TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 832-845

RECURRENCE RELATIONS FOR MOMENTS OF PROGRESSIVELY TYPE-II CENSORED FROM WEIBULL-RAYLEIGH DISTRIBUTION AND ITS CHARACTERIZATIONS

RASHAD M. EL-SAGHEER[1], NAGWA M. MOHAMED[2], §

Abstract. This paper is devoted to get new recurrence relations satisfied by the single and product moments based on Progressively Type-II Censored (ProgT-II) from the three parameters Weibull-Rayleigh distribution (WRD) and doubly truncated WRD. Finally characterizations of the WRD based on these recurrence relations, hazard rate function and truncated moments are discussed.

Keywords: Weibull-Rayleigh Distribution; Progressively Type-II Censored; Recurrence Relations; Characterization; Hazard Rate Function.

AMS Subject Classification: 47H17; 47H05; 47H09.

1. Introduction

Progressive censoring sampling scheme has a great importance in reliability and life time studies. This scheme enable the experimenters to obtain information about the quality of a product. This information leads the experimenters to reduce the number of failures and the total duration of the experiment. So, it allows them to save time and cost of the life-testing experiment.

The most common right censoring schemes are Type-I and Type-II censoring. This study deals only with ProgT-II censoring, in general a ProgT-II censored sample can be described as follows. Suppose that n independent items are put on a life test with continuous identically distributed failure times $X_1, X_2, ..., X_n$. Suppose further that a censoring scheme $(S_1, S_2, ..., S_r)$ is previously fixed such that immediately following the first failure X_1, S_1 surviving items are removed from the experiment at random, and immediately following the second failure X_2, S_2 surviving items are removed from the experiment at random. This process is repeated until $r-th$ observed failure X_r, the remaining S_r surviving items are removed from the test. The r ordered observed failure times denoted by $X_{1:r:n}^{(S_1,S_2,...,S_r)}$, $X_{2:r:n}^{(S_1,S_2,...,S_r)}$, ..., $X_{r:r:n}^{(S_1,S_2,...,S_r)}$are called ProgT-II right censored order statistics of size r from a sample of size n with progressive censoring scheme $(S_1, S_2, ..., S_r)$. It

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt.
e-mail: Rashadmath@azhar.edu.eg; ORCID: https://orcid.org/0000-0002-7828-3536.
[2] Department of Mathematics, Faculty of Science, Suez University, Suez, Egypt.
e-mail: Nagwa.Mohamed@sci.suezuni.edu.eg; ORCID: https://orcid.org/0000-0001-9800-3231.
§ Manuscript received: June 10, 2020; accepted: October 19, 2020.

is clear that $n = r + \sum_{p=1}^{r} S_p$. It is noted that if $S_1 = S_2 = \cdots = S_r = 0$, then $r = n$, the ProgT-II right censoring scheme reduces to ordinary order statistics (Oos).

The great discussion about ProgT-II censored is presented by Balakrishnan and Aggarwala [1,2,3,4]. Many authors have discussed the recurrence relations under ProgT-II censored using different lifetime distributions, see for example, Athar and Akhter [5], Athar et al. [6], Singh and Khan [7], Balakrishnan and Saleh [8], Balakrishnan et al.[9], Saran et al. [10], Saran and Pande [11], Mohamed et al. [12,13,14] and Mohie El Din et al. [15,16,17]. Hamedani [18,19,20] presented differenent methods to characterize many distributions as exponential distributions, continuous univariate and univariate Continuous Distributions II distributions. Kilany [21] and Nofal and El Gebaly [22] have been characterized Lindley and Pareto distributions. The joint probability density function for ProgT-II censored sample of size r from a sample of size n is given by

$$f_{x_{1:r:n},\ldots,x_{r:r:n}}(x_1, \ldots, x_r) = D_{n,S,\ldots,S_{r-1}} \prod_{p=1}^{r} f(x_p)\left[1 - F(x_p)\right]^{S_p}, \tag{1}$$
$$, -\infty < x_1 < x_2 < \ldots < x_r < \infty,$$

where

$$D_{n,S_1,\ldots,S_{r-1}} = n(n-1-S_1)(n-2-S_1-S_2)\ldots\left(n - \sum_{p=1}^{r-1}(S_p+1)\right). \tag{2}$$

For simplicity,we write $D_{n,S,\ldots,S_{r-1}} = D_{n,S_{r-1}}$; $1 \le r \le n$ and $D_{n,S_o} = n$.

The Weibull-Rayleigh distribution (WRD) was recently given by Afaq et al. [23] as a new generalization of the Rayleigh distribution. They estimated the distribution parameters using maximum likelihood, moments and L-moment methods. They derived the information matrix and use it get the confidence interval estimation of the parameters.

The probability density function (PDF),The cumulative distribution function (CDF), reliability function $M(x)$, and hazard rate function $h(x)$ of the WRD(a, b, c) are given respectively, by

$$f(x) = \frac{a}{b}\frac{x}{c^2}\left(\frac{x^2}{2bc^2}\right)^{a-1} \exp\left[-\left(\frac{x^2}{2bc^2}\right)^{a}\right], \quad a, b, c > 0, \; x > 0, \tag{3}$$

$$F(x) = 1 - \exp\left[-\left(\frac{x^2}{2bc^2}\right)^{a}\right], \tag{4}$$

$$M(x) = \exp\left[-\left(\frac{x^2}{2bc^2}\right)^{a}\right], \tag{5}$$

and

$$h(x) = \frac{a}{b}\frac{x}{c^2}\left(\frac{x^2}{2bc^2}\right)^{a-1}, \tag{6}$$

also, the characterizing differential equation is given by

$$f(x) = \frac{a}{b}\frac{x}{c^2}\left(\frac{x^2}{2bc^2}\right)^{a-1}\left[1 - F(x)\right]. \tag{7}$$

From Eq. (3), we see that if $a = 1$ and $b = 1$,then WRD reduces to Rayleigh distribution with parameter c. In addition, the density function of WRD is unimodal and for fixed a and c, it becomes more and more peaked as the value of b is decreased.

