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CERTAIN GENERATING FUNCTIONS INVOLVING THE INCOMPLETE I-FUNCTIONS

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Abstract. In this paper, we have derived a set of generating functions for incomplete I-functions. Bilateral along with linear generating relations are derived for incomplete I-functions. The results obtained are of a general nature, as special cases, the generating relations obtained for the incomplete $\overline{I}$-functions.

Keywords: I-function, Generating functions, Incomplete Gamma functions, Incomplete I-functions, Mellin-Barnes type contour.

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1. Introduction and Definitions

Working with a continuous function is sometimes much easier than working with a sequence, so linear, bilinear, and bilateral generating functions play a significant role in investigating the useful properties of the sequences they generate. Generating functions have useful applications in combinatorial problems, recurrence equations, physics and many fields of study. The extensive application of generating functions involving special functions has attracted the attention of many researchers. There are several methods available in the literature for generating functions involving special functions (see [6, 8, 9, 14, 12, 11]). Motivated by the above work, this paper sets out certain generating functions in the form of linear and bilateral functions involving incomplete I-functions and incomplete $\overline{I}$-functions. First of all, we recall some of the basic definitions available in the literature in order to establish the main results.

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In 1997, Rathie [7] introduced a new function which is known as I-function. The I-function is represented by the following Mellin-Barnes type contour integral

$$I^{a,b}_{c,d}(z) = I^{a,\,b}_{c,\,d}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1),...,(e_c,\rho_c;E_c) \\ (f_1,\sigma_1;F_1),...,(f_d,\sigma_d;F_d) \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \Psi(\xi)\, z^{\xi}\, d\xi, \tag{1}$$

where

$$\Psi(\xi) = \frac{\prod\limits_{j=1}^{a}[\Gamma(f_j-\sigma_j\xi)]^{F_j}\prod\limits_{j=1}^{b}[\Gamma(1-e_j+\rho_j\xi)]^{E_j}}{\prod\limits_{j=a+1}^{d}[\Gamma(1-f_j+\sigma_j\xi)]^{F_j}\prod\limits_{j=b+1}^{c}[\Gamma(e_j-\rho_j\xi)]^{E_j}}. \tag{2}$$

In the above definition, $z \neq 0$, a, b, c, d are positive integers satisfying $0 \leq b \leq c$, $0 \leq a \leq d$ with $\rho_j, E_j(j = 1, \cdots, c)$ and $\sigma_j, F_j(j = 1, \cdots, d) \in \mathbb{R}^+$. Also e_j, $f_j \in \mathbb{C}$ and an empty product is interpreted as unity.

We next recollect and define the lower and upper incomplete gamma functions $\gamma(\xi, z)$ and $\Gamma(\xi, z)$ as follow (see [1, 2, 10]):

$$\gamma(\xi,z) = \int_0^z y^{\xi-1}\, e^{-y}\, dy, \qquad \big(\Re(\xi) > 0;\ z \geq 0\big), \tag{3}$$

and

$$\Gamma(\xi,z) = \int_z^\infty y^{\xi-1}\, e^{-y}\, dy, \qquad \big(z \geq 0;\ \Re(\xi) > 0 \quad \text{if} \quad z = 0\big). \tag{4}$$

These functions fulfill the following relation

$$\gamma(\xi,z) + \Gamma(\xi,z) = \Gamma(\xi), \quad (\Re(\xi) > 0). \tag{5}$$

Using the above defined incomplete gamma functions, Jangid et al. [3] (see also, [4]) discovered and defined the incomplete I-functions ${}^{\gamma}I^{m,n}_{p,q}(z)$ and ${}^{\Gamma}I^{m,n}_{p,q}(z)$ in the following manner:

$${}^{\gamma}I^{a,\,b}_{c,\,d}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),(e_2,\rho_2;E_2),\cdots,(e_c,\rho_c;E_c) \\ (f_1,\sigma_1;F_1),(f_2,\sigma_2;F_2),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \theta(\xi,x)\, z^{\xi}\, d\xi, \tag{6}$$

and

$${}^{\Gamma}I^{a,\,b}_{c,\,d}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),(e_2,\rho_2;E_2),\cdots,(e_c,\rho_c;E_c) \\ (f_1,\sigma_1;F_1),(f_2,\sigma_2;F_2),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(\xi,x)\, z^{\xi}\, d\xi, \tag{7}$$

