TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 768-785

DETERMINATION OF A NONLINEAR SOURCE TERM IN A REACTION-DIFFUSION EQUATION BY USING FINITE ELEMENT METHOD AND RADIAL BASIS FUNCTIONS METHOD

H. ZEIDABADI[1], R. POURGHOLI[1], A. HOSSEINI[1], §

Abstract. In this paper, two numerical methods are presented to solve a nonlinear inverse parabolic problem of determining the unknown reaction term in the scalar reaction-diffusion equation. In the first method, the finite element method will be used to discretize the variational form of the problem and in the second method, we use the radial basis functions (RBFs) method for spatial discretization and finite-difference for time discretization. Usually, the matrices obtained from the discretization of the equations are ill-conditioned, especially in higher-dimensional problems. To overcome such difficulties, we use Tikhonov regularization method. In fact, this work considers a comparative study between the finite element method and radial basis functions method. As we will see, these methods are very useful and convenient tools for approximation problems and they are stable with respect to small perturbation in the input data. The effectiveness of the proposed methods are illustrated by numerical examples.

Keywords: Nonlinear inverse problem, Parabolic equations, Finite element method, Radial basis functions method, Least square method, Tikhonov regularization method, Stability analysis.

AMS Subject Classification: 65M32, 35K05.

1. Introduction

The inverse problem presents an interesting challenge in many areas of engineering and sciences. This concept has used widespread acceptance in applied mathematics. These problems appear in many significant scientific and technological fields and play a very important role, such as resources exploration, aerospace engineering, atmosphere measure, ocean engineering, quantum mechanics and etc [1]. Hence, analysis, design implementation and testing of inverse algorithms are also the great scientific and technological interest.

---

[1] School of Mathematics and Computer Science, Damghan University, P.O. Box 36715-364, Damghan, Iran.
e-mail: h.zeidabadi@yahoo.com; ORCID: http://orcid.org/0000-0001-5361-0396, corresponding author.
e-mail: pourgholi@du.ac.ir; ORCID: http://orcid.org/0000-0003-4111-5130.
e-mail: abbas.hosseini.k@gmail.com; ORCID: http://orcid.org/0000-0002-7366-7522.
§ Manuscript received: October 31, 2017; accepted: January 26, 2018.

In the process of transportation, diffusion, and conduction of natural materials, the following heat equation is induced:

$$u_t = a^2 \Delta u + F(x,t;u), \qquad (x,t) \in \Omega \times [0,T], \tag{1}$$

where a is the diffusion coefficient, Ω is a bounded domain in $\mathbb{R}^d$ and F denotes physical laws. In the context of heat conduction and diffusion when u represents temperature and concentration the unknown function $F(u)$ is interpreted as a heat and material source, respectively, while in a chemical or biochemical application F may be interpreted as a reaction term. Although the results in this paper apply to each of these interpretations.

There are many researches on such an inverse source problem from the 1970s [2–14]. Li et al [15] investigate a stability for the nonlinear source term's inversion. Isakov [16] discussed stability problem, and obtained some wonderful results for linear source term's inversion of parabolic equations. But for nonlinear source terms, there are fewer researches in the literatures we have. In 1982, Lorenzi [8] proved a stability of $W_\infty^\delta(\delta < 1/2)$ for nonlinear source $F(u)$, and this still seem a better result for inverse problems of a nonlinear source term.

In general, these problems belong to the class of problems called the ill-posed problems, i.e. small error in one's measurement may lead to big variation in the model determination. As a consequence, their solution does not satisfy the general requirement of existence, uniqueness and stability under small changes to the input data. Thus, due to importance, a variety of techniques for solving these problems have been proposed, where have been resulted from mathematical fields such as partial differential equations, numerical analysis, harmonic analysis, functional analysis, Fourier analysis and etc. Among the most versatile methods, the followings can be mentioned: Tikhonov regularization [17, 18], iterative regularization [19], mollification [20], BFM (Base Function Method) [21], SFDM (Semi Finite Difference Method) [22] and the FSM (Function Specification Method) [23].

The finite element method (FEM) is known as very powerful tool for solving differential equation which was first introduced by Courant in 1943 [24]. This method is applied for solving linear and nonlinear problems by many researchers. Milos Zlamal [25] used FEMs for nonlinear parabolic equations. In [26], El-Azab and Abdelgaber obtained finite element solution of nonlinear diffusion problems. Volker John and Ellen Schmeyer [27] applied this method for time-dependent convection-diffusion-reaction equations with small diffusion. Wolfgang Bangerth [28] applied adaptive FEMs for nonlinear inverse problems. Larisa Beilina and Johnson [29] used a hybrid finite element/difference method for an inverse scattering problem. Larisa Beilina [30] applied adaptive finite element/difference method for inverse elastic scattering waves. In [31], Xianwu Ling used a non-iterative FEM for inverse heat conduction problems.

In the last decays, the radial basis functions method is one of the most often applied meshless methods in modern approximation theory. Since Kansa [32,33] extended Hardy's MQ [34] and proposed radial basis functions method (RBFs) to solve partial differential equations (PDEs) the approach has been applied to solve many different problems, such as nonlinear Burgers equation [35] with shockwave, shallow water equations for tide and currents simulation [36], heat transfer problems [37]. Fasshauer [38] implemented this method with MATLAB.

Radial basis functions are used actively for solving inverse problems. Hon and Wu [39] utilized Hermite-Birkhoff collocation method in which the RBFs were replaced by harmonic functions with shift invariability to determine an unknown boundary. Li [40, 41] used the conical-type RBFs to solve the inverse boundary value problems of the elliptic equation and biharmonic equation. Cheng and Cabral [42] choose the inverse multiquadric

(IMQ) as the RBFs to solve several types of ill-posed inverse boundary problems for the Laplace equation. Ma and Wu [43] used radial basis functions method to solve semi linear inverse problem.

It is worth mentioning that the shape parameter in radial basis functions plays an important role in the accuracy of the method. Hence, selecting a suitable shape parameter is a challengeable issue. Bayona et al. [44] obtained the optimal value of the constant parameter which is called Optimal Variable Shape Parameter (OVSP) for some problems, but the optimum selection of the constant shape parameter is still an open problem. For more descriptions see [45] and references therein.

The rest of the paper is organized as follows. Section 2 is devoted to formulate inverse problem. The variational formulation, discretization of inverse problem and solving it by finite element method are proposed in Section 3. Furthermore regularizing the resultant ill-conditioned linear system of equations, the least square minimization technique and Tikhonov regularization are also illustrated in Section 3. Section 4 introduce the RBFs method and present the numerical scheme for solving inverse problem. In Section 5, we prove the stability of the finite element method. To illustrate the effectiveness and compare of the presented methods, Section 6 gives some examples with analytical solution. Section 7 ends this paper with a brief conclusion.