This paper is organized as follows: In Sections (2,3) the recurrence relations for single and product moments of ProgT-II right censored order statistics from WRD are derived. Section 4 is devoted to get recurrence relations for the single and product moments of ProgT-II right censored order statistics from the doubly truncated WRD. Finaly, in Section 5 characterizations of WRD are discussed.

2. Recurrence Relations for the Single Moments

Let $\underline{X} = X_{1:r:n},\ X_{2:r:n}, ..., X_{r:r:n}$ be the ProgT-II censored sample drawn from WRD population whose PDF and CDF are given by (3) and (4), with censoring scheme $(S_1, S_2, ..., S_r)$. The single moments of the ProgT-II can be written as

$$\begin{aligned}\phi_{p:r:n}^{(S_1,S_2,...,.S_r)^{(k)}} &= D_{n,S_{r-1}} \int\int \int\limits_{0<x_1<x_2<...<x_r<\infty} x_p^k f(x_p)\,[1-F(x_p)]^{S_p} \\ &\quad \times f(x_1)\,[1-F(x_1)]^{S_1}\cdots f(x_r)\,[1-F(x_r)]^{S_r}\,dx_1\cdots dx_r, \qquad (8)\end{aligned}$$

where $D_{n,S_{r-1}}$ is defined in (2) and $k \geqslant 1$. The following recurrence relations gives the single moments of ProgT-II censored.

Theorem 2.1. *For* $2 \leq p \leq r-1$ *and* $b, c > 0$,

$$\begin{aligned}&\phi_{p:r:n}^{(S_1,..,S_r)^{(k)}} \\ &= \frac{a}{bc^2\,(k+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[(n-S_1-S_2-\cdots-S_p-p)\,\phi_{p:r-1:n}^{(S_1,...S_{p-1},S_p+S_{p+1}+1,S_{p+2}...,S_r)^{(k+2a)}} \\ &\quad -(n-S_1-S_2-\cdots-S_{p-1}-p+1)\,\phi_{p-1:r-1:n}^{(S_1,S_2,...S_{p-2},S_{p-1}+S_p+1,S_{p+1}...,S_r)^{(k+2a)}} \\ &\quad +(S_p+1)\,\phi_{p:r:n}^{(S_1,...,S_r)^{(k+2a)}}\Big]. \qquad (9)\end{aligned}$$

Proof. Making use of (7) and (8), yields

$$\begin{aligned}\phi_{p:r:n}^{(S_1,S_2,...,S_r)^{(k)}} &= D_{n,S_{r-1}} \int\int \int\limits_{0<x_1<x_2<...<x_{p-1}<x_{p+1}<\cdots x_r<\infty} G\,(x_{p-1}, x_{p+1}) \\ &\quad \times f(x_1)\,[1-F(x_1)]^{S_1}\cdots f(x_{p-1})\,[1-F(x_{p-1})]^{S_{p-1}}\, f(x_{p+1})\,[1-F(x_{p+1})]^{S_{p+1}} \\ &\quad \times\cdots f(x_r)\,[1-F(x_r)]^{S_r}\,dx_1\cdots dx_{p-1}dx_{p+1}\cdots dx_r, \qquad (10)\end{aligned}$$

where

$$G\,(x_{p-1}, x_{p+1}) = \frac{a}{bc^2}\left(\frac{1}{2bc^2}\right)^{a-1}\int\limits_{x_{p-1}}^{x_{p+1}} x_p^{k+2a-1}\,[1-F(x_p)]^{S_p+1}\,dx_p. \qquad (11)$$

Integrating the innermost integral by parts, gives

$$\begin{aligned} G\left(x_{p-1}, x_{p+1}\right) &= \frac{a}{bc^2\left(k+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[x_{p+1}^{k+2a}\left[1-F(x_{p+1})\right]^{S_p+1} \\ &\quad - x_{p-1}^{k+2a}\left[1-F(x_{p-1})\right]^{S_p+1} \\ &\quad + (S_p+1)\int\limits_{x_{p-1}}^{x_{p+1}} x_p^{k+2a}\left[1-F(x_p)\right]^{S_p} f(x_p)dx_p\Big]. \end{aligned} \tag{12}$$

Upon substituting Eq.(12) into Eq.(10), and simplifying the resulting equation, Eq.(9) can be obtained. □

Theorem 2.2. *For* $2 \le r \le n$ *and* $b,\ c > 0$,

$$\begin{aligned} \phi_{1:r:n}^{(S_1,S_2,\ldots,S_r)^{(k)}} &= \frac{a}{bc^2\left(k+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[(n-S_1-1)\phi_{1:r-1:n}^{(S_1+S_2+1,S_3,\ldots,S_r)^{(k+2a)}} \\ &\quad + (S_1+1)\phi_{1:r:n}^{(S_1,S_2,\ldots,S_r)^{(k+2a)}}\Big]. \end{aligned} \tag{13}$$

Proof. The proof is omitted. □

Theorem 2.3. *For* $2 \le r \le n$ *and* $b,\ c > 0$,

$$\begin{aligned} \phi_{r:r:n}^{(S_1,S_2,\ldots,S_r)^{(k)}} &= \frac{a}{bc^2\left(k+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[-(n-S_1-S_2-\cdots-S_{r-1}-r+1) \\ &\quad \times\phi_{r-1:r-1:n}^{(S_1,S_2,\ldots,S_{r-2},S_{r-1}+S_r+1)^{(k+2a)}} + \left(S_r+1\right)\phi_{r:r:n}^{(S_1,S_2,\ldots,S_r)^{(k+2a)}}\Big]. \end{aligned} \tag{14}$$