for all $z \neq 0$, where

$$\theta(\xi,x) = \frac{[\gamma(1-e_1+\rho_1\xi,x)]^{E_1}\prod\limits_{j=1}^{a}[\Gamma(f_j-\sigma_j\xi)]^{F_j}\prod\limits_{j=2}^{b}[\Gamma(1-e_j+\rho_j\xi)]^{E_j}}{\prod\limits_{j=a+1}^{d}[\Gamma(1-f_j+\sigma_j\xi)]^{F_j}\prod\limits_{j=b+1}^{c}[\Gamma(e_j-\rho_j\xi)]^{E_j}}, \tag{8}$$

and

$$\Theta(\xi,x) = \frac{[\Gamma(1-e_1+\rho_1\xi,x)]^{E_1}\prod\limits_{j=1}^{a}[\Gamma(f_j-\sigma_j\xi)]^{F_j}\prod\limits_{j=2}^{b}[\Gamma(1-e_j+\rho_j\xi)]^{E_j}}{\prod\limits_{j=a+1}^{d}[\Gamma(1-f_j+\sigma_j\xi)]^{F_j}\prod\limits_{j=b+1}^{c}[\Gamma(e_j-\rho_j\xi)]^{E_j}}. \tag{9}$$

For $E_1 = 1$, the definitions (6) and (7) at once yield the following relation

$$ {}^{\gamma}I_{c,d}^{a,b}(z) + {}^{\Gamma}I_{c,d}^{a,b}(z) = I_{c,d}^{a,b}(z), \ (E_1 = 1), \tag{10}$$

where $I_{c,d}^{a,b}(z)$ is defined in (1).

The incomplete I-functions defined in (6) and (7) exist for all $x \geq 0$, under the set of conditions given by Rathie [7], with

$$\Delta > 0, |\arg (z)| < \Delta \frac{\pi}{2},$$

where

$$\Delta = \sum_{j=1}^{a} F_j\sigma_j - \sum_{j=a+1}^{d} F_j\sigma_j + \sum_{j=1}^{b} E_j\rho_j - \sum_{j=b+1}^{c} E_j\rho_j.$$

2. Main Results

In this section, we established certain new generating functions for the incomplete I-functions.

Theorem 2.1. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \leq b \leq c,\ 0 \leq a \leq d,\ \rho_j, E_j\ (j = 1, \cdots, c)$, $\sigma_j, F_j\ (1, \cdots, d) \in \mathbb{R}^+$ and $e_j,\ f_j,\ \lambda \in \mathbb{C}$, then the following linear generating relation holds:*

$$\sum_{l=0}^{\infty} \binom{\lambda+l-1}{l} {}^{\Gamma}I_{c,\,d}^{a,\,b}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),(e_j,\rho_j;E_j)_{2,c-1},(\lambda+l,0;1) \\ (\lambda+l,1;1),(f_j,\sigma_j;F_j)_{2,d} \end{matrix}\right] y^l$$
$$= (1-y)^{-\lambda}\, {}^{\Gamma}I_{c,\,d}^{a,\,b}\left[z(1-y) \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),(e_j,\rho_j;E_j)_{2,c-1},(\lambda,0;1) \\ (\lambda,1;1),(f_j,\sigma_j;F_j)_{2,d} \end{matrix}\right]. \tag{11}$$

Proof. To prove the result, let us take the left-hand side of (11)

$$LHS = \sum_{l=0}^{\infty} \binom{\lambda+l-1}{l} {}^{\Gamma}I_{c,\,d}^{a,\,b}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),(e_j,\rho_j;E_j)_{2,c-1},(\lambda+l,0;1) \\ (\lambda+l,1;1),(f_j,\sigma_j;F_j)_{2,d} \end{matrix}\right] y^l.$$

Write incomplete I-function in terms of Mellin-Barnes type contour integral, we get

$$LHS = \sum_{l=0}^{\infty} \binom{\lambda+l-1}{l} \left(\frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(\xi,x) z^{\xi}\, d\xi\right) y^l.$$

Now, change the order of the summation and integration, then after some calculation, we have

$$LHS = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{[\Gamma(1-e_1+\rho_1\xi,x)]^{E_1}\prod_{j=1}^{a}[\Gamma(f_j-\sigma_j\xi)]^{F_j}\prod_{j=2}^{b}[\Gamma(1-e_j+\rho_j\xi)]^{E_j}}{\prod_{j=a+1}^{d}[\Gamma(1-f_j+\sigma_j\xi)]^{F_j}\prod_{j=b+1}^{c-1}[\Gamma(e_j-\rho_j\xi)]^{E_j}}\, z^{\xi}$$
$$\times \sum_{l=0}^{\infty} \binom{\lambda+l-\xi-1}{l} \frac{\Gamma(\lambda-\xi)}{\Gamma(\lambda)}\, y^l\, d\xi$$