2. Inverse Scalar Reaction-Diffusion Problem

Here we consider a simple model equation, that is a scalar reaction-diffusion equation of one-space dimension. Suppose $u(x,t)$ satisfies

$$u_t(x,t) = u_{xx}(x,t) + F(u(x,t)), \qquad 0 < x < 1, \quad 0 < t < T, \tag{2a}$$
$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{2b}$$
$$u(0,t) = g_0(t), \qquad 0 \le t \le T, \tag{2c}$$
$$u(1,t) = g_1(t), \qquad 0 \le t \le T, \tag{2d}$$

and the overspecified data

$$u(x^*,t) = g_s(t), \qquad 0 < x^* < 1, \quad 0 \le t \le T, \tag{2e}$$

where T represents the final time, $\omega = \{(x,t) : x \in [0,1] = \Omega,\, t \in [0,T = J\}$ and f, g_0, g_1 and g_s are given continuous functions. The reaction term $F(u(x,t))$ is unknown and is, in fact, to be determined from the overspecified data. We seek both the functions $u(x,t)$ and $F(u(x,t))$.

It is assumed that $F(u)$ is smooth on $\mathbb{R}$ and there exist the constant $M \in \mathbb{R}$ such that

$$|F'(u)| \le M, \ \ for \ \ u \in \mathbb{R} \tag{3}$$

and

$$F(0) = F(1) = 0. \tag{4}$$

For an unknown reaction term $F(u)$ we must therefore provide additional information $(2e)$ to provide a unique solution $(u, F(u))$ to the inverse problem (2), [3, 5, 46].

In this paper the unknown function $F(u)$ is approximated as

$$\tilde{F}(u) = u(a_1 + a_2 u + a_3 u^2 + a_4 u^3 + \cdots + a_q u^{q-1}), \tag{5}$$

where $\{a_1, a_2, \ldots, a_q\}$ are constants which remain to be determined [47].

3. Implementation of Finite Element Method

First, we convert the boundary conditions such that to be homogeneous. For this purpose we use $w(x,t) = u(x,t) + A(t) * x + B(t)$. Then we have the homogeneous form of Eq. (2) as follows:

$$w_t(x,t) = w_{xx}(x,t) + F_1(x,t,w(x,t)), \qquad 0 < x < 1, \quad 0 < t < T, \tag{6a}$$

$$w(x,0) = f^*(x), \qquad 0 \le x \le 1, \tag{6b}$$

$$w(0,t) = 0, \qquad 0 \le t \le T \tag{6c}$$

$$w(1,t) = 0, \qquad 0 \le t \le T, \tag{6d}$$

and the overspecified data

$$w(x^*,t) = g_s^*(t), \qquad 0 < x^* < 1, \quad 0 \le t \le T, \tag{6e}$$

Let $V = H_0^1(\Omega)$. The variational formulation of problem (6) can be obtained as [48],

$$\begin{aligned} (w_t, v) + (w_x, v_x) &= (F_1(w), v), \qquad \forall v \in V, \\ w(x,0) &= f^*, \end{aligned} \tag{7}$$

where

$$(w_x, v_x) = \int_0^1 \frac{\partial w}{\partial x}\frac{\partial v}{\partial x}dx, \quad (w_t, v) = \int_0^1 \frac{\partial w}{\partial t} v dx. \tag{8}$$

Since V is a infinite dimensional space, we choose a subspace of V with finite dimension and call it V_h. So the problem is converted to find $w_h \in V_h$ such that

$$\begin{aligned} (w_{h,t}, v) + (w_{h,x}, v_x) &= (F_1(w_h), v), \qquad \forall v \in V_h, \\ w_h(0) &= f_h^*, \end{aligned} \tag{9}$$

where $f_h^* \in V_h$ is some approximation of f^*. We consider a set of basis continuous piecewise polynomials functions such as $\{\phi_i\}_{i=1}^M$, where $V_h = span\{\phi_1, \phi_2, \dots, \phi_M\}$. We choose M nodes in interval $\Omega = [0,1]$ and denote these nodes by $\{x_1, x_2, \dots, x_M\}$. Corresponding to each node, we construct a basis function, such that satisfies the following properties

i) $\phi_i(x_j) = \delta_{ij}, \qquad i,j = 1,2,\dots,M,$

ii) $\phi_i|_{\Omega_e} = \psi_i^{(e)}, \qquad \psi_i^{(e)}(x_j) = \delta_{ij}, \qquad i,j = 1,2,\dots,M,$

where $\psi_i^{(e)}$ are called local functions.

Representing the solution by $w_h = \sum_{j=1}^M \alpha_j(t)\phi_j(x)$. For $i = 1,\dots,M$, we take $v = \phi_i(x)$, hence by substituting w_h and v in variational formulation (9)

$$\sum_{j=1}^M \dot{\alpha}_j(t)(\phi_j,\phi_i) + \sum_{j=1}^M \alpha_j(t)(\frac{\partial \phi_j}{\partial x}, \frac{\partial \phi_i}{\partial x}) = (F_1(x,t,\sum_{l=1}^M \alpha_l(t)\phi_l), \phi_i), \tag{10}$$

where the notation " $\dot{}$ " means differentiation with respect to t. This yields the following system for the weight $\alpha(t)$

$$B\dot{\alpha} + A\alpha = F_1(\alpha), \quad for\ t > 0, \quad with\ \ \alpha(0) = \gamma, \tag{11}$$

where γ is the vector of nodal values of f_h^*,

$$\begin{aligned} B &= (b_{ji}), & b_{ji} &= (\phi_j, \phi_i), \\ A &= (a_{ji}), & a_{ji} &= (\frac{\partial \phi_j}{\partial x}, \frac{\partial \phi_i}{\partial x}), \\ F_1(\alpha) &= (F_{1,i}(\alpha)), & F_{1,i}(\alpha) &= (F_1(x,t,\sum_{l=1}^{M} \alpha_l(t)\phi_l), \phi_i), \end{aligned} \tag{12}$$

and $\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_M(t))^T$, where "$T$" means transpose, is the vector of unknown functions $\alpha_j(t)$, $j = 1, \dots, M$.

By solving the system (11), the coefficients α_j are obtained, and with these coefficients, the approximate solution can be obtained.

Now, let us divide the interval $[0, T]$ into N equal parts of length $\Delta t = \tau = \frac{T}{N}$ and denote $t_n = n\Delta t$, $n = 0, 1, 2, \dots, N$, and W^n be the approximation of exact solution $w(t)$ at $t = t_n$. The nonlinear system (11) can be linearized by allowing the nonlinearities to lag one time step behind. Thus the backward Euler method takes the form

$$(\frac{\bar{\partial} W^n}{\partial t}, v) + (\frac{\partial \widehat{W}^n}{\partial x}, \frac{\partial v}{\partial x}) = (F_1(W^{n-1}), v), \qquad \forall v \in V_h, \tag{13}$$

where $W^0 = \phi_h$, $\frac{\bar{\partial} W^n}{\partial t} = \frac{(W^n - W^{n-1})}{k}$ and $\widehat{W}^n = \frac{W^n + W^{n-1}}{2}$. Expressing W^n in terms of the basis functions as $W^n(x) = \sum_{j=1}^{M} \alpha_j^n \phi_j(x)$, with B and A as before, furthermore we use a polynomial approximation for function F_1 as (5), this equation may be written

$$(B + \frac{k}{2}A)\alpha^n = (B - \frac{k}{2}A)\alpha^{n-1} + k\widehat{F}(\alpha^{n-1}), \qquad for \;\; t_n \in J. \tag{14}$$

