as

$$a_{ij} = \begin{cases} 1; & if\ i \sim j \\ 0; & otherwise. \end{cases}$$

Definition 1.1. [2] *The integer sequence* $\{F_n\}_{n=0}^{\infty}$ *with initial values* $F_0 = 0$, $F_1 = 1$ *and the recurrence relation* $F_n = F_{n-1} + F_{n-2}$ *is called Fibonacci sequence.*

Definition 1.2. [2] *The integer sequence* $\{L_n\}_{n=0}^{\infty}$ *with initial values* $L_0 = 2$, $L_1 = 1$ *and the recurrence relation* $L_n = L_{n-1} + L_{n-2}$ *is called Lucas sequence.*

Definition 1.3. [5] *For each* $n \geq 1$, *the graph* $G_n = (V, E)$ *is defined as Fibonacci-sum graph with the vertex set* $V = \{1, 2, \ldots, n\}$ *and the edge set* $E = \{ij : i, j \in V, i \neq j, i + j$ *is a Fibonacci number*$\}$.

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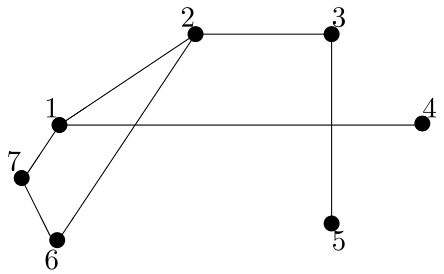
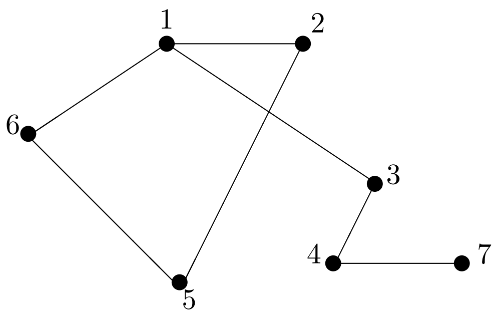
Example 1.1. *A Fibonacci-sum graph for $n = 7$ is as follows*



Definition 1.4. [3] *For each $n \geq 1$, the graph $H_n = (V, E)$ is defined as Lucas-sum graph with the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{ij : i, j \in V, i \neq j, i + j$ is a Lucas number$\}$.*

Example 1.2. *A Lucas-sum graph for $n = 7$ is as follows*



Lemma 1.1. [4] *Let A be an $n-$square nonnegative matrix. Then $\min R_i \leq \lambda(A) \leq \max R_i$. Here $R_i = \sum_{j=1}^{n} a_{ij}$ (i th row sum).*

In other words, the spectral radius of a nonnegative square matrix is between the smallest row sum and the largest row sum.

Adjacency matrix $A(G)$ of a graph G is a symmetric matrix and the sum of i.th row/column is the degree of i th vertex. It means that $R_i = d(i)$.

Adjacency matrix $A(G)$ of a Fibonacci-sum graph for $n = 21$ is as follows

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1		1		1			1					1								1	
2	1		1			1					1								1		
3		1			1					1								1			
4	1			0					1								1				
5			1					1								1					
6		1					1								1						
7	1					1								1							
8					1								1								
9				1								1									
10			1								1										
11		1								1											
12	1								1												
13								1													1
14							1													1	
15						1													1		
16					1													1			
17				1													0				
18			1													1					
19		1													1						
20	1													1							
21													1								

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Theorem 1.1. [1] *Let $n \geq 1$ and $x \in [1, n]$. Let $k \geq 2$ satisfy $F_k \leq x \leq F_{k+1}$ and $l \geq k$ satisfy $F_l \leq x + n \leq F_{l+1}$. Then the degree of x in Fibonacci-sum graph G_n is*

$$\deg_{G_n}(x) = \begin{cases} l-k; & \text{if } 2x \text{ is not a Fibonacci number} \\ l-k-1; & \text{if } 2x \text{ is a Fibonacci number.} \end{cases}$$

2. The bounds for the largest eigenvalues of Fibonacci-sum and Lucas-sum graphs

Theorem 2.1. *If $\lambda(A(G_n))$ is the largest eigenvalue of the adjacency matrix of a Fibonacci-sum graph G_n with n vertices then, $1 \leq \lambda(A(G_n)) \leq l-3$. Here, l satisfy $F_l \leq 2+n \leq F_{l+1}$.*

Proof. In Fibonacci-sum graph G_n, the degree of at least one point which is the first Fibonacci number less than n is 1. It means that, $\min R_i = 1$ in $A(G_n)$. Also, the maximum row sum is $\deg_{G_n}(2)$. By Theorem 1.1, $\deg_{G_n}(2) = l-k$ is not a Fibonacci number by the reason of $2 \times 2 = 4$. Since $F_k = 2 \leq 2 \leq F_{k+1} = 3$ then, we get $k = 3$. The value of l changes according to n. For any l satisfy $F_l \leq 2+n \leq F_{l+1}$, we get $1 \leq \lambda \leq l-3$. □

Theorem 2.2. *Let $n \geq 1$ and $x \in [1, n]$. Let $k \geq 2$ and $L_k \leq x \leq L_{k+1}$ satisfy $L_t \leq x+n \leq L_{t+1}$. Then the degree of x in Lucas-sum graph H_n is*

$$\deg_{H_n}(x) = \begin{cases} t-k-1; & \textit{if } 2x \textit{ is a Lucas number} \\ t-k; & \textit{if } 2x \textit{ is not a Lucas number.} \end{cases}$$

Proof. For any $s \in [1, n]$ we have $L_k < x + s \le x + n < L_{t+1}$. If $x + s$ is a Lucas number then, $k < t$ and $x + s \in \{L_{k+1}, ..., L_t\}$. So we have,

$$\deg_{H_n}(x) = |\{s \in [1, n] : s \neq x, x + s \in \{L_{k+1}, ..., L_t\}\}| .$$

Then, $\deg_{H_n}(x) \le t - k$. If $s = x$ then, $2x$ is a Lucas number. Since H_n has no loop then $x + x$ does not give an edge. By the way, $\deg_{H_n}(x) \le t - k - 1$. On the other hand, if $2x$ is not a Lucas number, we have $\deg_{H_n}(x) \le t - k$. □

Theorem 2.3. *If $\mu(A(H_n))$ is the largest eigenvalue of the adjacency matrix of a Lucas-sum graph H_n with n vertices then, $1 \le \mu(A(H_n)) \le t-1$. Here, t satisfy $L_t \le 2+n \le L_{t+1}$.*

Proof. In Lucas-sum graph H_n, the degree of the point which is the first Lucas number less than n, is 1. It means that, $\min R_i = 1$ in $A(H_n)$. Also, the maximum row sum is $\deg_{H_n}(2)$. By Theorem 2.2, $\deg_{H_n}(2) = t - k - 1$ is a Lucas number by the reason of $2 \times 2 = 4$. Since $L_k = 1 \le 2 \le L_{k+1} = 3$, we get $k = 1$. The value of t changes according to n. For any t satisfy $L_t \le 2 + n \le L_{t+1}$, we get $1 \le \mu \le t - 1$. □

3. Conclusion

In this work, we obtain the degree of each vertex of Lucas-sum graph. Also, we give upper and lower bounds for the spectral radius of Fibonacci-sum and Lucas-sum graphs using the degree of these graph vertices.

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