Using the following generalized binomial expansion formula;

$$\sum_{k=0}^{\infty}\binom{s+k-1}{k} x^k = (1-x)^s$$

We have

$$LHS = \frac{(1-y)^{-\lambda}}{2\pi i}\int_{\mathcal{L}} \frac{[\Gamma(1-e_1+\rho_1\xi, x)]^{E_1}\prod_{j=1}^{a}[\Gamma(f_j-\sigma_j\xi)]^{F_j}\prod_{j=2}^{b}[\Gamma(1-e_j+\rho_j\xi)]^{E_j}}{\prod_{j=a+1}^{d}[\Gamma(1-f_j+\sigma_j\xi)]^{F_j}\prod_{j=b+1}^{c-1}[\Gamma(e_j-\rho_j\xi)]^{E_j}}$$
$$\times \frac{\Gamma(\lambda-\xi)}{\Gamma(\lambda)}\left(z(1-y)\right)^{\xi}\, d\xi.$$

Finally, using (7) we obtain the right-hand side of the assertion (11) of Theorem 2.1. □

Continuing to follow the related Theorem 2.1 process and then using the (6) definition, we get the following theorem concerning the lower incomplete I-functions:

Theorem 2.2. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \le\ b \le\ c,\ 0 \le\ a \le\ d,\ \rho_j, E_j\,(j=1,\cdots,c)$, $\sigma_j, F_j\,(1,\cdots,d) \in \mathbb{R}^+$ and $e_j,\ f_j,\ \lambda \in \mathbb{C}$, then the following linear generating relation holds:*

$$\sum_{l=0}^{\infty}\binom{\lambda+l-1}{l}\, {}^{\gamma}I^{a,\,b}_{c,\,d}\left[z\,\middle|\,\begin{matrix}(e_1,\rho_1;E_1:x),(e_j,\rho_j;E_j)_{2,c-1},(\lambda+l,0;1)\\(\lambda+l,1;1),(f_j,\sigma_j;F_j)_{2,d}\end{matrix}\right] y^l$$
$$= (1-y)^{-\lambda}\, {}^{\gamma}I^{a,\,b}_{c,\,d}\left[z(1-y)\,\middle|\,\begin{matrix}(e_1,\rho_1;E_1:x),(e_j,\rho_j;E_j)_{2,c-1},(\lambda,0;1)\\(\lambda,1;1),(f_j,\sigma_j;F_j)_{2,d}\end{matrix}\right]. \tag{12}$$

Theorem 2.3. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \le\ b \le\ c,\ 0 \le\ a \le\ d,\ \rho_j, E_j\,(j=1,\cdots,c)$, $\sigma_j, F_j\,(1,\cdots,d) \in \mathbb{R}^+$ and $\alpha,\ \beta,\ e_j,\ f_j \in \mathbb{C}$, then the following bilateral generating relation holds:*

$$\sum_{n=0}^{\infty} P^{(\alpha,\,\beta)}_{n,\,m}(x,y)\frac{t^n}{n!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m]$$
$$\times\ {}^{\Gamma}I^{a,\,b}_{c,\,d}\left[y(1+\zeta)^{\epsilon}\,\middle|\,\begin{matrix}(e_1,\rho_1;E_1:z),(e_j,\rho_j;E_j)_{2,c}\\(f_j,\sigma_j;F_j)_{1,d}\end{matrix}\right], \tag{13}$$

where, $\zeta = t(1+\zeta)^{\beta+1}$, $\Phi[z] = \sum_{n=0}^{\infty}\Omega_n\, z^n$, $\{\Omega_n\}_{n=0}^{\infty}$ is an arbitrary complex sequence and

$$P^{(\alpha,\,\beta)}_{n,\,m}(x,y) = \sum_{k=0}^{[n/m]}(-n)_{mk}\,\Omega_k\, x^k$$
$$\times\ {}^{\Gamma}I^{a,b+1}_{c+1,d+1}\left[y\,\middle|\,\begin{matrix}(e_1,\rho_1;E_1:z),(-\alpha-(\beta+1)n,\epsilon;1),(e_j,\rho_j;E_j)_{2,c}\\(f_j,\sigma_j;F_j)_{1,d},(-\alpha-\beta n-mk,\epsilon;1)\end{matrix}\right].$$

Proof. To prove the result, we first use the Mellin-Barnes type contour integral form of the incomplete I-function defined in (7), we have

$$P_{n,\,m}^{(\alpha,\,\beta)}(x,y) = \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(\xi,z) y^{\xi} \left\{ \sum_{k=0}^{[n/m]} \frac{\Gamma(1+\alpha+(\beta+1)n+\epsilon\xi)}{\Gamma(1+\alpha+\beta n+mk+\epsilon\xi)} (-n)_{mk}\, \Omega_k\, x^k \right\} d\xi, \tag{14}$$

where, $\Theta(\xi,z)$ is given in (9).