It is must be mentioned that, in each time step α^n are achieved based on unknown coefficients a_i. To obtain the unknown coefficients $a_i, i = 1, 2, \dots, q$ and so on α^n, we use the least square method when the squares of the deviations between the exact and numeric value of w, at $x = x^*$ must be minimized, i.e. the error function E which defined as follows:

$$E(a_1, a_2, \dots, a_q) = (w(x^*, t_n) - w_h(x^*, t_n))^2 = (g_s^*(t_n) - w_h(x^*, t_n))^2 . \tag{15}$$

must be minimized. To determine the unknown set of expansion coefficients , we minimize (15) with respect to each unknown coefficient, i.e.,

$$\frac{\partial E}{\partial a_k} = 0, \qquad k = 1, 2, \dots, q. \tag{16}$$

Now the linear system corresponding to coefficients a_i of Eq. (16) can be expressed as

$$C\Theta = \beta. \tag{17}$$

The matrix C is ill-conditioned, the estimate of Θ by (17) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors. The Tikhonov regularized solution Θ to the system of linear algebraic equation $C\Theta = \beta$ is defined as the solution of following minimization problem [?, ?, ?]

$$\min_{\Theta}\{\| C\Theta - \beta \|^2 + \alpha^2 \| R^{(s)}\Theta \|^2\}, \tag{18}$$

where $\| \cdot \|$ denotes the usual Euclidean norm and α is called the regularization parameter. Choosing a suitable value of the regularization parameter α is crucial for the accuracy of the numerical solution and is still under intensive research [?]. In our computation, we use the GCV scheme to determine an appropriate value of α [?, ?, ?].

On the case of the zeroth-, first-, and second-order Tikhonov regularization method the matrix $R^{(s)}$, for $s = 0, 1, 2$, is given by, see e.g. [54]

$$R^{(0)} = I_{M\times M} \in \mathbb{R}^{M\times M},$$

$$R^{(1)} = \begin{pmatrix} -1 & 1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 1 & 0 \\ 0 & 0 & \ldots & 0 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(M-1)\times M},$$

$$R^{(2)} = \begin{pmatrix} 1 & -2 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{(M-2)\times M}.$$

Therefore, we obtain the Tikhonov regularized solution of the regularized equation as

$$\Theta = \left[C^T C + \alpha \left(R^{(s)}\right)^T R^{(s)}\right]^{-1} C^T \beta. \tag{19}$$

4. Radial Basis Functions Method

In this section, RBFs methods have been introduced for interpolation of scattered data. Some well-known RBFs are listed in Table 1. RBF spaces are generated by the shifts of a radial function $\phi_j(\cdot) = \phi\left(\| \cdot - x_j \|\right)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a given, continuous univariate function, and $\{x_j\}$ are some nodes in the domain of the problem. Let the set $X = \{x_j\}_{j=1}^M$, where M is the number of data points. Given data $\{x_j, u(x_j)\}_{j=1}^M$, the interpolant is schemed as follows

$$s(x) = \sum_{j=1}^{M} \lambda_j \phi\left(\| x - x_j \|\right), \qquad x \in \mathbb{R}^d,$$

where the λ_j are real coefficients that satisfy the interpolation conditions $s|_X = u|_X$, i.e. $s(x_i) = u(x_i)$ for $i = 1, \ldots, M$, which result in the following linear system of equations:

$$\Phi\Lambda = \mathbf{u},$$

where $\Lambda = [\lambda_1, \ldots, \lambda_M]^T, \mathbf{u} = [u_1, \ldots, u_M]^T$ and $\Phi = \left(\phi\left(\| x_k - x_j \|\right)\right)$ is the coefficient matrix.

Table 1. Some well-known functions that generate RBFs.

Name of function	Definition
Gaussian (GA)	$\phi(r) = e^{\frac{-r^2}{2c^2}}$
Hardy multiquadrics(MQ)	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse multiquadrics(IMQ)	$\phi(r) = \frac{1}{\sqrt{r^2+c^2}}$
Inverse quadric(IQ)	$\phi(r) = \frac{1}{r^2+c^2}$

The matrix Φ has been shown to be positive definite (and therefore, nonsingular) for distinct interpolation points for GA, IMQ and IQ by Schoenberg's Theorem [?]. Additionally, by using the Micchelli Theorem [?] we can show that Φ is invertible for distinct sets of the scattered points in the case of MQ. For the existence, uniqueness and convergence proofs the interested readers are referred to [57–61].

Although the matrix Φ is nonsingular in the above cases, usually it is very ill-conditioned, i.e. the condition number of

$$\kappa_s(\Phi) = \| \Phi \|_s \| \Phi^{-1} \|_s, \qquad s = 1, 2, \dots$$

is a very large number. Therefore, a small perturbation in initial data may produce a large amount of perturbation in the solution. Thus we have to use more precision arithmetic than the standard floating-point arithmetic in our computation. For a fixed number of interpolation points the condition number of Φ depends on the shape parameter c, support of the RBFs and minimum separation distance of interpolation points. Furthermore, the condition number grows with M for definite values of shape parameter c. In practice, the shape parameter c must be adjusted to the number of interpolating points in order to produce an interpolation matrix which is well conditioned enough to be inverted in finite precision arithmetic [?].

4.1. **RBFs Method for nonlinear inverse problem.** In this section, we apply RBFs method for solving the problem 2. At the first way we use of finite-difference and radial basis functions for discretizing of time and space variables, respectively. For discretization of time variable, we need some preliminary. We define

$$t_n = n\tau, \qquad n = 0, \dots, N, \tag{20}$$

where $\tau = \frac{T}{N}$ is the step size of time variable. In this section, we discretize the time variable by applying a simple one-step forward difference formula to the time domain t, for the first-order derivative on time variable. Moreover, assume that M is the number of collocation nodes in the space domain. We discretizing the Eq. ($2a$) in point (x, t_n) and obtain

$$\frac{u^{n+1} - u^n}{\tau} = \Delta u^{n+1} + F(u^n), \quad n = 0, \dots, N-1, \tag{21}$$

by simplification we have

$$u^{n+1} - \tau \Delta u^{n+1} = u^n + \tau F(u^n), \quad n = 0, \dots, N-1. \tag{22}$$

Furthermore, the unknown function $F(u)$ is approximated by Eq. 5 and $\{a_1, a_2, \dots, a_q\}$ are constants which remain to be determined.

