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SHEARLET SHRINKAGE WITH INTEGRO-DIFFERENTIAL EQUATIONS USING 3-DIMENSIONAL CONTINUOUS SHEARLET TRANSFORM

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Abstract. The generalization of continuous wavelet, a directional multiscale is known as continuous shearlet which is able to study the directional functions and distributions. Many useful features do not carry from 2-dimensional to 3-dimensional cases due to the complexity of singularity sets defined on surfaces rather than along curves. Therefore, we obtained a relation between 3-dimensional continuous shearlet transform and sum of smoothed partial derivative operators. The transform has been explained as a weighted average of pseudo-differential equations. Our results are applicable in medical and seismic imaging related problems.

Keywords: Directional multiscale transform, shearlet shrinkage, integro-differential equations and smoothed partial operators.

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1. Introduction

Due to the arbitrary choice of the scale and translation, the continuous wavelet transform has wide applications in the fields of pattern recognition, feature extraction and detection in comparison of discrete wavelet transform. The continuous wavelet transform described local regularity of the functions and distribution and detect the location of singularity points through its decay at fine scales but it does not provide additional information about geometry of the set of singularities. To overcome this problem, various constructions have been made. Among them the curvelets [2] and shearlets [7] achieved this additional flexibility by defining a collection of analyzing functions ranging over various scales, locations, orientations and with highly anisotropic supports. In one dimension, there are wavelets not generating from an MRA. The most successful generalizations of the wavelet transform is a directional multiscale transform such as shearlet transform which is able to analyze functions and distributions not only in terms of the locations and scales, but also to their directional informations. For 2-dimensional shearlet transform, several

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techniques have been introduced to study the asymptotic decay at fine scales and singularities. Due to the additional complexity of dealing with singularity sets, which are defined on surfaces rather than along curves, the above mentioned features do not carry over from the 2-dimension to 3-dimension setting. For application point of view, the 3-dimensional case is of much interest such as medical and seismic imaging, where important phenomena are related with surfaces of discontinuities. Various authors ([1], [3], [8-10], [12], [15-18]) obtained several results in this direction but our results are different from those authors.

Wavelets can be defined by replacing dyadic dilations with dilations by $r > 1$, where $r \notin$ the set of integers. In this situation the construction of the orthonormal bases associated with the wavelet may require more than one generators, namely, if $r = \frac{p}{q} > 1$ and p, q are relatively prime, then p-q generators are required. To avoid multiple wavelet generators in higher dimensions we restrict our attention to 3×3 real matrices $M^{(1)}_{a,s} \in GL(3, \mathbb{R})$ where $GL(3, \mathbb{R})$ denote the general linear group of 3×3 invertible real matrices acts on $\mathbb{R}$ by linear transformations. The $|detM^{(1)}_{a,s}| = a^2$ where the dilation set $\{2^k : k \in \mathbb{N}\}$ is replaced by $\{(M^{(1)}_{a,s})^k : k \in \mathbb{N}\}$ and

$$\{M^{(1)}_{a,s} = \begin{pmatrix} a & a^{\frac{1}{2}}s_1 & a^{\frac{1}{2}}s_2 \\ 0 & a^{\frac{1}{2}} & 0 \\ 0 & 0 & a^{\frac{1}{2}} \end{pmatrix} : a > 0, s = (s_1, s_2) \in \mathbb{R}^2\}.$$

Shearlet system is constructed by applying different operators to a generator function $\varphi \in L^2(\mathbb{R}^3)$ obtained elements of the form

$$\varphi^1_{a,s,p}(x) = |detM^{(1)}_{a,s}|^{-\frac{1}{2}}\varphi((M^{(1)}_{a,s})^{-1}(x - p)), \quad for \quad p, x \in \mathbb{R}^3. \tag{1.1}$$

Following [11] we have the following definition of admissible shearlet.