Putting (14) into the LHS of the bilateral generating relation (13), and after changing the order of the summation and integration, we obtain

$$I = \sum_{n=0}^{\infty} P_{n,\,m}^{(\alpha,\,\beta)}(x,y)\frac{t^n}{n!} = \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(\xi,z) y^{\xi} \times \left\{ \sum_{n=0}^{\infty} \binom{\alpha+\epsilon\xi+(\beta+1)n}{n} t^n \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}\,\Omega_k\, x^k}{(1+\alpha+\epsilon\xi+\beta n)_{mk}} \right\} d\xi. \tag{15}$$

Now, use the following result given by Srivastava [8]

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}\, C_k}{(1+\alpha+\beta n)_{mk}} \frac{x^k}{k!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} \sum_{k=0}^{\infty} C_k \frac{[x(-\zeta)^m]^k}{k!},$$

with $\alpha \to \alpha+\epsilon\xi$ and $C_k \to k!\,\Omega_k, k \geq 0$, using (15) we get

$$I = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} \sum_{k=0}^{\infty} \Omega_k\, x^k\, (-\zeta)^{mk} \times \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(\xi,z) y^{\xi} (1+\zeta)^{\epsilon\xi} d\xi, \tag{16}$$

where $\zeta = t(1+\zeta)^{\beta+1}$.

At last, use the definition (7) and $\Phi[z]$, we get the desired result (13). $\square$

Theorem 2.4. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \leq\ b \leq\ c,\ 0 \leq\ a \leq\ d,\ \rho_j, E_j\, (j = 1,\cdots,c)$, $\sigma_j, F_j\, (1,\cdots,d) \in \mathbb{R}^+$ and $\alpha,\ \beta,\ e_j,\ f_j, \in \mathbb{C}$, then the following bilateral generating relation holds:*

$$\sum_{n=0}^{\infty} P_{n,\,m}^{(\alpha,\,\beta)}(x,y)\frac{t^n}{n!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m] \times {}^{\gamma}I_{c,\,d}^{a,\,b}\left[y(1+\zeta)^{\epsilon} \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:z),(e_j,\rho_j;E_j)_{2,c} \\ (f_j,\sigma_j;F_j)_{1,d} \end{matrix} \right], \tag{17}$$

where, $\zeta = t(1+\zeta)^{\beta+1}$, $\Phi[z] = \sum\limits_{n=0}^{\infty} \Omega_n\, z^n$, $\{\Omega_n\}_{n=0}^{\infty}$ is an arbitrary complex sequence and

$$P_{n,\,m}^{(\alpha,\,\beta)}(x,y) = \sum_{k=0}^{[n/m]} (-n)_{mk}\,\Omega_k\, x^k \times {}^{\gamma}I_{c+1,d+1}^{a,b+1}\left[y \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:z),(-\alpha-(\beta+1)n,\epsilon;1),(e_j,\rho_j;E_j)_{2,c} \\ (f_j,\sigma_j;F_j)_{1,d},(-\alpha-\beta n-mk,\epsilon;1) \end{matrix} \right].$$

Theorem 2.5. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \le b \le c,\ 0 \le a \le d,\ \rho_j, E_j\,(j = 1, \cdots, c)$, $\sigma_j, F_j\,(1, \cdots, d) \in \mathbb{R}^+$ and $\alpha,\ \beta,\ e_j,\ f_j,\ \omega \in \mathbb{C}$, then the following bilateral generating relation holds:*

$$\sum_{n=0}^{\infty} Q_{n,m}^{(\alpha,\beta)}(\omega; x, y)\frac{t^n}{n!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m(1+\zeta)^{\omega}] \times {}^{\Gamma}I_{c,\,d}^{a,\,b}\left[y(1+\zeta)^{\epsilon}\ \middle|\ \begin{matrix}(e_1,\rho_1;E_1:z),(e_j,\rho_j;E_j)_{2,c}\\(f_j,\sigma_j;F_j)_{1,d}\end{matrix}\right], \tag{18}$$

where, $\zeta = t(1+\zeta)^{\beta+1}$, $\Phi[z] = \sum\limits_{n=0}^{\infty}\Omega_n\, z^n$, $\{\Omega_n\}_{n=0}^{\infty}$ is an arbitrary complex sequence and

$$Q_{n,\,m}^{(\alpha,\,\beta)}(\omega; x, y) = \sum_{k=0}^{[n/m]}(-n)_{mk}\,\Omega_k\, x^k \times {}^{\Gamma}I_{c+1,d+1}^{a,b+1}\left[y\ \middle|\ \begin{matrix}(e_1,\rho_1;E_1:z),(-\alpha-(\beta+1)n-\omega k,\epsilon;1),(e_j,\rho_j;E_j)_{2,c}\\(f_j,\sigma_j;F_j)_{1,d},(-\alpha-\beta n-(\omega+m)k,\epsilon;1)\end{matrix}\right].$$