A radial basis approximation in the space domain for every time t, has the form

$$\tilde{u}(x,t) = \sum_{j=1}^{M} \lambda_j(t)\phi(x - x_j), \tag{23}$$

where $(x,t) \in [0,1] \times [0,T]$, and $\phi(x)$ is a positive definite radial function [43]. Then, by putting the $\tilde{u}(x,t)$ and $\tilde{F}$ into Eq. 22, we obtain the following equation

$$\sum_{j=1}^{M} \lambda_j(t_{n+1})\phi(x - x_j) - \tau \sum_{j=1}^{M} \lambda_j(t_{n+1})\phi''(x - x_j) = \sum_{j=1}^{M} \lambda_j(t_n)\phi(x - x_j) + \tau \tilde{F}\left(\sum_{j=1}^{M} \lambda_j(t_n)\phi(x - x_j)\right). \tag{24}$$

By simplification of the above equation and substituting each x_k for x, we have the following iterative system of equations

$$\sum_{j=1}^{M} \lambda_j(t_{n+1}) \left(\phi(x_k - x_j) - \tau\phi''(x_k - x_j)\right) = \sum_{j=1}^{M} \lambda_j(t_n)\phi(x_k - x_j) + \tau\tilde{F}\left(\sum_{j=1}^{M} \lambda_j(t_n)\phi(x_k - x_j)\right). \quad (25)$$

For convenience, we denote the vector $\left(\tilde{u}(x_1, t_n), \ldots, \tilde{u}(x_M, t_n)\right)^T$ by $\tilde{U}(t_n)$ and $\left(\lambda_1(t_n), \ldots, \lambda_M(t_n)\right)^T$ by $\Lambda(t_n)$. So in matrix form we have

$$(\Phi - \tau\Phi'')\Lambda(t_{n+1}) = \tilde{U}(t_n) + \tau\tilde{F}(\tilde{U}(t_n)), \quad (26)$$

in which

$$\Phi = \begin{pmatrix} \phi(0) & \phi(x_1 - x_2) & \cdots & \phi(x_1 - x_M) \\ \phi(x_2 - x_1) & \phi(0) & \cdots & \phi(x_2 - x_M) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(x_{M-1} - x_1) & \phi(x_{M-1} - x_2) & \cdots & \phi(x_{M-1} - x_M) \\ \phi(x_M - x_1) & \phi(x_M - x_2) & \cdots & \phi(0) \end{pmatrix},$$

and similarly, $(\Phi')_{ij} = \phi'(x_i - x_j)$ and $(\Phi'')_{ij} = \phi''(x_i - x_j)$, for $i, j = 1, \ldots, M$.

Obviously, in each time step, $\Lambda(t_{n+1})$ are achieved based on unknown coefficients a_i. To obtain the unknown coefficients $a_i, i = 1, 2, \ldots, q$, we use the least square method when the squares of the deviations between the exact and numeric value of u, at $x = x^*$ must be minimized, i.e. the error function E which defined as follows:

$$E(a_1, a_2, \ldots, a_q) = (u(x^*, t_{n+1}) - \tilde{u}(x^*, t_{n+1}))^2 = (g_s(t_{n+1}) - \tilde{u}(x^*, t_{n+1}))^2 . \quad (27)$$

must be minimized. The linear system corresponding to coefficients a_i of Eq. (27) can be expressed as

$$C\Theta = \beta. \quad (28)$$

The condition number of the resulting linear system depends directly on the shape parameter c. Generally, the obtained linear system is ill-conditioned. To overcome such difficulties, we use Tikhonov regularization method that is investigated in detail in the previous section.

5. Stability Analysis

In this section, stability of the time discrete numerical scheme defined by Eq. (13) will be investigated in details. We begin by considering function $F(u)$ satisfy the Lipschitz condition with respect to u

$$|F_1(u) - F_1(\tilde{u})| \leq L_1|u - \tilde{u}|, \qquad \forall u, \tilde{u}, \quad (29)$$

where $\mathcal{L}_1$ is Lipschitz constant. Also, we assume that

$$\frac{2}{1 + 2L_1} \leq \tau \leq \frac{1}{L_1}.$$

Now, we introducing the functional spaces endowed with standard norms and inner products [?],

$$H^1(\Omega) = \left\{ v \in L^2(\Omega), \quad \frac{dv}{dx} \in L^2(\Omega) \right\},$$
$$H^1_0(\Omega) = \left\{ v \in H^1(\Omega), \quad v|_{\partial\Omega} = 0 \right\},$$

where $L^2(\Omega)$ is the space of measurable functions whose square is Lebesgue integrable in Ω. The standard inner products of $L^2(\Omega)$ and $H^1(\Omega)$ are defined, respectively by

$$(u, v) = \int_\Omega uv d\Omega, \qquad (u, v)_1 = (u, v) + (\nabla \cdot u, \nabla \cdot v),$$

and the corresponding norms are defined as

$$\|v\| = (v, v)^{\frac{1}{2}}, \qquad \|v\|_1 = (v, v)_1^{\frac{1}{2}}, \qquad |v|_1 = (\nabla \cdot v, \nabla \cdot v)^{\frac{1}{2}}.$$

Also, in this paper we use the following weighted H^1-norm

$$\|v\|_{*,1} = \left[\|v\|^2 + |v|_1^2\right]^{\frac{1}{2}}.$$

Furthermore, for any functions $f(x)$ and $g(x)$ we have

$$(\nabla \cdot |f|, g) \leq |f|_1 \|g\|. \tag{30}$$

Theorem 5.1. *For* $\frac{2}{1+2L_1} \leq \tau \leq \frac{1}{L_1}$, *the time discrete numerical scheme defined by Eq.(13) is stable in* H^1*-norm.*

Proof. We assume that W^{n+1} and $\tilde{W}^{n+1}$ are exact and approximation solutions of Eq.(13), respectively. The roundoff error has the following form

$$(\frac{\overline{\partial} e^{n+1}}{\partial t}, v) + (\frac{\partial \widehat{e}^{n+1}}{\partial x}, \frac{\partial v}{\partial x}) = (F_1(W^n) - F_1(\tilde{W}^n), v), \qquad \forall v \in V_h, \tag{31}$$

in which

$$e^{n+1} = W^{n+1} - \tilde{W}^{n+1}, \quad n = 0, \dots, N-1.$$

By using $v = e^{n+1}$ we obtain

$$(\frac{e^{n+1} - e^n}{\tau}, e^{n+1}) + (\frac{\partial}{\partial x}(\frac{e^{n+1} + e^n}{2}), \frac{\partial e^{n+1}}{\partial x}) = (F_1(W^n) - F_1(\tilde{W}^n), e^{n+1}), \tag{32}$$

Now, using the Schwarz inequality and relations (29) and (30), we can write

$$\|e^{n+1}\|^2 + \frac{\tau}{2}|e^{n+1}|_1^2 \leq \frac{1}{2}\|e^n\|^2 + \frac{1}{2}\|e^{n+1}\|^2 + \frac{\tau}{4}(|e^n|_1^2 + |e^{n+1}|_1^2) + \frac{\tau L_1}{2}\|e^n\|^2 + \frac{\tau L_1}{2}\|e^{n+1}\|^2, \tag{33}$$

and by simplification of the above relation we obtain

$$(2 - 2\tau L_1)\|e^{n+1}\|^2 + \tau|e^{n+1}|_1^2 \leq (2 + 2\tau L_1)\|e^n\|^2 + \tau|e^n|_1^2. \tag{34}$$