Definition 1.1. A function $\varphi \in L^2(\mathbb{R}^3)$ be an admissible shearlet if

$$C_\varphi = \int_{\mathbb{R}^2}\int_{\mathbb{R}^+} |\hat{\varphi}(M^{(1)}_{a,s})^T\xi|^2 a^{-2} dads = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \frac{|\hat{\varphi}(w)|^2}{|w_1|^3} dw_1 dw_2 dw_3 < \infty.$$

Example 1.1. Let φ_1 be a continuous wavelet with $\hat{\varphi}_1 \in \mathbb{C}^\infty(\mathbb{R})$ and supp$\hat{\varphi}_1 \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, and let φ_2 be such that $\hat{\varphi}_2 \in \mathbb{C}^\infty(\mathbb{R}^2)$ defined by

$$\hat{\varphi}(w) = \hat{\varphi}(w_1, \tilde{w}) = \hat{\varphi}_1(w_1)\hat{\varphi}_2(\frac{\tilde{w}}{w})$$

is a continuous shearlet. The support of $\hat{\varphi}$ is obtained for $w_1 \geq 0$.

Let us consider the subspace of $L^2(\mathbb{R}^3)$ given by $L^2(P^1_{u_1,u_2,u_3}) = \{f \in L^2(\mathbb{R}^3) : supp\ \hat{f} \subset P^1_{u_1,u_2,u_3}\}$, where $P^1_{u_1,u_2,u_3}$ is the pyramid for $u_1, u_2, u_3 > 0$ in the frequency plane given by

$$P^1_{u_1,u_2,u_3} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_1| \geq u_1, |\frac{\xi_2}{\xi_1}| \leq u_2 \ \ and \ \ |\frac{\xi_3}{\xi_1}| \leq u_3\}.$$

Definition 1.2 (Continuous Wavelet Transform). For $G = \mathbb{R}_+ \times \mathbb{R}^3$ the continuous wavelet transform of $f \in L^2(\mathbb{R}^3)$ is defined as

$$(W_\varphi f)(a, p) = (W_\varphi f)(aI_3, p) = a^{-\frac{3}{2}} \int_{\mathbb{R}^3} f(x)\overline{\varphi(a^{-1}(x - p))}dx = (\tilde{\varphi}_a * f)(x),$$

where I_3 is the 3×3 identity matrix and $\varphi_a(x) = a^{-\frac{3}{2}}\varphi(\frac{x}{a}), \tilde{\varphi}(x) = \overline{\varphi(-x)}, x \in \mathbb{R}^3, a > 0, p \in \mathbb{R}^3$.

Definition 1.3 (Continuous Shearlet Transform). The function $\varphi^{(1)}_{a,s,p}(x)$ are continuous shearlets for $P^1_{u_1,u_2,u_3}$ and the corresponding continuous shearlet transform on $P^1_{u_1,u_2,u_3}$ be defined as:

$$\begin{aligned}(SH^{(1)}_{\varphi} f(a,s,p) &= (SH^{(1)}_{\varphi} f)(M^{(1)}_{a,s}(p)) \\ &= a^{-1}\int_{\mathbb{R}^3} f(x)\overline{\varphi(M^{(1)}_{a,s})^{-1}(x-p)}dx = (\widetilde{\varphi}_{a,s} * f)(x),\end{aligned}$$

where $a > 0, s \in \mathbb{R}^2, \varphi_{a,s}(x) = a^{-1}(M^{(1)}_{a,s})^{-1}(x)$ and $\widetilde{\varphi}_{a,s}(x) = \overline{\varphi_{a,s}(x)}$. The transform is able to detect not only the location of singularity points through its decay at fine scale but also the geometric information of the singularity set. The index (1) used above in the notation of shearlet system (1.1) indicates that the system (1.1) has frequency support in the pyramidal region $P^1_{u_1,u_2,u_3}$, similar shearlet systems defined in the two other complementary pyramidal regions of $\mathbb{R}^3$ are defined as:

$$P^2_{u_1,u_2,u_3} = \{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : |\xi_1| \geq u_1, |\frac{\xi_2}{\xi_1}| > u_2 \ \ and \ \ |\frac{\xi_3}{\xi_1}| \leq u_3\},$$