Proof. To prove the result, we first use the Mellin-Barnes type contour integral form of the incomplete I-function defined in (7), we have

$$Q_{n,m}^{(\alpha,\beta)}(\omega; x, y) = \frac{1}{2\pi i}\int_{\mathcal{L}}\Theta(\xi,z)y^{\xi}\left\{\sum_{k=0}^{[n/m]}\frac{\Gamma(1+\alpha+(\beta+1)n+\omega k+\epsilon\xi)}{\Gamma(1+\alpha+\beta n+(\omega+m)k+\epsilon\xi)}(-n)_{mk}\,\Omega_k\, x^k\right\}d\xi, \tag{19}$$

where, $\Theta(\xi, z)$ is given in (9).

Putting (19) into the LHS of the bilateral generating relation (18), and after changing the order of the summation and integration, we obtain

$$I = \sum_{n=0}^{\infty}Q_{n,\,m}^{(\alpha,\,\beta)}(\omega; x, y)\frac{t^n}{n!} = \frac{1}{2\pi i}\int_{\mathcal{L}}\Theta(\xi,z)y^{\xi} \times \left\{\sum_{n=0}^{\infty}\binom{\alpha+\epsilon\xi+(\beta+1)n}{n}t^n\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(1+\alpha+\epsilon\xi+(\beta+1)n)_{\omega k}\,\Omega_k\, x^k}{(1+\alpha+\epsilon\xi+\beta n)_{(\omega+m)k}}\right\}d\xi. \tag{20}$$

Now, use the following result given by Srivastava and Buschman [9],

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}t^n\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(1+\alpha+(\beta+1)n)_{\omega k}}{(1+\alpha+\beta n)_{(\omega+m)k}}\Omega_k\, x^k = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m(1+\zeta)^{\omega}], \tag{21}$$

with $\alpha \to \alpha + \epsilon\xi$, we have from (20)

$$I = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m(1+\zeta)^{\omega}] \times \frac{1}{2\pi i}\int_{\mathcal{L}}\Theta(\xi,z)y^{\xi}(1+\zeta)^{\epsilon\xi}d\xi, \tag{22}$$

where, $\zeta = t(1+\zeta)^{\beta+1}$ and $\Theta(\xi, z)$ is defined in (9).

At last, use the definition (7) and $\Phi[z]$, we reached at the desired bilateral generating relation (18). □

Theorem 2.6. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \le b \le c,\ 0 \le a \le d,\ \rho_j, E_j\,(j = 1, \cdots, c)$, $\sigma_j, F_j\,(1, \cdots, d) \in \mathbb{R}^+$ and $\alpha,\ \beta,\ e_j,\ f_j,\ \omega \in \mathbb{C}$, then the following bilateral generating relation holds:*

$$\sum_{n=0}^{\infty} Q_{n,m}^{(\alpha,\beta)}(\omega; x, y)\frac{t^n}{n!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m(1+\zeta)^{\omega}] \times {}^{\gamma}I_{c,\,d}^{a,\,b}\left[y(1+\zeta)^{\epsilon} \,\middle|\, \begin{matrix} (e_1, \rho_1; E_1 : z), (e_j, \rho_j; E_j)_{2,c} \\ (f_j, \sigma_j; F_j)_{1,d} \end{matrix} \right], \tag{23}$$

where, $\zeta = t(1+\zeta)^{\beta+1}$, $\Phi[z] = \sum\limits_{n=0}^{\infty} \Omega_n\, z^n$, $\{\Omega_n\}_{n=0}^{\infty}$ *is an arbitrary complex sequence and*

$$Q_{n,\,m}^{(\alpha,\,\beta)}(\omega; x, y) = \sum_{k=0}^{[n/m]} (-n)_{mk}\,\Omega_k\, x^k \times {}^{\gamma}I_{c+1,d+1}^{a,b+1}\left[y \,\middle|\, \begin{matrix} (e_1, \rho_1; E_1 : z), (-\alpha-(\beta+1)n-\omega k, \epsilon; 1), (e_j, \rho_j; E_j)_{2,c} \\ (f_j, \sigma_j; F_j)_{1,d}, (-\alpha-\beta n-(\omega+m)k, \epsilon; 1) \end{matrix} \right].$$