Proof. The proof is omitted. □

Special Cases:
Setting $S_1 = S_2 = ...S_r = 0$, Eqs.(9,13) and Eq.(14) reduce respectively to

$$\begin{aligned} \phi_{p:n}^{(S_1,..,S_r)^{(k)}} &= \frac{a}{bc^2\left(k+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[\left(n-p\right)\phi_{p:n-1:n}^{(0,,..0,1,0...,0)^{(k+2a)}} \\ &\quad + \left(n-p+1\right)\phi_{p-1:n-1:n}^{(0,..0,1,0,...,0)^{(k+2a)}} \\ &\quad + \phi_{p:n}^{(k+2a)}\Big], \end{aligned} \tag{15}$$

$$\phi_{1:n}^{(k)} = \frac{a}{bc^2\left(k+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[(n-1)\phi_{1:n-1:n}^{(1,0,...,0)^{(k+2a)}} + \phi_{1:n}^{(k+2a)}\right], \tag{16}$$

and

$$\phi_{n:n}^{(k)} = \frac{a}{bc^2\left(k+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[-\left(n-r+1\right)\phi_{n-1:n-1:n}^{(0,...0,1)^{(k+2a)}} + \phi_{n:n}^{(k+2a)}\right]. \tag{17}$$

3. Recurrence Relations for the Product Moments

Theorem 3.1. *For* $1 \le p < q \le r-1,\ r \le n$ *and* $b, c > 0$,

$$\begin{aligned}
&\phi_{p,q:r:n}^{(S_1,\ldots,S_r)^{(k_1,k_2)}}\\
=\ &\frac{a}{bc^2\,(k_2+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[(n-S_1-\ldots S_q-q)\,\phi_{p,q:r-1:n}^{(S_1,\ldots S_{q-1},S_q+S_{q+1}+1,S_{q+2}\ldots,S_r)^{(k_1,k_2+2a)}}\\
&-(n-S_1-\cdots-S_{q-1}-q+1)\,\phi_{p,q-1:r-1:n}^{(S_1,\ldots S_{q-2},S_{q-1}+S_q+1,S_{q+1},\ldots,S_r)^{(k_1,k_2+2a)}}\\
&+(S_q+1)\phi_{p,q:r:n}^{(S_1,\ldots,S_r)^{(k_1,k_2+2a)}}\Big].
\end{aligned} \tag{18}$$

Proof. From (8), we have

$$\begin{aligned}
\phi_{p,q:r:n}^{(S_1,S_2,\ldots,S_r)^{(k_1,k_2)}} &= E\left[X_{p:r:n}^{(S_1,S_2,\ldots,S_r)^{(k_1)}}\left\{X_{q:r:n}^{(S_1,S_2,\ldots,S_r)}\right\}^{(k_2)}\right]\\
&= D_{n,S_{r-1}}\int\cdots\underset{0<x_1<x_2<\ldots.<x_{p-1}<x_{p+1}\ldots<x_r<\infty}{\int}\cdots\int x_p^{k_1}G(x_{q-1},x_{q+1})\\
&\quad\times f(x_1)\left[1-F(x_1)\right]^{S_1}\cdots f(x_{q-1})\left[1-F(x_{q-1})\right]^{S_{q-1}}\\
&\quad\times f(x_{q+1})\left[1-F(x_{q+1})\right]^{S_{q+1}}\\
&\quad\times\cdots f(x_r)\left[1-F(x_r)\right]^{S_r}dx_1dx_2\cdots dx_{q-1}dx_{q+1}\cdots dx_r,
\end{aligned} \tag{19}$$

where

$$G(x_{q-1},x_{q+1}) = \frac{a}{bc^2}\left(\frac{1}{2bc^2}\right)^{a-1}\int_{x_{q-1}}^{x_{q+1}} x_q^{k_2+2a-1}\left[1-F(x_q)\right]^{S_q+1}dx_q. \tag{20}$$

Integrating Eq.(20) by parts, gives

$$\begin{aligned}
G(x_{q-1},x_{q+1}) &= \frac{a}{bc^2\,(k_2+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[x_{q+1}^{k_2+2a}\left[1-F(x_{q+1})\right]^{S_q+1}\\
&\quad -x_{q-1}^{k_2+2a}\left[1-F(x_{q-1})\right]^{S_q+1}\\
&\quad +(S_q+1)\int_{x_{q-1}}^{x_{q+1}} x_q^{k_2+2a}\left[1-F(x_q)\right]^{S_q}f(x_q)dx_q\Big].
\end{aligned} \tag{21}$$

Then substituting Eq.(21) into Eq.(19) and simplifying, Eq.(18) can be deduced. □

Theorem 3.2. *For* $1 \le p \le r-1$ *and* $r \le n$ *and* $b,\ c > 0$,

$$\begin{aligned}
\phi_{p,r:r:n}^{(S_1,S_2,\ldots,S_r)^{(k_1,k_2)}} &= \frac{a}{bc^2\,(k_2+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[-(n-S_1-S_2-\ldots-S_{r-1}-r+1)\\
&\quad\times\phi_{p,r-1:r-1:n}^{(S_1,\ldots S_{r-2},S_{r-1}+S_r+1)^{(k_1,k_2+2a)}}+(S_r+1)\,\phi_{p,r:r:n}^{(S_1,\ldots S_r)^{(k_1,k_2+2a)}}\Big]
\end{aligned} \tag{22}$$