$$P^3_{u_1,u_2,u_3} = \{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : |\xi_1| \geq u_1, |\frac{\xi_2}{\xi_1}| \leq u_2 \ \ and \ \ |\frac{\xi_3}{\xi_1}| > u_3\},$$

$$\{M^{(2)}_{a,s} = \begin{pmatrix} a^{\frac{1}{2}} & 0 & 0 \\ a & a^{\frac{1}{2}}s_1 & a^{\frac{1}{2}}s_2 \\ 0 & 0 & a^{\frac{1}{2}} \end{pmatrix} : a > 0, s = (s_1,s_2) \in \mathbb{R}^2\},$$

and

$$\{M^{(3)}_{a,s} = \begin{pmatrix} a^{\frac{1}{2}} & 0 & 0 \\ 0 & a^{\frac{1}{2}} & 0 \\ a & a^{\frac{1}{2}}s_1 & a^{\frac{1}{2}}s_2 \end{pmatrix} : a > 0, s = (s_1,s_2) \in \mathbb{R}^2\}.$$

Example 1.2. [6]. Let $H(x_1,x_2,x_3) = \chi_{x_1>0}(x_1,x_2,x_3)$ be the 3-dimensional Heaviside function. We see that $\frac{\partial}{\partial x_1}H = \delta_1$, where δ_1 is the delta distribution defined by

$$< \delta_1, \phi >= \int\int \phi(0,x_2,x_3)dx_1dx_2$$

where $\phi \in S(\mathbb{R}^3)$ (Schwartz class) and the notation of inner product $<,>$ denote the functional on S.

Now

$$\hat{H}(\xi_1,\xi_2,\xi_3) = (2\pi i\xi_1)^{-1}\hat{\delta}_1(\xi_1,\xi_2,\xi_3),$$

$\hat{\delta}_1$ satisfying $< \hat{\delta}_1, \hat{\phi} >= \int\int \hat{\phi}(\xi_1, 0, 0)d\xi_1$.
The continuous shearlet transform of H is obtained as

$$\begin{aligned} SH^{(1)}_{\varphi} H(a, s, p) &=< H, \varphi_{a,s,p} > \\ &= \int_{\mathbb{R}^3} (2\pi i\xi_1)^{-1}\hat{\delta}_1(\xi)\overline{\hat{\varphi}_{a,s,p}(\xi)}d\xi \\ &= \int_{\mathbb{R}} (2\pi i\xi_1)^{-1}\overline{\hat{\varphi}_{a,s,p}(\xi_1, 0, 0)}d\xi_1 \\ &= \int_{\mathbb{R}} a(2\pi i\xi_1)^{-1}\hat{\varphi}_1(a\xi_1)\overline{\hat{\varphi}_2}(a^{-\frac{1}{2}}s_1)\overline{\hat{\varphi}_2}(a^{-\frac{1}{2}}s_2)e^{2\pi i\xi_1 p_1}d\xi_1 \\ &= a(2\pi i)^{-1}\overline{\hat{\varphi}_2}(a^{-\frac{1}{2}}s_1)\overline{\hat{\varphi}_2}(a^{-\frac{1}{2}}s_2)\int_{\mathbb{R}} \overline{\hat{\varphi}_1}(u)e^{2\pi iu\frac{p_1}{a}}\frac{du}{u}, \end{aligned}$$

where p_1 is the first component of $p \in \mathbb{R}^3$.
Example 1.3. [8]. Let $\Omega \subset \mathbb{R}^3$ be a solid region with smooth boundary surface $S = \partial\Omega$ having non vanishing Gaussian curvature and let $B = \chi_\Omega$. Using divergence theorem, we have

$$\hat{B}(\xi) = \hat{\chi}_S(\xi) = -\frac{1}{2\pi i|\xi|^2}\int_S e^{-2\pi i\xi x}\xi.\vec{n}(x)d\sigma(x),$$

where $\vec{n}$ is the outward normal vector to S at x. By taking $\xi \in \mathbb{R}^3$ and using spherical coordinates as $\xi = \rho\Theta$, where $\rho \in \mathbb{R}^+$ and $\Theta = \Theta(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$, now we can write

$$\hat{B}(\rho, \theta, \phi) = -\frac{1}{2\pi i\rho}\int_S e^{-2\pi i\rho\Theta(\theta,\phi).x}\Theta(\theta, \phi).\vec{n}(x)d\sigma(x).$$