Remark 2.1. *If we set $\omega = 0$ in* (18) *and* (23) *we get the relations* (13) *and* (17), *respectively.*

Remark 2.2. *If we set $\alpha = \lambda - 1, \beta = 0$ and $\zeta = t/(1-t)$ in the relations* (13) *and* (17), *respectively, we get the below Theorems 2.7 and 2.8.*

Theorem 2.7. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \le\ b \le\ c,\ 0 \le\ a \le\ d,\ \rho_j, E_j\,(j = 1, \cdots, c)$, $\sigma_j, F_j\,(1, \cdots, d) \in \mathbb{R}^+$ and $\lambda,\ e_j,\ f_j \in \mathbb{C}$, then the following bilateral generating relation holds:*

$$\sum_{n=0}^{\infty} \sigma_n^m(x)\, {}^{\Gamma}I_{c+1,\,d}^{a,\,b+1}\left[y \,\middle|\, \begin{matrix} (e_1, \rho_1; E_1 : z), (1-\lambda-n, \epsilon; 1), (e_j, \rho_j; E_j)_{2,c} \\ (f_j, \sigma_j; F_j)_{1,d} \end{matrix} \right]\frac{t^n}{n!}$$
$$= (1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{\Omega_k}{(mk)!}\left[x\left(\frac{t}{1-t}\right)^m\right]^k \times {}^{\Gamma}I_{c+1,\,d}^{a,\,b+1}\left[\frac{y}{(1-t)^{\epsilon}} \,\middle|\, \begin{matrix} (e_1, \rho_1; E_1 : z), (1-\lambda-mk, \epsilon; 1), (e_j, \rho_j; E_j)_{2,c} \\ (f_j, \sigma_j; F_j)_{1,d} \end{matrix} \right], \tag{24}$$

where, $\sigma_n^m(x) = \sum\limits_{k=0}^{[n/m]} \binom{n}{mk}\,\Omega_k\, x^k$ *and* $\{\Omega_k\}_{n=0}^{\infty}$ *is an arbitrary complex sequence.*

Theorem 2.8. *If $a,\ b,\ c,\ d \in \mathbb{N}_0$ with $0 \le\ b \le\ c,\ 0 \le\ a \le\ d,\ \rho_j, E_j\,(j = 1, \cdots, c)$, $\sigma_j, F_j\,(1, \cdots, d) \in \mathbb{R}^+$ and $\lambda,\ e_j,\ f_j \in \mathbb{C}$, then the following bilateral generating relation holds:*

$$\sum_{n=0}^{\infty} \sigma_n^m(x)\, {}^{\gamma}I_{c+1,\,d}^{a,\,b+1}\left[y \,\middle|\, \begin{matrix} (e_1, \rho_1; E_1 : z), (1-\lambda-n, \epsilon; 1), (e_j, \rho_j; E_j)_{2,c} \\ (f_j, \sigma_j; F_j)_{1,d} \end{matrix} \right]\frac{t^n}{n!}$$
$$= (1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{\Omega_k}{(mk)!}\left[x\left(\frac{t}{1-t}\right)^m\right]^k \times {}^{\gamma}I_{c+1,\,d}^{a,\,b+1}\left[\frac{y}{(1-t)^{\epsilon}} \,\middle|\, \begin{matrix} (e_1, \rho_1; E_1 : z), (1-\lambda-mk, \epsilon; 1), (e_j, \rho_j; E_j)_{2,c} \\ (f_j, \sigma_j; F_j)_{1,d} \end{matrix} \right], \tag{25}$$

where, $\sigma_n^m(x) = \sum\limits_{k=0}^{[n/m]} \binom{n}{mk} \Omega_k\, x^k$ *and* $\{\Omega_k\}_{n=0}^{\infty}$ *is an arbitrary complex sequence.*

3. Special Cases

In this section, we derive the generating functions for the incomplete $\overline{I}$-functions.