Proof. The proof is omitted. □

Special Cases:
Setting $S_p = 0$, $p = 1, 2, ...r$, Eqs.(18, 22) reduce respectively to

$$\begin{aligned}\phi_{p,q:n}^{(k_1,k_2)} &= \frac{a}{bc^2\left(k_2+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[(n-q)\,\phi_{p,q:n-1:n}^{(0,...0,1,0...,0)^{(k_1,k_2+2a)}} \\ &\quad +(n-q+1)\phi_{p,q-1:n-1:n}^{(0,...,0,1,0,...,0)^{(k_1,k_2+2a)}}+\phi_{p,q:n}^{(k_1,k_2+2a)}\Big], \qquad (23)\end{aligned}$$

and

$$\phi_{p,n:n}^{(k_1,k_2)} = \frac{a}{bc^2\left(k_2+2a\right)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[-\phi_{p,n-1:n-1:n}^{(0,...,0,1)^{(k_1,k_2+2a)}} + \phi_{p,n:n}^{(k_1,k_2+2a)}\right]. \qquad (24)$$

4. The Doubly Truncated Weibull-Rayleigh Distribution

In this section, we present the recurrence relations for the single and product moments of ProgT-II censored order statistics from the doubly truncated WRD. The PDF of the doubly truncated WRD is given by

$$f_t\left(x\right) = \frac{1}{\left(U-V\right)}\frac{a}{b}\frac{x}{c^2}\left(\frac{x^2}{2bc^2}\right)^{a-1}\exp\left[-\left(\frac{x^2}{2bc^2}\right)^{a}\right], \quad 0 < v_1 < x < u_1, ..., a, b, c > 0, x \geq 0, \qquad (25)$$

where

$$U = F_t\left(u_1\right) = 1-\exp\left[-\left(\frac{u_1^2}{2bc^2}\right)^{a}\right],$$

and

$$V = F_t\left(v_1\right) = 1-\exp\left[-\left(\frac{v_1^2}{2bc^2}\right)^{a}\right]. \qquad (26)$$

Here, $1-U$ is the proportion of right truncation on the WRD and V is the proportion of the left truncation.

Thus CDF of doubly truncated WRD can be put in the form

$$F_t(x) = \frac{1}{U-V}\left[\exp\left[-\left(\frac{v_1^2}{2bc^2}\right)^{a}\right]-\exp\left[-\left(\frac{x^2}{2bc^2}\right)^{a}\right]\right]. \qquad (27)$$

The relation between the PDF and CDF can be written as

$$f_t\left(x\right) = \frac{a}{b}\frac{x}{c^2}\left(\frac{x^2}{2bc^2}\right)^{a-1}\left[\frac{1-U}{U-V}+\left[1-F_t\left(x\right)\right]\right]. \qquad (28)$$

4.1. Recurrence relations for the single moments.

Relation 4.1. *For* $2 \leq r \leq n-1$, $S_1 > -1$ *and* b, $c > 0$

$$
\begin{aligned}
&\phi_{1:r:n}^{(S_1,\dots,S_r)^{(k)}}\\
= \quad & \frac{a}{bc^2(k+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[\left(\frac{1-U}{U-V}\right)\left\{\frac{n\,(n-S_1-1)}{(n-1)}\phi_{1:r-1:n-1}^{(S_1+S_2,S_3,\dots,S_r)^{(k+2a)}}\right.\right.\\
&\left.-nv_1^{k+2a}+\frac{nS_1}{(n-1)}\phi_{1:r:n-1}^{(S_1-1,S_2,\dots,S_r)^{(k+2a)}}\right\}+\left\{(n-S_1-1)\,\phi_{1:r-1:n}^{(S_1+S_2+1,S_3,\dots,S_r)^{(k+2a)}}\right.\\
&\left.\left.-nv_1^{k+2a}+(S_1+1)\phi_{1:r:n}^{(S_1,S_2\dots,S_r)^{(k+2a)}}\right\}\right]. \qquad (29)
\end{aligned}
$$

Relation 4.2. *For* $2 \le r \le n-1$, $S_p > -1$ *and* $b,\ c > 0$,

$$
\begin{aligned}
&\phi_{p:r:n}^{(S_1,..,S_r)^{(k)}}\\
= \quad & \frac{a}{bc^2(k+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[\left(\frac{1-U}{U-V}\right)\left\{\frac{D_{n,S_p}}{D_{n-1,S_{p-1}}}\phi_{p:r-1:n-1}^{(S_1,\dots S_{p-1},S_p+S_{p+1},S_{p+2}\dots,S_r)^{(k+2a)}}\right.\right.\\
&\left.-\frac{D_{n,S_{p-1}}}{D_{n-1,S_{p-2}}}\phi_{p:r-1:n-1}^{(S_1,\dots S_{p-2},S_{p-1}+S_p,S_{p+1}\dots,S_r)^{(k+2a)}}+S_p\phi_{p:r:n-1}^{(S_1,\dots,S_p-1,\dots,S_r)^{(k+2a)}}\right\}\\
&+\left\{(n-S_1-\cdots-S_p-p)\,\phi_{p:r-1:n}^{(S_1,\dots S_{p-1}.S_p+S_{p+1}+1,S_{p+2}\dots,S_r)^{(k+2a)}}\right.\\
&-(n-S_1-S_2-\cdots-S_{p-1}-p+1)\,\phi_{p-1:r-1:n}^{(S_1,S_2,\dots S_{p-2},S_{p-1}+S_p+1,\dots,S_r)^{(k+2a)}}\\
&\left.+(S_p+1)\phi_{p:r:n}^{(S_1,\dots,S_r)^{(k+2a)}}\right\}. \qquad (30)
\end{aligned}
$$