Thus, regarding to $\frac{2}{1+2L_1} \leq \tau \leq \frac{1}{L_1}$, we can write

$$(2 - 2\tau L_1)(\|e^{n+1}\|^2 + |e^{n+1}|_1^2) \leq (2 + \tau(2L_1 + 1))(\|e^n\|^2 + |e^n|_1^2) \tag{35}$$

i.e.

$$\left\{\|e^{n+1}\|^2 + |e^{n+1}|_1^2\right\} \leq \left(\frac{2 + \tau(2L_1 + 1)}{2 - 2\tau L_1}\right)\left\{\|e^n\|^2 + |e^n|_1^2\right\},$$

also, using the weighted H^1-norm, yields

$$\begin{aligned}
\|e^{n+1}\|^2_{*,1} &\leq \left(\frac{2+\tau(2L_1+1)}{2-2\tau L_1}\right)\|e^n\|^2_{*,1} \\
&\leq \left(\frac{2+\tau(2L_1+1)}{2-2\tau L_1}\right)^2\|e^{n-1}\|^2_{*,1} \\
&\vdots \\
&\leq \left(\frac{2+\tau(2L_1+1)}{2-2\tau L_1}\right)^{n+1}\|e^0\|^2_{*,1}, \quad n=0,\ldots,N-1.
\end{aligned}$$

Furthermore, we can write

$$\lim_{n\to\infty}\left(\frac{2+\tau(2L_1+1)}{2-2\tau L_1}\right)^{n+1} = \lim_{n\to\infty}\left(\frac{1+\frac{T(2L_1+1)}{2(n+1)}}{1-\frac{TL_1}{n+1}}\right)^{n+1} = \frac{e^{T(L_1+\frac{1}{2})}}{e^{-TL_1}} = e^{T(2L_1+\frac{1}{2})}, \quad (36)$$

therefore we obtain

$$\|e^{n+1}\|_{*,1} \leq \sqrt{e^{T(2L_1+\frac{1}{2})}}\|e^0\|_{*,1}, \quad n=0,\ldots,N-1,$$

which completes the proof. □ □

6. Numerical examples

In this section, to illustrate the description above and demonstrating the accuracy and efficiency of the presented method to solve nonlinear inverse problems with unknown source function, we include two numerical examples. As expected the inverse problems are ill-posed and therefore, it is necessary to investigate the stability of the proposed scheme. Thus we consider following examples with noisy data (noisy data=input data+(0.001)rand(1)) [64].

Remark 6.1. *In an inverse problem, there are two sources of error in the estimation. The first source is the unavoidable bias deviation or deterministic error, and the second source of error is the variance due to the amplification of measurement errors or stochastic error. The global effect of deterministic and stochastic errors is considered in the root mean square or total error [?]. Therefore, we compare exact and approximate solutions by considering total error RMS defined by*

$$RMS = \left[\frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\Psi_i}-\Psi_i\right)^2\right]^{\frac{1}{2}} \quad (37)$$

where N, $\hat{\Psi}$ and Ψ are the number of estimated values, the estimated values and the exact values, respectively.

Note that, we suppose $u(x,t)$ and $F(u(x,t))$ be exact solutions of (2) and $u^*(x,t)$, $F^*(u(x,t))$ be solutions obtained by applying the given methods. Also, we consider $T=1$, $\Delta x=\frac{1}{20}$, $\Delta t=\frac{1}{100}$ and $q=6$ with noisy data (noisy data=input data+(0.001) rand(1)). In RBF method case, we compute the numerical solutions using Gaussian RBF.

Example 6.1. *In this example, let us consider the following nonlinear inverse reaction-diffusion problem*

$$u_t = u_{xx} + F(u), \qquad 0 < x < 1, \quad 0 < t < T,$$

and overspecified data

$$u(0.5,t) = \frac{\exp(\frac{1}{2\sqrt{2}} + (\frac{1}{2}-\mu)t) + \mu\exp(\frac{\mu}{2\sqrt{2}} - (\mu - \frac{\mu^2}{2})t)}{1+\exp(\frac{1}{2\sqrt{2}} + (\frac{1}{2}-\mu)t) + \mu\exp(\frac{\mu}{2\sqrt{2}} - (\mu - \frac{\mu^2}{2})t)}, \qquad 0 \le t \le T,$$

where $\mu = 0.6$.

The exact solutions are

$$u(x,t) = \frac{\exp(\frac{x}{\sqrt{2}} + (\frac{1}{2}-\mu)t) + \mu\exp(\frac{\mu x}{\sqrt{2}} - (\mu - \frac{\mu^2}{2})t)}{1+\exp(\frac{x}{\sqrt{2}} + (\frac{1}{2}-\mu)t) + \exp(\frac{\mu x}{\sqrt{2}} - (\mu - \frac{\mu^2}{2})t)}$$

and

$$F(u) = u(1-u)(u-\mu).$$

The comparison between exact and approximate values for $u(x,t)$ and $F(u(x,t))$ at points $x = 0.3$ and $x = 0.8$ obtained by methods presented in this paper, are given in Tables $2-5$, where execution time (s) and regularization parameter (R.p) and condition number (Cn) are also given in the tables.

time	Exact solution		FE method		RBF method	
t	$u(0.3,t)$	$F(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$
0.1	0.566678	−0.008182	0.566678	−0.009224	0.566900	−0.008440
0.2	0.564598	−0.008702	0.564599	−0.009748	0.564837	−0.009004
0.3	0.562483	−0.009232	0.562484	−0.010281	0.562728	−0.009594
0.4	0.560333	−0.009772	0.560334	−0.010824	0.560582	−0.010212
0.5	0.558149	−0.010320	0.558150	−0.011376	0.558403	−0.010860
0.6	0.555933	−0.010878	0.555933	−0.011937	0.556189	−0.011540
0.7	0.553684	−0.011445	0.553685	−0.012507	0.553945	−0.012257
0.8	0.551404	−0.012020	0.551405	−0.013084	0.551671	−0.013011
0.9	0.549095	−0.012603	0.549096	−0.013670	0.549366	−0.013807
1	0.546756	−0.013194	0.546757	−0.014262	0.547030	−0.014646
RMS			7.159×10^{-7}	1.060×10^{-3}	2.888×10^{-4}	4.126×10^{-3}
s			24.8		37.7	
R.p			5.210×10^{-15}		3.900×10^{-17}	
Cn			Inf		9.071×10^{24}	

TABLE 2. The comparison between exact and numerical solutions of Example 6.1 at $(0.3,t)$ with noisy data.

time	Exact solution		FE method		RBF method	
t	$u(0.8,t)$	$F(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$
0.1	0.623781	0.005580	0.623780	0.004627	0.623558	0.008843
0.2	0.622484	0.005283	0.622483	0.004330	0.622255	0.008224
0.3	0.621144	0.004975	0.621143	0.004021	0.620911	0.007606
0.4	0.619761	0.004656	0.619760	0.003702	0.619523	0.006989
0.5	0.618336	0.004327	0.618335	0.003372	0.618093	0.006375
0.6	0.616868	0.003986	0.616868	0.003031	0.616621	0.005765
0.7	0.615360	0.003635	0.615359	0.002679	0.615108	0.005160
0.8	0.613810	0.003273	0.613810	0.002316	0.613553	0.004562
0.9	0.612221	0.002901	0.612220	0.001942	0.611960	0.003971
1	0.610593	0.002518	0.610592	0.001557	0.610326	0.003387
RMS			6.935×10^{-7}	9.605×10^{-4}	1.595×10^{-4}	4.126×10^{-3}