If we set $F_j\,(j=1,\cdots,a)=1$ in the definitions (6) and (7), respectively, we obtain the incomplete $\overline{I}$-functions ${}^{\gamma}\overline{I}^{a,\,b}_{c,\,d}(z)$ and ${}^{\Gamma}\overline{I}^{a,\,b}_{c,\,d}(z)$ in the following manner

$$ {}^{\gamma}\overline{I}\,^{a,\,b}_{c,\,d}(z) = {}^{\gamma}\overline{I}\,^{a,\,b}_{c,\,d}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),\cdots,(e_b,\rho_b;E_b), \\ (f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1), \end{matrix} \begin{matrix} (e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c) \\ (f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right], \tag{26} $$

and

$$ {}^{\Gamma}\overline{I}\,^{a,\,b}_{c,\,d}(z) = {}^{\Gamma}\overline{I}\,^{a,\,b}_{c,\,d}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),\cdots,(e_b,\rho_b;E_b), \\ (f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1), \end{matrix} \begin{matrix} (e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c) \\ (f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right]. \tag{27} $$

It is interesting to note that

$$ {}^{\gamma}\overline{I}\,^{a,\,b}_{c,\,d}(z) + {}^{\Gamma}\overline{I}\,^{a,\,b}_{c,\,d}(z) = \overline{I}\,^{a,\,b}_{c,\,d}(z), \tag{28} $$

where, $\overline{I}\,^{a,\,b}_{c,\,d}(z)$ denotes the familiar $\overline{I}$-function given by Rathie[7].

Now, if we set $F_j\,(j=1,\cdots,a)=1$ in Theorem 2.1, Theorem 2.3, Theorem 2.5 and Theorem 2.7 then we obtain the linear and bilateral generating relations involving incomplete ${}^{\Gamma}\overline{I}\,^{a,b}_{c,d}(z)$ function and given as corollaries:

Corollary 3.1. *If* $a,\ b,\ c,\ d \in \mathbb{N}_0$ *with* $0 \le\ b \le\ c,\ 0 \le\ a \le\ d,\ \rho_j, E_j\,(j=1,\cdots,c)$, $\sigma_j, F_j\,(1,\cdots,d) \in \mathbb{R}^+$ *and* $e_j,\ f_j,\ \lambda \in \mathbb{C}$, *then the following linear generating relation holds:*

$$ \sum_{l=0}^{\infty} \binom{\lambda+l-1}{l}\, {}^{\Gamma}\overline{I}\,^{a,\,b}_{c,\,d}\left[z \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b), \\ (\lambda+l,1;1),(f_2,\sigma_2;1),\cdots,(f_a,\sigma_a;1), \end{matrix} \begin{matrix} (e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_{c-1},\rho_{c-1};E_{c-1}),(\lambda+l,0;1) \\ (f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right] y^l $$
$$ = (1-y)^{-\lambda}\, {}^{\Gamma}\overline{I}\,^{a,\,b}_{c,\,d}\left[z(1-y) \,\middle|\, \begin{matrix} (e_1,\rho_1;E_1:x),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b), \\ (\lambda,1;1),(f_2,\sigma_2;1),\cdots,(f_a,\sigma_a;1), \end{matrix} \begin{matrix} (e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_{c-1},\rho_{c-1};E_{c-1}),(\lambda,0;1) \\ (f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right]. \tag{29} $$

Corollary 3.2. *If* $a,\ b,\ c,\ d \in \mathbb{N}_0$ *with* $0 \le\ b \le\ c,\ 0 \le\ a \le\ d,\ \rho_j, E_j\,(j=1,\cdots,c)$, $\sigma_j, F_j\,(1,\cdots,d) \in \mathbb{R}^+$ *and* $\alpha,\ \beta,\ e_j,\ f_j \in \mathbb{C}$, *then the following bilateral generating relation* *holds:*

$$\sum_{n=0}^{\infty} P_{n,\,m}^{(\alpha,\,\beta)}(x,y)\frac{t^n}{n!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m] \times {}^{\Gamma}\overline{I}\,{}_{c,\,d}^{a,\,b}\left[y(1+\zeta)^{\epsilon}\left|\begin{matrix}(e_1,\rho_1;E_1:x),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b),\\(f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1),\end{matrix}\right.\right. \left.\begin{matrix}(e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c)\\(f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d)\end{matrix}\right], \tag{30}$$

where, $\zeta = t(1+\zeta)^{\beta+1}$, $\Phi[z] = \sum\limits_{n=0}^{\infty}\Omega_n\, z^n$, $\{\Omega_n\}_{n=0}^{\infty}$ *is an arbitrary complex sequence and*

$$P_{n,\,m}^{(\alpha,\,\beta)}(x,y) = \sum_{k=0}^{[n/m]}(-n)_{mk}\,\Omega_k\,x^k \times {}^{\Gamma}\overline{I}\,{}_{c+1,\,d+1}^{a,\,b+1}\left[y\left|\begin{matrix}(e_1,\rho_1;E_1:x),(-\alpha-(\beta+1)n,\epsilon;1),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b),\\(f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1),\end{matrix}\right.\right. \left.\begin{matrix}(e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c)\\(f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d),(-\alpha-\beta n-mk,\epsilon;1)\end{matrix}\right].$$