Relation 4.3. *For* $2 \le r \le n-1$, $S_r > -1$ *and* $b,\ c > 0$,

$$
\begin{aligned}
&\phi_{r:r:n}^{(S_1,..,S_r)^{(k)}}\\
= \quad & \frac{a}{bc^2(k+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[\left(\frac{1-U}{U-V}\right)\left\{-\frac{D_{n,S_{r-1}}}{D_{n-1,S_{r-2}}}\phi_{r-1:r-1:n-1}^{(S_1,S_2,\dots S_{r-2},S_{r-1}+S_r)^{(k+2a)}}\right.\right.\\
&\left.+S_r\frac{D_{n,S_{r-1}}}{D_{n-1,S_{r-1}}}\phi_{r:r:n-1}^{(S_1,S_2,\dots .S_r-1)^{(k+2a)}}\right\}\\
&\left\{-(n-S_1-\cdots-S_{r-1}-r+1)\phi_{r-1:r-1:n}^{(S_1,S_2,\dots S_{r-2},S_{r-1}+S_r+1)^{(k+2a)}}\right.\\
&\left.\left.+(S_r+1)\phi_{r:r:n}^{(S_1,S_2,\dots,S_r)^{(k+2a)}}\right\}\right]. \qquad (31)
\end{aligned}
$$

Remark 4.1. *Setting* $U = 1$ *and* $V = 0$ *in Relations (29)-(31), Theorems (9), (13) and (14) can be obtained.*

Remark 4.2. *Setting* $S_1 = S_2 = ... = S_r = 0$, $U = 1$ *and* $V = 0$, *the relations (29)-(31) reduce to the corresponding recurrence relations based on the usual order statistics.*

4.2. **Recurrence relations for the product moments.**

Relation 4.4. *For* $1 \le p < q \le r-1$, $r \le n-1$, $S_q \ge 1$ *and* $b,\ c > 0$,

$$
\begin{aligned}
&\phi_{p,q:r:n}^{(S_1,\ldots,S_r)^{(k_1,k_2)}} \\
&= \frac{a}{bc^2(k_2+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[\left(\frac{1-U}{U-V}\right)\left\{\frac{D_{n,S_q}}{D_{n-1,S_{q-1}}}\phi_{p,q:r-1:n-1}^{(S_1,\ldots S_{q-1},S_q+S_{q+1},S_{q+2},\ldots,S_r)^{(k_1,k_2+2a)}}\right.\right. \\
&\left. -\frac{D_{n,S_{q-1}}}{D_{n-1,S_{q-2}}}\phi_{p,q-1:r-1:n-1}^{(S_1,\ldots S_{q-2},S_{q-1}+S_q,S_{q+1}\ldots,S_r)^{(k_1,k_2+2a)}} + S_q\phi_{p,q:r:n-1}^{(S_1,\ldots S_{q-1},S_q-1,S_{q+1}\ldots,S_r)^{(k_1,k_2+2a)}}\right\}\Bigg] \\
&+\Bigg\{(n-S_1-\cdots-S_q-q)\,\phi_{p,q:r-1:n}^{(S_1,\ldots S_{q-1},S_q+S_{q+1}+1,S_{q+2}\ldots,S_r)^{(k_1,k_2+2a)}} \\
&-(n-S_1-\cdots-S_{q-1}-q+1)\,\phi_{p,q-1:r-1:n}^{(S_1,\ldots S_{q-2},S_{q-1}+S_q+1,S_{q+1}\ldots,S_r)^{(k_1,k_2+2a)}} \\
&\left.\left.+(S_q+1)\phi_{p,q:r:n}^{(S_1,\ldots,S_r)^{(k_1,k_2+2a)}}\right\}\right].
\end{aligned}
\tag{32}
$$

Relation 4.5. *For* $1\le p<r\le n-1$, $r\le n-1$, $S_r\ge 1$ *and* $b,c>0$,

$$
\begin{aligned}
&\phi_{p,r:r:n}^{(S_1,S_2,\ldots,S_r)^{(k_1,k_2)}} \\
&= \frac{a}{bc^2(k_2+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\left[\left(\frac{1-U}{U-V}\right)\left\{-\frac{D_{n,S_{r-1}}}{D_{n-1,S_{r-2}}}\phi_{p,r-1:r-1:n-1}^{(S_1,\ldots,S_{r-2},S_{r-1}+S_r)^{(k_1,k_2+2a)}}\right.\right. \\
&\left.+S_r\frac{D_{n,S_{r-1}}}{D_{n-1,S_{r-1}}}\phi_{p,r:r:n-1}^{(S_1,\ldots,S_{r-1},S_r-1)^{(k_1,k_2+2a)}}\right\} \\
&-\Bigg\{(n-S_1-,\ldots-S_{r-1}-r+1)\phi_{p,r-1:r-1:n}^{(S_1,\ldots,S_{r-2},S_{r-1}+S_r+1)^{(k_1,k_2+2a)}} \\
&\left.\left.-(S_r+1)\,\phi_{p,r:r:n}^{(S_1,S_2,\ldots,S_r)^{(k_1,k_2+2a)}}\right\}\right].
\end{aligned}
\tag{33}
$$

Remark 4.3. *Setting* $U=1$ *and* $V=0$ *in Relations (32) and (33), we obtain Theorems (18) and (22).*

Remark 4.4. *Setting* $S_1=S_2=....=S_r=0$, $U=1$ *and* $V=0$, *so the relations (32) and (33) reduce to the corresponding recurrence relations based on the usual order statistics.*

5. Characterizations of Weibull-Rayleigh Distribution

The characterizations of some distributions studied by many authors see for example Hamedani [18,19,20], Kilany [21] and Nofal and El Gebaly [22]. In this section, we discuss the characterizations of WRD based on the the following methods: (i) recurrence relations for the single and product moments ii) hazard rate function and (iii) truncated moments of functions of the random variable.