TABLE 3. The comparison between exact and numerical solutions of Example 6.1 at $(0.8,t)$ with noisy data.

time	Exact solution		FE method		RBF method	
t	$u(0.3,t)$	$F(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$
0.1	0.566678	−0.008182	0.566678	−0.008224	0.566793	−0.008334
0.2	0.564598	−0.008702	0.564599	−0.008748	0.564721	−0.008865
0.3	0.562483	−0.009232	0.562484	−0.009281	0.562609	−0.009416
0.4	0.560333	−0.009772	0.560333	−0.009824	0.560462	−0.009988
0.5	0.558149	−0.010320	0.558150	−0.010376	0.558280	−0.010582
0.6	0.555933	−0.010878	0.555933	−0.010937	0.556065	−0.011201
0.7	0.553684	−0.011445	0.553685	−0.011507	0.553817	−0.011845
0.8	0.551404	−0.012020	0.551405	−0.012084	0.551538	−0.012519
0.9	0.549095	−0.012603	0.549096	−0.012670	0.549228	−0.013222
1	0.546756	−0.013194	0.546757	−0.013262	0.546888	−0.013958
RMS			7.027×10^{-7}	5.617×10^{-5}	1.263×10^{-4}	3.770×10^{-4}

TABLE 4. The comparison between exact and numerical solutions of Example 6.1 at $(0.3,t)$ without noisy data.

Example 6.2. *Consider the following nonlinear inverse problem*

$$\begin{aligned}
u_t &= u_{xx} + F(u), && 0 < x < 1, \quad 0 < t < T,\\
u(x,0) &= (-\frac{1}{2}\tanh(\frac{3}{4}x - \frac{1}{2}) + \frac{1}{2})^{\frac{1}{3}}, && 0 \le x \le 1,\\
u(0,t) &= (\frac{1}{2}\tanh(\frac{1}{2} + \frac{15}{8}t) + \frac{1}{2})^{\frac{1}{3}}, && 0 \le t \le T,\\
u(1,t) &= (\frac{1}{2}\tanh(-\frac{1}{4} + \frac{15}{8}t) + \frac{1}{2})^{\frac{1}{3}}, && 0 \le t \le T,
\end{aligned}$$

and overspecified data

$$u(0.5,t) = (\frac{1}{2}\tanh(\frac{1}{8} + \frac{15}{8}t) + \frac{1}{2})^{\frac{1}{3}}, \qquad 0 \le t \le T.$$

time	Exact solution		FE method		RBF method	
t	$u(0.8,t)$	$F(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$
0.1	0.623781	0.005580	0.623781	0.005627	0.623648	0.008666
0.2	0.622484	0.005283	0.622484	0.005330	0.622343	0.008075
0.3	0.621144	0.004975	0.621144	0.005021	0.620998	0.007484
0.4	0.619761	0.004656	0.619761	0.004702	0.619610	0.006892
0.5	0.618336	0.004327	0.618335	0.004372	0.618179	0.006300
0.6	0.616868	0.003986	0.616868	0.004031	0.616704	0.005710
0.7	0.615360	0.003635	0.615359	0.003679	0.615188	0.005121
0.8	0.613810	0.003273	0.613810	0.003316	0.613630	0.004536
0.9	0.612221	0.002901	0.612221	0.002942	0.612032	0.003955
1	0.610593	0.002518	0.610593	0.002557	0.610393	0.003378
RMS			6.800×10^{-7}	4.461×10^{-5}	1.604×10^{-4}	2.154×10^{-3}

TABLE 5. The comparison between exact and numerical solutions of Example 6.1 at $(0.8,t)$ without noisy data.

time	Exact solution		FE method		RBF method	
t	$u(0.3,t)$	$F(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$
0.1	0.894642	0.435922	0.894475	0.434551	0.898466	0.431865
0.2	0.922806	0.352939	0.922526	0.351008	0.926421	0.353808
0.3	0.944377	0.274463	0.944052	0.272479	0.947539	0.278050
0.4	0.960438	0.206587	0.960111	0.204844	0.963046	0.211162
0.5	0.972135	0.151618	0.971841	0.150156	0.974178	0.156265
0.6	0.980514	0.109192	0.980271	0.107936	0.982065	0.113545
0.7	0.986444	0.077559	0.986254	0.076428	0.987597	0.081536
0.8	0.990604	0.054548	0.990461	0.053482	0.991448	0.058180
0.9	0.993504	0.038095	0.993399	0.037063	0.994117	0.041452
1	0.995517	0.026475	0.995441	0.025458	0.995962	0.029622
RMS			2.320×10^{-4}	1.440×10^{-3}	1.425×10^{-3}	6.813×10^{-3}
(s)			26.3		36.1	
R.p			4.225×10^{-14}		4.155×10^{-16}	
Cn			Inf		4.6883×10^{16}	

TABLE 6. The comparison between exact and numerical solutions of Example 6.2 at $(0.3,t)$ with noisy data.

The exact solutions in a closed form are given by

$$u(x,t) = (\frac{1}{2}\tanh(-\frac{3}{4}x + \frac{15}{8}t + \frac{1}{2}) + \frac{1}{2})^{\frac{1}{3}}$$

and

$$F(u) = u(1 - u^6).$$

The results obtained by presented methods for $u(x,t)$ and $F(u(x,t))$ with noisy data are presented in Tables 6 − 9, where execution time and regularization parameter are also given in the tables.

time	Exact solution		FE method		RBF method	
t	$u(0.8,t)$	$F(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$
0.1	0.816150	0.574942	0.816202	0.575218	0.811981	0.568416
0.2	0.859133	0.513651	0.859321	0.512951	0.855090	0.504346
0.3	0.894642	0.435922	0.894897	0.433438	0.891021	0.431865
0.4	0.922806	0.352939	0.923075	0.349168	0.919759	0.353808
0.5	0.944377	0.274463	0.944628	0.270191	0.941960	0.278050
0.6	0.960438	0.206587	0.960652	0.202421	0.958593	0.211162
0.7	0.972135	0.151618	0.972307	0.147886	0.970777	0.156265
0.8	0.980514	0.109192	0.980646	0.106000	0.979539	0.113545
0.9	0.986444	0.077559	0.986541	0.074882	0.985758	0.081536
1	0.990604	0.054548	0.990674	0.052305	0.990124	0.058180
RMS			1.870×10^{-4}	3.037×10^{-3}	1.425×10^{-3}	6.813×10^{-3}

Table 7. The comparison between exact and numerical solutions of Example 6.2 at $(0.8,t)$ with noisy data.

time	Exact solution		FE method		RBF method	
t	$u(0.3,t)$	$F(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$	$u^*(0.3,t)$	$F^*(u(0.3,t))$
0.1	0.894642	0.435922	0.894475	0.435555	0.897258	0.425286
0.2	0.922806	0.352939	0.922527	0.352010	0.925315	0.345059
0.3	0.944377	0.274463	0.944052	0.273482	0.946581	0.268468
0.4	0.960438	0.206587	0.960111	0.205846	0.962262	0.201891
0.5	0.972135	0.151618	0.971841	0.151157	0.973583	0.147983
0.6	0.980514	0.109192	0.980271	0.108937	0.981620	0.106550
0.7	0.986444	0.077559	0.986254	0.077429	0.987257	0.075869
0.8	0.990604	0.054548	0.990461	0.054483	0.991194	0.053620
0.9	0.993504	0.038095	0.993399	0.038064	0.993953	0.037623
1	0.995517	0.026475	0.995441	0.026458	0.995861	0.026364
RMS			2.322×10^{-4}	5.368×10^{-4}	1.641×10^{-3}	5.565×10^{-3}

Table 8. The comparison between exact and numerical solutions of Example 6.2 at $(0.3,t)$ without noisy data.