Let $p \in \mathbb{R}^3$. For $\epsilon > 0$, let $\beta_\epsilon(p)$ be the ball with radius ϵ and center p and let $P_\epsilon = S \cap \beta_\epsilon(p)$. Using this notation, we break up above equation into a component close to p and another component away from p as

$$\hat{B}(\rho, \theta, \phi) = T_1(\rho, \theta, \phi) + T_2(\rho, \phi, \theta),$$

where

$$T_1(\rho, \theta, \phi) = -\frac{1}{2\pi i\rho}\int_{P_\epsilon(p)} e^{-2\pi i\rho\Theta(\theta,\phi).x}\Theta(\theta, \phi).\vec{n}(x)d\sigma(x),$$

$$T_2(\rho, \theta, \phi) = -\frac{1}{2\pi i\rho}\int_{S\backslash P_\epsilon(p)} e^{-2\pi i\rho\Theta(\theta,\phi).x}\Theta(\theta, \phi).\vec{n}(x)d\sigma(x).$$

It follows that

$$SH^{(1)}_{\varphi} B(a, s_1, s_2, p) =< B, \varepsilon_{a,s_1,s_2,p} >= I_1(a, s_1, s_2, p) + I_2(a, s_1, s_2, p),$$

where

$$I_1(a, s_1, s_2, p) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} T_1(\rho, \theta, \phi)\overline{\hat{\varphi}_{a,s_1,s_2,p}}(\rho, \theta, \phi)\rho^2 \sin\phi d\rho d\phi d\theta,$$

$$I_2(a, s_1, s_2, p) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} T_2(\rho, \theta, \phi)\overline{\hat{\varphi}_{a,s_1,s_2,p}}(\rho, \theta, \phi)\rho^2 \sin\phi d\rho d\phi d\theta.$$

2. Wavelet Shrinkage

The concept of wavelet shrinkage has been introduced by Donoho and Johnstone [5] and the main idea is to transform the data to reduce the noise in a straight forward way, particularly by diminishing the modulus of the wavelet coefficients. Due to low computational complexity of the wavelet transform this approach became more interesting for applications in signal and image processing. In this approach a nonlinear shrinkage function $S : \mathbb{R}^3 \to \mathbb{R}^3$ is applied to the shearlet transform $Sh^{(1)}_\varphi f(a, s, p)$, with some assumptions on S as $x > 0 \Rightarrow S(x) \geq 0, S(-x) = -S(x)$ and $|S(x)| \leq |x|$. for all $x \in \mathbb{R}^3$. The shrinkage function S depends on a parameter λ^* which determines the amount of shrinkage. Due to simplifications of notations λ^* is omitted here.

A function can be reconstructed from its shearlet transform by the following formula

$$\begin{aligned} u(x) = SH^*_\varphi(SoSH_\varphi f) &= \int_{\mathbb{R}^2}\int_0^\infty (\varphi_{a,s} * S(SH_\varphi f(.,a,s)))\frac{dads}{a^4} \\ &= \int_{\mathbb{R}^2}\int_0^\infty \frac{dads}{a^4}\int_{\mathbb{R}^3} S(SH_\varphi(f)a^{-1}\varphi(M^{(1)}_{a,s})^{-1}(x-p))dp \\ &= \int_{\mathbb{R}^2}\int_0^\infty (\varphi_{a,s} * S(\widetilde{\varphi}_{a,s} * f))\frac{dads}{a^4} \end{aligned}$$

where $SH^*_\varphi f(a, s, p)$ is the adjoint operator the inverse of $SH_\varphi f(a, s, p)$.