5.1. Characterization of WRD based on recurrence relation for the single moments.

Theorem 5.1. *Eq. (9) holds if and only if*

$$F(x_p)=1-\exp\left[-\left(\frac{x_p^2}{2bc^2}\right)^a\right]. \tag{34}$$

Proof. The first part follows from Eqs.(7,8). On the other hand if the recurrence relation in Eq.(9) is satisfied, then

$$\begin{aligned}
&\phi_{p:r:n}^{(S_1,..,S_r)^{(k)}} \\
&= \frac{a}{bc^2\,(k+2a)}\left(\frac{1}{2bc^2}\right)^{a-1}\Big[(n-S_1-S_2-\cdots-S_p-p)\,\phi_{p:r-1:n}^{(S_1,\ldots S_{p-1},S_p+S_{p+1}+1,S_{p+2}\ldots,S_r)^{(k+2a)}} \\
&-(n-S_1-S_2-\cdots-S_{p-1}-p+1)\,\phi_{p-1:r-1:n}^{(S_1,S_2,\ldots S_{p-2},S_{p-1}+S_p+1,S_{p+1}\ldots,S_r)^{(k+2a)}} \\
&+(S_p+1)\,\phi_{p:r:n}^{(S_1,\ldots,S_r)^{(k+2a)}}\Big], \qquad (35)
\end{aligned}$$

where

$$\begin{aligned}
&\phi_{p:r:n}^{(S_1,S_2,\ldots,S_r)^{(k+2a)}} \\
= \;& D_{n,S_{r-1}}\int\int\int\limits_{0<x_1<x_2<\ldots<x_{p-1}<x_{p+1}<\cdots x_r<\infty} G\left(x_{p-1},x_{p+1}\right) \\
&\times f(x_1)\left[1-F(x_1)\right]^{S_1}\cdots f(x_{p-1})\left[1-F(x_{p-1})\right]^{S_{p-1}} f(x_{p+1})\left[1-F(x_{p+1})\right]^{S_{p+1}} \\
&\cdots f(x_r)\left[1-F(x_r)\right]^{S_r} dx_1\cdots dx_{p-1}dx_{p+1}\cdots dx_r, \qquad (36)
\end{aligned}$$

and

$$G\left(x_{p-1},x_{p+1}\right)=\int\limits_{x_{p-1}}^{x_{p+1}} x_p^{k+2a}\left[1-F(x_p)\right]^{S_p} f(x_p)dx_p. \qquad (37)$$

Integrating the innermost integral by parts,gives

$$\begin{aligned}
G\left(x_{p-1},x_{p+1}\right) = \; & \frac{1}{(S_p+1)}\Big[x_{p+1}^{k+2a}\left[1-F(x_{p+1})\right]^{S_p+1} \\
& -x_{p-1}^{k+2a}\left[1-F(x_{p-1})\right]^{S_p+1} \\
& +(k+2a)\int\limits_{x_{p-1}}^{x_{p+1}} x_p^{k+2a-1}\left[1-F(x_p)\right]^{S_p+1}dx_p\Big]. \qquad (38)
\end{aligned}$$

Upon substituting Eq.(38) into Eq.(36),and simplifying the resulting equation, yields

$$D_{n,S_{r-1}}\int\int\int\limits_{0<x_1<x_2<\ldots<x_{p-1}<x_{p+1}<\cdots x_r<\infty} x_p^k\left[f(x_p)-\frac{a}{bc^2}\left(\frac{1}{2bc^2}\right)^{a-1}x_p^{2a-1}\left[1-F(x_p)\right]\right]$$

$$\times\left[1-F(x_p)\right]^{S_p} f(x_1)\left[1-F(x_1)\right]^{S_1}\ldots\times f(x_r)\left[1-F(x_r)\right]^{S_r}dx_1dx_2\ldots dx_r=0. \qquad (39)$$

Applying the Muntz-Szasz theorem, see Hwang and Lin [24], so

$$f\left(x_p\right)=\frac{a}{b}\frac{x_p}{c^2}\left(\frac{x_p^2}{2bc^2}\right)^{a-1}\left[1-F(x_p)\right],$$

and this completes the proof. □

5.2. **Characterization of WRD based on recurrence relation for the product moments.**

Theorem 5.2. *Eq. (18) holds if and only if*

$$F\left(x_q\right) = 1 - \exp\left[-\left(\frac{x_q^2}{2bc^2}\right)^a\right]. \tag{40}$$

Proof. The first part is proved using Eqs.(7,8). On the other hand if the recurrence relation in Eq.(18) is satisfied, then

$$\begin{aligned}\phi_{p,q:r:n}^{(S_1,S_2,\ldots,S_r)(k_1,k_2)} &= = D_{n,S_{r-1}} \int \cdots \int\limits_{0<x_1<x_2<\ldots.<x_{p-1}<x_{p+1}<\ldots<x_r<\infty} \cdots \int x_p^{k_1} G(x_{q-1}, x_{q+1}) \\ &\times f(x_1)\left[1-F(x_1)\right]^{S_1} \cdots f(x_{q-1})\left[1-F(x_{q-1})\right]^{S_{p-1}} \\ &\times f(x_{q+1})\left[1-F(x_{q+1})\right]^{S_{q+1}} \ldots \\ &\times \cdots f(x_r)\left[1-F(x_r)\right]^{S_r} dx_1 dx_2 \cdots dx_{q-1} dx_{q+1} \cdots dx_r,\end{aligned} \tag{41}$$

where

$$G(x_{q-1}, x_{q+1}) = \int\limits_{x_{q-1}}^{x_{q+1}} x_q^{k_2+2a} f(x_q)\left[1-F(x_q)\right]^{S_q} dx_q. \tag{42}$$

Integrating Eq.(42) by parts, yeilds

$$\begin{aligned}G(x_{q-1}, x_{q+1}) &= \frac{1}{(S_q+1)}\left[x_{q+1}^{k_2+2a}\left[1-F(x_{q+1})\right]^{S_q+1}\right. \\ &-x_{q-1}^{k_2+2a}\left[1-F(x_{q-1})\right]^{S_q+1} \\ &\left.+(k_2+2a)\int\limits_{x_{q-1}}^{x_{q+1}} x_q^{k_2+2a-1}\left[1-F(x_q)\right]^{S_q+1} dx_q\right].\end{aligned} \tag{43}$$