7. Conclusion

In this paper, the inverse problem of determining an unknown reaction term in a homogeneous parabolic equation was considered. The following results are obtained.

- The present study, successfully applies the numerical methods to inverse problems.
- Numerical examples also verified the efficiency and accuracy of the method that can be obtained within a couple of minutes CPU time at Core($i5$)–2.67 GHz PC.
- The present methods has been found stable with respect to small perturbation in the input data.
- The comparison between methods presented in this work, shows that the finite element method have better accuracy than the RBFs method.

time	Exact solution		FE method		RBF method	
t	$u(0.8,t)$	$F(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$	$u^*(0.8,t)$	$F^*(u(0.8,t))$
0.1	0.816150	0.574942	0.816203	0.576219	0.813586	0.571615
0.2	0.859133	0.513651	0.859322	0.513953	0.856617	0.509097
0.3	0.894642	0.435922	0.894898	0.434440	0.892456	0.436967
0.4	0.922806	0.352939	0.923076	0.350169	0.921041	0.358710
0.5	0.944377	0.274463	0.944629	0.271192	0.943019	0.282351
0.6	0.960438	0.206587	0.960653	0.203421	0.959438	0.214558
0.7	0.972135	0.151618	0.972308	0.148887	0.971431	0.158626
0.8	0.980514	0.109192	0.980647	0.107000	0.980031	0.114954
0.9	0.986444	0.077559	0.986542	0.075882	0.986105	0.082248
1	0.990604	0.054548	0.990674	0.053305	0.990403	0.058144
RMS			1.871×10^{-4}	2.207×10^{-3}	1.593×10^{-3}	6.011×10^{-3}

Table 9. The comparison between exact and numerical solutions of Example 6.2 at $(0.8,t)$ without noisy data.

References

[1] Kabanikhin, S. I., (2011), Inverse and Ill-Posed Problems, Theory and Applications, De Gruyter.

[2] Bushuyev, I., (1995), Global uniqueness for inverse parabolic problems with final observation, Inverse Problems, 11, (4), pp. 11-16.

[3] Cannon, J. R. and DuChateau, P., (1998), Structural identification of an unknown source term in a heat equation, Inverse Problems, 14, (3), pp. 535-551.

[4] Choulli, M. and Yamamoto, M., (1997), An inverse parabolic problem with non-zero initial condition, Inverse Problems, 13, (1), pp. 19-27.

[5] DuChateau, P. and Rundell, W., (1985), Unicity in an inverse problem for an unknown reaction term in a reaction-diffusion equation, J. Differential Equations, 59, (2), pp. 155-164.

[6] Gatti, S., (1998), An existence result for an inverse problem for a quasilinear parabolic equation, Inverse Problems, 14, (1), pp. 53-65.

[7] Isakov, V., (1999), Some inverse problems for the diffusion equation, Inverse Problems, 15, (1), pp. 3-10.

[8] Lorenzi, A., (1982), An inverse problem for a semilinear parabolic equation, Ann. Mat. Pura Appl., 131, (4), pp. 145-166.

[9] Nanda, A. and Das, P. C., (1996), Determination of the source term in the heat conduction equation, Inverse Problems, 12, (3), pp. 325-339.

[10] Pilant, M. and Rundell, W., (1988), Fixed point methods for a nonlinear parabolic inverse coefficient problem, Comm. Partial Differential Equations, 13, (4), pp. 469-493.

[11] Riganti, R. and Savateev, E., (1994), Solution of an inverse problem for the nonlinear heat equation, Comm. Partial Differential Equations, 19, (9), pp. 1611-1628.

[12] Rundell, W., (1987), The determination of a parabolic equation from initial and final data, Proc. Amer. Math. Soc., 99, (4), pp. 637-642.

[13] Savateev, E., G. and Riganti, R., (1995), Inverse problem for the nonlinear heat equation with the final overdetermination, Math. Comput. Modelling, 22, (1), pp. 29-43.

[14] Tadi, M., (1997), Inverse heat conduction based on boundary measurement, Inverse Problems, 13, (6), pp. 1585-1605.

[15] Li, G., Ma, Y. and Li, K., (2004), Determination of a nonlinear source term in a diffusion equation with final observations, Int. J. Math. Mathe. Sci., 2004, (14), pp. 741-753.

[16] Isakov, V., (1998), Inverse Problems for Partial Differential Equations, App. Math. Sci., 127, Springer-Verlag, New York.

[17] Tikhonov, A. N. and Arsenin, V. Y., (1977), On The Solution of Ill-Posed Problems, Wiley, New York.