3. Continuous Shearlet Transforms and The Sum of Smoothed Partial Derivative Operators

Definition 3.1. [14]. The mother wavelet $\varphi(x)$ has fast decay if for any exponent $r \in \mathbb{N}$ there exists a constant C_r such that for all $x \in \mathbb{R}^3$

$$|\varphi(x)| \leq \frac{C_r}{1 + |x|^r}.$$

Definition 3.2. [14]. The mother wavelet $\varphi(x)$ has d (a finite number $d \in \mathbb{N}$) order vanishing moments if

$$\int_{\mathbb{R}^3} x^k \varphi(x)dx = 0 \texttt{ for } k = (k_1, k_2, k_3) \in \mathbb{N}^3, 0 \leq |k| < d.$$

If the mother wavelet $\varphi(x)$ satisfies Definitions 3.1 and 3.2 we get the following theorem about the shearlet transform and the smoothed partial derivative operators.

Theorem 3.1. *Wavelet $\varphi(x)$ with a fast decay has a vanishing moments if and only if there exist a group of functions $\theta^k(x)(|k| = d)$ with fast decay such that*

$$\varphi(x) = (-1)^d \sum_{|k|=d} \frac{\partial^k \theta^k(x)}{\partial x^k}, x \in \mathbb{R}^3. \tag{3.1}$$

and

$$SH_\varphi f(a, s, p) = a^{2d} \sum_{|k|=d} \frac{\partial^k}{\partial p^k}\left(f * \widetilde{\theta}^k_{a,s}\right)(p)$$

where

$$\widetilde{\theta}^k_{a,s}(x) = a^{-d}\theta^k\left((M^{(1)}_{a,s})^{-1}(-x)\right).$$

It has no more than vanishing moments if and only if

$$\sum_{|k|=d}(-i)^d C_d^k k! \int_{\mathbb{R}^3} \theta^k(x)dx \neq 0. \quad |k| = k_1 + k_2 + k_3 < d.$$

Proof. Following the lines of the proof of Lui, Feng and Li [13, Thm. A] the proof of Theorem 3.1 is immediate. □

4. Three Dimensional Shearlet Shrinkage and Evolution Equation

We shall investigate the relationship between 3-dimensional shearlet shrinkage and an integro-differential evolution equations involving the data on a continuous spectrum of scales. From (3.1), we obtain

$$\widetilde{\varphi}_{a,s} * f = a^{2d} \sum_{|k|=d} \frac{\partial^k}{\partial x^k}(\widetilde{\theta}^k_{a,s} * f) = a^{2d} \left(\sum_{|k|=d} \frac{\partial^k \widetilde{\theta}^k_{a,s}}{\partial x^k} * f \right)$$

and

$$\varphi_{a,s} * f = (-1)^d a^{2d} \sum_{|k|=d} \frac{\partial^k}{\partial x^k}(\widetilde{\theta}^k_{a,s} * f) = (-1)^d a^{2d} \left(\sum_{|k|=d} \frac{\partial^k \widetilde{\theta}^k_{a,s}}{\partial x^k} * f \right).$$

For $n = 3$, shearlet transform $SH_\varphi f(a,s,p)$ and the adjoint transform $SH^*_\varphi f(a,s,p)$ are expressed as

$$\begin{aligned} SH_\varphi f(a,s,p) &= SH_\varphi f(M^{(1)}_{a,s}(p)) \\ &= \widetilde{\varphi}_{a,s} * f = a^{2d} \sum_{|k|=d} \frac{\partial^k}{\partial t^k}(\widetilde{\theta}^k_{a,s} * f), \end{aligned} \tag{4.1}$$

$$SH^*_\varphi f(a,s,p) = \int_{\mathbb{R}^2}\int_0^\infty (\varphi_{a,s} * f)\frac{dads}{a^4} = \int_{\mathbb{R}^2}\int_0^\infty (-1)^d (a)^{2d} \sum_{|k|=d} \frac{\partial^k}{\partial p^k}(\theta^k_{a,s} * f)\frac{dads}{a^4}. \tag{4.2}$$