Then substituting, Eq.(43) into Eq.(41) and simplifying, yields

$$D_{n,S_{r-1}} \int\int \int\limits_{0<x_1<x_2<\ldots\prec x_{p-1}<x_{p+1}<\cdots x_n<\infty} x_p^{k_1} x_q^{k_2}\left[f(x_q) - \frac{a}{bc^2}\left(\frac{1}{2bc^2}\right)^{a-1} x_q^{2a-1}\left[1-F(x_q)\right]\right]$$

$$\times\left[1-F(x_q)\right]^{S_q} f(x_1)\left[1-F(x_1)\right]^{S_1} \ldots \times f(x_r)\left[1-F(x_r)\right]^{S_r} dx_1 dx_2 \ldots dx_r = 0. \tag{44}$$

Applying the Muntz-Szasz theorem, see Hwang and Lin [24] gives

$$f\left(x_q\right) = \frac{a}{b}\frac{x_q}{c^2}\left(\frac{x_q^2}{2bc^2}\right)^{a-1}\left[1-F(x_q)\right]$$

and this completes the proof. □

5.3. **Characterization of WRD based on hazard rate function.** This subsection presents the basic defintion in order to characterize the WRD based on hazard rate function.

Definition 5.1. *Let F be absolutely continuous distribution with the corresponding PDF f. The hazard function corresponding to F is donoted by h_f and its defined by*

$$h_F(x) = \frac{f(x)}{1-F(x)}, \qquad x \in \sup F, \tag{45}$$

where sup F is the support of F. Also, the hazard function of twice differentiable function satisfies the first order differential equation,

$$\frac{h_F^{'}(x)}{h_F(x)} - h_F(x) = q(x),$$

where $q(x)$ is an appropiate integrable function. Also, this differentiable equation has an obvious form,

$$\frac{f^{'}(x)}{f(x)} = \frac{h_F^{'}(x)}{h_F(x)} - h_F(x). \tag{46}$$

Note: Our CDF should be twice differentiable.

Proposition 5.1. *Let $X : \upsilon \to (0, \infty)$ be a continuous random variable. The PDF of X is given by Eq.(3) if and only if its $h_F(x)$ satisfied the following differential equation*

$$xh_F^{'}(x) = \frac{a(2a-1)}{bc^2} x \left(\frac{x^2}{2bc^2}\right)^{a-1}, \qquad 0 < x < \infty, \tag{47}$$

with initial condition

$$h_F(x_o) = \frac{a}{bc^2} x_0 \left(\frac{x_0^2}{2bc^2}\right)^{a-1}.$$

Proof. If X has PDF (3), then cleary Eq.(47) holds. Now, if Eq.(47) holds, then

$$xh_F^{'}(x) = \frac{a(2a-1)}{bc^2} x \left(\frac{x^2}{2bc^2}\right)^{a-1},$$

from which we have

$$\frac{d}{dx}\left[xh_F(x)\right] = \frac{2a^2}{bc^2}\left(\frac{1}{2bc^2}\right)^{a-1} x^{2a-1}. \tag{48}$$

Integrating both sides of Eq.(48) from x_o to x, and using the initial condition on $h_F(x)$, yields

$$h_F(x) = \frac{f(x)}{1-F(x)} = \frac{a}{bc^2}\left(\frac{1}{2bc^2}\right)^{a-1} x^{2a-1}.$$

Now, integrating both sides of the last equation from 0 to x, gives

$$F(x) = 1 - \exp\left[-\left(\frac{x^2}{2bc^2}\right)^{a}\right], \qquad x > 0. \tag{49}$$

□

5.4. **Characterization based on truncated moments of functions of the random variable.** In this subsection, we need to the following proposition which proved by Hamedani [18].

Proposition 5.2. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable with CDF $F(x)$. Let $\Upsilon(x)$ and $r(x)$ be two differentiable functions on $(0, \infty)$ such that $\int\limits_a^b \frac{r'(t)}{[r(t)-\Upsilon(t)]}dt = \infty$. Then*

$$E[\Upsilon(x) \mid X > x] = q(x), \quad a < x < b, \tag{50}$$

implies

$$F(x) = 1 - \exp\left[-\int\limits_a^b \frac{r'(t)}{[r(t) - \Upsilon(t)]}dt\right], \quad a \leq x \leq b\ . \tag{51}$$

Upon using proposition (5.2) we can characterize WRD based on truncated moments of functions of the random variable as follows:

Setting $\Upsilon(x) = \exp\left[-\left(\frac{x^2}{2bc^2}\right)^a\right]$ and $r(x) = \frac{1}{2}\Upsilon(x)$ for $0 < x < \infty$, yields CDF of WRD.

6. Conclusion

In this paper, based on ProgT-II we establish new recurrence relations satisfied by the single and product moments from WRD and doubly truncated WRD. Characterizations of the WRD based on these recurrence relations, hazard rate function and truncated moments are discussed. So, this paper investigates the importance of studying recurrence relations as we can express the higher order moments of order statistics in terms of the lower order moments and hence make the evaluation of higher order moments. In addition, these are useful in checking the computation of the moments of order statistics. Finally, the characterization of distributions can be obtained based on these recurrence relations.

Acknowledgement. The authors would like to extend their gratitude to the referees and the Editor-in-Chief for many valuable suggestions and comments which have helped to improve the paper significantly.

References

[1] Balakrishnan, N. and Aggarwala, R., (2000). Progressive Censoring – Theory, Methods, and Applications, Birkhauser, Boston, MA.

[2] Balakrishnan, N. and Aggarwala, R., (1998). Some properties of progressive censored order statistics from arbitrary and uniform distribution with application to inference and simulation. J. Statist. Plann. Inf., 70 (1), pp. 35–49.

[3] Saran, J., Nain, K. and Bhattacharya, A.P., (2018). Recurrence relations for single and product Moments of Progressive Type-II right censored order statistics from left logistic distribution with application to inference, Int. J. Math. Stat., 19 (1), pp. 113-136.