[18] Tikhonov, A. N. and Arsenin, V. Y., (1977), Solution of Ill-Posed Problems, Winston and Sons, Washington, DC.
[19] Alifanov, O. M., (1994), Inverse Heat Transfer Problems, Springer, New York.
[20] Murio, D. A., (1993), The Mollification Method and the Numerical Solution of Ill-Posed Problems, Wiley-Interscience, New York.
[21] Pourgholi, R. and Rostamian, M., (2010), A numerical technique for solving IHCPs using Tikhonov regularization method, Appl. Math. Model, 34, pp. 2102-2110.
[22] Molhem, H. and Pourgholi, R., (2008), A numerical algorithm for solving a one-dimensional inverse heat conduction problem, J. Math. Stat., 4, pp. 60-63.
[23] Beck, J. V., Blackwell, B. and Clair, C. R. St., (1985), Inverse Heat Conduction: Ill-Posed Problems, Wiley-Interscience, New York.
[24] Courant, R., (1943), Variational methods for the solution of problems of equilibrium and vibrations, Bull. Amer. Math. Soc., 49, pp. 1-23.
[25] Zlamal, M., (1977), Finite element methods for nonlinear parabolic equations, Num. Analysis, 11, pp. 93-107.
[26] El-Azab, M. S. and Abdelgaber, K. M., (2011), Finite element solution of nonlinear diffusion problems, App. Math. Comp, 217, pp. 6198-6205.
[27] John, V. and Schmeyer, E., (2008), Finite element methods for time dependent convection diffusion reaction equations with small diffusion, Comput. Methods Appl. Mech. Engrg. 198, pp. 475-494.
[28] Bangerth, W. and Joshi, A., (2009), Adaptive finite element methods for nonlinear inverse problems, ACM, pp. 8-12.
[29] Beilina, L. and Johnson, C., (2001), A hybrid finite element/difference method for an inverse scattering problem. In Num. Math. Adv. Appl., 2001, Springer-Verlag.
[30] Beilina, L., (2003), Adaptive finite element/difference method for inverse elastic scattering waves, Appl. Comput. Math. 2, pp. 119-134.
[31] Ling, Xia., Russell, Keanini, G. and Cherukuri, H. P., (2003), A non-iterative finite element method for inverse heat conduction problems, Int. J. Numer. Meth. Engng, 56, pp. 1315-1334.
[32] Kansa, E. J., (1990), Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics-I, Comput. Math. Appl., 19, pp. 127-145.
[33] Kansa, E. J., (1990), Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics-II, Comput. Math. Appl., 19, pp. 147-161.
[34] Hardy, R. L., (1971), Multiquadric equations of topography and other irregular surfaces, J. Geophysc. Res., 76, pp. 1905-1915.
[35] Hon, Y. C. and Mao, X. Z., (1998), An efficient numerical scheme for Burgers equation, Appl. Mathe. Compu., 95, pp. 37-50.
[36] Hon, Y. C., Cheung, K. F., Mao, X. Z. and Kansa, E. J., (1999), Multiquadric solution for shallow water equations, J. Hydr. Eng., 125, pp. 524-533.
[37] Zerroukat, M., Power, H. and Chen, C. S., (1992), A numerical method for heat transfer problem using collocation and radial basis functions, Int. J. Numer. Methods Eng., 42, pp. 1263-1278.
[38] G. Fasshauer, G., (2007), Meshfree Approximation Methods with Matlab, Inte. Math. Sci., 6.
[39] Hon, Y. C. and Wu, Z., (2000), A numerical computation for inverse boundary determination problem, Eng. Analys. Bound. Element., 24, pp. 599-606.
[40] Li, J., (2004), A radial basis meshless method for solving inverse boundary value problems, Commun. Num. Meth. Engi. 20, pp. 51-60.
[41] Li, J., (2005), Application of radial basis meshless methods to direct and inverse biharmonic boundary value problems, Commun. Num. Meth. Engi., 21, pp. 169-182.
[42] Cheng, AH-D. and Cabral, JJSP., (2005), Direct solution of ill-posed boundary value problems by radial basis function collocation method, Int. J. Num. Meth. Engi., 64, pp. 45-64.
[43] Ma, L. and Wu, Z., (2011), Radial basis functions method for parabolic inverse problem, Int. J. Comp. Math., 88, (2), pp. 384-395.
[44] Bayona, V., Moscoso, M., Carretero, M., Kindelan, M., (2010), RBF-FD formulas and convergence properties, J. Comput. Phys, 229, pp. 8281-8295.
[45] Fasshauer, G. E. and Zhang, J. G., (2007), On choosing "optimal" shape parameters for RBF approximation, Numer. Algor, 45, pp. 345-368.
[46] Chen, Xi. and Guo, J., (2005), Existence and uniqueness of entire solutions for a reaction-diffusion equation, J. Diff. Equ., 212, pp. 62-84.
[47] Ozisik, M. N. and Orland, H. R. B., (2000), Inverse Heat transfer: fundamentals and applications, Taylor and Francis.

[48] Thomee, V., (1997), Galerkin Finite Element Methods for Parabolic Problems, Second edition, Springer.
[49] Hansen, P. C., (1992), Analysis of discrete ill-posed problems by means of the L-curve, SIAM Rev. 34, pp. 561-580.
[50] Lawson, C. L. and Hanson, R. J., (1995), Solving Least Squares Problems, SIAM, Philadelphia.
[51] Elden, L., (1984), A note on the computation of the generalized cross-validation function for ill-conditioned least squares problems, BIT, 24, pp. 467-472.
[52] Golub, G. H., Heath, M. and Wahba, G., (1979), Generalized cross-validation as a method for choosing a good ridge parameter, Technometrics, 21, pp. 215-223.
[53] Wahba, G., (1990), Spline Models for Observational Data, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 59.
[54] Aster, R. C., Borchers, B. and Thurber, C. H., (2012), Parameter Estimation and Inverse Problems, Second edition, Academic Press.
[55] Schoenberg, I. J., (1938), Metric spaces and completely monotone functions, Ann. Mathe., 39, pp. 811-841.
[56] Micchelli, C. A., (1986), Interpolation of scattered data: Distance matrices and conditionally positive definite functions, Constructive Approximation, 2, pp. 11-22.
[57] Buhmann, M. D., (2003), Radial Basis Functions: Theory and Implementations, Cambridge University Press, Cambridge, UK.
[58] Wendland, H., (2005), Scattered Data Approximation, Cambridge University Press, Cambridge, UK.
[59] Madych, W. R. and Nelson, S. A., (1986), Multivariate interpolation and conditionally positive definite functions, Constr. Approx. 2, pp. 11-22.
[60] Powell, M. D. J., (1987), Radial basis functions for multivariable interpolation: A review, Numerical Analysis, Longman Scientific and Technical, pp. 223-241.
[61] Wu, Z. M. and Schaback, R., (1993), Local error estimates for radial basis function interpolation of scattered data, IMA J. Num. Anal. 13, pp. 13-27.
[62] Sarra, S. A., (2005), Adaptive radial basis function method for time dependent partial differential equations, App. Num. Math., 54, pp. 79-94.
[63] Reddy, B. D., (1986), Functional analysis and boundary-value problems: an introductory treatment, John Wiley & Sons. Inc.
[64] Pourgholi, R., Abbasi Molai, A. and Houlari, T., (2013), Resolution of an inverse parabolic problem using Sinc-Galerkin method, TWMS J. App. Eng. Math., 3, (2), pp. 160-181.
[65] Cabeza, J. M. G., Garcia, J. A. M. and Rodriguez, A. C., (2005), A Sequential Algorithm of Inverse Heat Conduction Problems Using Singular Value Decomposition, Int. J. Ther. Sci., 44, pp. 235-244.

Hamed Zeidabadi received his B.Sc degree in applied mathematics from Semnan University (2010), M.Sc. degree in applied mathematics from Ferdowsi University, (2013) and Ph.D. degree in applied mathematics from Damghan University, (2018). His area of research is numerical solution of inverse parabolic problems, partial differential and integral equations.

Reza Pourgholi received his M.Sc. degree in 2001 and Ph.D. degree in 2007 both in applied mathematics from IUST, Iran. He is a professor of applied mathematics at Damghan University. His area of research is numerical solution of inverse parabolic problems and inverse heat conduction problems.

Abbas Hosseini received his B.S. and M.S. degrees both in applied mathematics from Islamic Azad University, Karaj Branch, Karaj, Iran, in 2006 and 2011 respectively, and Ph.D. degree in applied mathematics (numerical analysis) from Damghan University, Damghan, Iran, in 2018. His research interests are numerical solutions of inverse parabolic problems and meshless methods in general.