The equations (4.1) and (4.2) can be summarized as 3-dimensional shearlet transform $SH_\varphi f(a,s,p)$ is equivalent to taking a sum of a smoothed partial derivative with an additional factor a^{2d}, and the adjoint inverse transform $SH^*_\varphi f(a,s,p)$ is an additional integration over all scales a^2.

Taking an unitary function g such that [3]

$$S(x) = x - g(|x|^2)x.$$

In view of (2.1), the above expression is equivalent to

$$u(x) - f(x) = -\int_{\mathbb{R}^2}\int_0^\infty \varphi_{a,s} * (g(|\widetilde{\varphi}_{a,s} * f|^2)(\widetilde{\varphi}_{a,s} * f))\frac{dads}{a^4}.$$

In the consequence of (4.1) and (4.2) the following equivalent expression of shearlet shrinkage in 3-dimension as integro-differential equation is obtain

$$u - f = (-1)^{d+1} \int_{\mathbb{R}^2} \int_0^{\infty} \sum_{|k|=d} \frac{\partial^k \theta^k_{a,s}}{\partial x^k} * \left(g \left(\left| a^{2d} \sum_{|k|=d} \frac{\partial^k \theta^k_{a,s}}{\partial x^k} * f \right|^2 \right) \cdot \left(\sum_{|k|=d} \frac{\partial^k \theta^k_{a,s}}{\partial x^k} * f \right) \right) \frac{dads}{a^4}. \tag{4.3}$$

Similar to the discussion in [4], (4.3) can be understood as a single step time-explicit approximation to the evolution equation:

$$\partial_t u = (-1)^{d+1} \int_{\mathbb{R}^2} \int_0^{\infty} \sum_{|k|=d} \frac{\partial^k \theta^k_{a,s}}{\partial x^k} * \left(g \left(\left| a^{2d} \sum_{|k|=d} \frac{\partial^k \theta^k_{a,s}}{\partial x^k} * f \right|^2 \right) \cdot \left(\sum_{|k|=d} \frac{\partial^k \theta^k_{a,s}}{\partial x^k} * f \right) \right) \frac{dads}{a^4}. \tag{4.4}$$

If we compare (4.4) to the higher order nonlinear diffusion equation for the multivariate function

$$\partial_t u = (-1)^{d+1} \left(\sum_{|k|=d} \frac{\partial^k}{\partial x^k} g \left(\left| a^{2d} \sum_{|k|=d} \frac{\partial^k f}{\partial x^k} \right|^2 \right) \sum_{|k|=d} \frac{\partial^k f}{\partial x^k} \right),$$

we find the following two main differences:

(1) All appearing partial operators in the equation are pre-smoothed by convolution with scaled and mirrored versions of a set of kernel functions θ^k.
(2) The right hand side is not only considered at one single scale but there is an integration over all scales with additional weight factors.

To cover whole space $\mathbb{R}^3$, we define admissible shearlet as:

$$\varphi_{a,s,p} = a^{-1} \varphi^i((M^{(i)})^{-1}(x-p)), i = 2, 3,$$

where $\varphi^i = \varphi o \mathbb{R}^{i-1}$ with

$$\{R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \}.$$

For $i = 2, 3$, all above results can be obtain after a simple modifications.

5. Conclusions

It has been noticed that several techniques have been introduced to study the asymptotic decay at fine scales and singularities for two dimensional shearlet transform. The singularity sets defined on surfaces rather than along curves have additional complexity, therefore, all features do not carry over from two dimensional to three dimensional setting. It is significant to mention here that three dimensional study has more interest for application point of view in medical and seismic imaging. Hence it is reasonable to study the shearlet transform in three dimensional setting.

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