[4] Pushkarna, N., Saran, J., Nain, J. and Verma, K., (2020). Progressively Type-II right censored order statistics from Hjorth distribution and related inference, Stat. optim. inf. comput., 8 (2), pp. 481-498.

[5] Athar, H. and Akhter, Z., (2016). Recurrence relations between moments of progressive type-II right censored order statistics from doubly truncated Weibull distribution, Stat Pap., 57, pp. 419–428.

[6] Athar, H., Akhter, Z. and Saran, J., (2014). Moments of Progressive Type-II Right Censored Order Statistics from Lindley Distribution, Stat Probab Lett., 3 (1), pp. 1-6.

[7] Singh, B. and Khan, R. U., (2019). Moments of Progressively type-II right censored order statistics from additive Weibull distribution, ProbStat Farum., 12, pp. 36-46.

[8] Balakrishnan, N. and Saleh, H. M., (2013). Recurrence relations for single and product moments of progressively Type-II censored order statistics from a generalized halflogistic distribution with application to inference, J. Stat. Comput. Simul., 83 (9), pp.1704-1721.

[9] Balakrishnan, N., AL-Hussaini, E. K. and Saleh, H. M., (2011). Recurrence relations for moments of progressively censored order statistics from logistic distribution with applications to inference, J. Statist. Plann. Inf., 141 (1), pp. 17-30.

[10] Saran, S., Athar, H. and Abdel-Aty, Y., (2019). Relations Moments of generalized order statistics from power lomax distribution, J. Stat. Appl. Pro. Lett., 6 (1), pp. 29-36.

[11] Saran, J and Pande, V., (2012). Recurrence relations for moments of progressive type-II right censored order statistics from half logistic distribution, J. of Stat. Theory. Appl., 11 (1), pp. 87-96.

[12] Mahmoud, M. A.W., Yhiea, N. M. and Shymaa M. EL-Said., (2014). Recurrence relations for single and product moments of progressive type-II right censored order statistics from doubly truncated exponentiated Pareto distribution, J. Egypt Math. Soc., 22 (3), pp. 496–500.

[13] Mahmoud, M. A.W., Moshref, M. E., Yhiea, N. M. and Mohamed, N. M., (2014). Progressively censored data from the Weibull gamma distribution moments and estimation, J. Stat. Appl. Prob., 3 (1), pp. 45-60.

[14] Mahmoud. M. A.W., EL-Sagheer, R. M. and Mohamed, N. M., (2016). Recurrence relations for moments of progressive type-II right censored from modified Weibull distribution and its characterizations, J. Appl. Statist. Sc., 22 (3-4), pp. 357-374.

[15] Mohie El-Din, M. M., Sadek, A., Marwa M. Mohie. El-Din. and A. M. Sharawy., (2017). Characterization of the generalized Pareto distribution by general progressive type-II right censored order statistics, J. Egypt Math. Soc., 25 (4), pp. 369-374

[16] Mohie El-Din, M. M., Sadek, A. and Marwa M. Mohie. El-Din., (2017). Characterization for the linear failure rate distribution by general progressive type-II right censored order statistics, Amer.J. Theor. Appl. Stat., 6 (3), pp. 129-140.

[17] Mohie El-Din, M. M., Sadek, A. and Marwa M. Mohie. El-Din., (2017). Characterization for Gompertz distribution based on general progressively type-II right censored order statistics, Int. J. Adv. Stat. Prob., 5 (1), pp. 52-56.

[18] Hamedani, G.G., (2013). Characterizations of exponential distributions, Pakistan J. Stat. Oper. Res., 9 (1), pp. 17-24.

[19] Hamedani, G.G., (2010). Characterizations of continuous univariate distributions based on the truncated moments of functions of order statistics, Studia Sci. Math. Hangar, 47, pp. 462-484.

[20] Hamedani, G. G., (2017). Characterizations and Infinite Divisibility of Certain 2016-2017 Univariate Continuous Distributions II, Int. Math. Forum, 12 (12), pp. 565 - 609.

[21] Kilany, N. M., (2017). Characterization of Lindley Distribution Based on Truncated Moments of Order Statistics, J. Stat. Appl. Prob., 6 (2), pp. 355-360.

[22] Nofal, Z. M. and El Gebaly, Y. M., (2017). New Characterizations of the Pareto Distribution, Pakistan J. Stat. Oper. Res., 13 (1), pp. 63-74.

[23] Afaq, A., Ahmad, S. P. and Ahmad, A., (2017). Characterization and estimation of Weibull-Rayleigh distribution with applivation to life time data, Appl. Math. Inf. Sci., 5 (2), pp. 71-79.

[24] Hwang, J. S. and Lin, G. D., (1984). On a generalized moments problems II, Proc. Am. Math. Soc., 91 (4), pp. 577-580.

Rashad M. EL-Sagheer is a lecturer of Mathematical Statistics in Mathematics Department, Faculty of Science AL-Azhar University, Cairo Egypt. He received his B.A. degree in Mathematics from Faculty of Science AL-Azhar University Cairo Egypt in 2004. He received Master's and PhD degrees in Mathematical Statistics from Faculty of Science Sohag University Egypt in 2011 and 2014 respectively. His areas of research include statistical inference, theory of estimation, bayesian inference, order statistics, records, theory of reliability, censored data, life testing and distribution theory, statistical computing and reliability analysis.

Nagwa M. Mohamedis a lecturer of Mathematical Statistics in Mathematics Department, Faculty of Science at Suez University in Egypt. She received her B.A. degree in Mathematics from the Faculty of Science Suez Canal University Egypt in 2007. She received her Master's and PhD degrees in Mathematical Statistics from the same university in 2012. Her areas of research include statistical inference, theory of estimation, bayesian inference, order statistics, records, theory of reliability, censored data, life testing and distribution theory, statistical computing and reliability analysis.