Corollary 3.3. *If* $a,\ b,\ c,\ d \in \mathbb{N}_0$ *with* $0 \le b \le c,\ 0 \le a \le d,\ \rho_j, E_j\,(j=1,\cdots,c)$, $\sigma_j, F_j\,(1,\cdots,d) \in \mathbb{R}^+$ *and* $\alpha,\ \beta,\ e_j,\ f_j,\ \omega \in \mathbb{C}$, *then the following bilateral generating relation holds:*

$$\sum_{n=0}^{\infty} Q_{n,m}^{(\alpha,\beta)}(\omega;x,y)\frac{t^n}{n!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi[x(-\zeta)^m(1+\zeta)^{\omega}] \times {}^{\Gamma}\overline{I}\,{}_{c,\,d}^{a,\,b}\left[y(1+\zeta)^{\epsilon}\left|\begin{matrix}(e_1,\rho_1;E_1:z),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b),\\(f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1),\end{matrix}\right.\right. \left.\begin{matrix}(e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c)\\(f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d)\end{matrix}\right], \tag{31}$$

where, $\zeta = t(1+\zeta)^{\beta+1}$, $\Phi[z] = \sum\limits_{n=0}^{\infty}\Omega_n\, z^n$, $\{\Omega_n\}_{n=0}^{\infty}$ *is an arbitrary complex sequence and*

$$Q_{n,\,m}^{(\alpha,\,\beta)}(\omega;x,y) = \sum_{k=0}^{[n/m]}(-n)_{mk}\,\Omega_k\,x^k \times {}^{\Gamma}\overline{I}\,{}_{c+1,\,d+1}^{a,\,b+1}\left[y(1+\zeta)^{\epsilon}\left|\begin{matrix}(e_1,\rho_1;E_1:z),(-\alpha-(\beta+1)n-\omega k,\epsilon;1),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b),\\(f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1),\end{matrix}\right.\right. \left.\begin{matrix}(e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c)\\(f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d),(-\alpha-\beta n-(\omega+m)k,\epsilon;1)\end{matrix}\right].$$

Corollary 3.4. *If* $a,\ b,\ c,\ d \in \mathbb{N}_0$ *with* $0 \le b \le c,\ 0 \le a \le d,\ \rho_j, E_j\,(j=1,\cdots,c)$, $\sigma_j, F_j\,(1,\cdots,d) \in \mathbb{R}^+$ *and* $\lambda,\ e_j,\ f_j \in \mathbb{C}$, *then the following bilateral generating relation* *holds:*

$$\sum_{n=0}^{\infty} \sigma_n^m(x)\ {}^{\Gamma}\overline{I}\,{}_{c+1,\,d}^{a,\,b+1}\left[y \left| \begin{matrix} (e_1,\rho_1;E_1:z),(1-\lambda-n,\epsilon;1),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b), \\ (f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1), \end{matrix} \right.\right.$$
$$\left.\begin{matrix} (e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c) \\ (f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right] \frac{t^n}{n!}$$
$$= (1-t)^{-\lambda} \sum_{k=0}^{\infty} \frac{\Omega_k}{(mk)!}\left[x\left(\frac{t}{1-t}\right)^m\right]^k$$
$$\times\ {}^{\Gamma}\overline{I}\,{}_{c+1,\,d}^{a,\,b+1}\left[\frac{y}{(1-t)^{\epsilon}} \left| \begin{matrix} (e_1,\rho_1;E_1:z),(1-\lambda-mk,\epsilon;1),(e_2,\rho_2;E_2),\cdots,(e_b,\rho_b;E_b), \\ (f_1,\sigma_1;1),\cdots,(f_a,\sigma_a;1), \end{matrix} \right.\right.$$
$$\left.\begin{matrix} (e_{b+1},\rho_{b+1};E_{b+1}),\cdots,(e_c,\rho_c;E_c) \\ (f_{a+1},\sigma_{a+1};F_{a+1}),\cdots,(f_d,\sigma_d;F_d) \end{matrix}\right], \tag{32}$$

where, $\sigma_n^m(x) = \sum_{k=0}^{[n/m]} \binom{n}{mk} \Omega_k\, x^k$ *and* $\{\Omega_k\}_{n=0}^{\infty}$ *is an arbitrary complex sequence.*

Remark 3.1. *If we set* $E_j\,(j = b+1,\cdots,c) = 1$ *and* $F_j\,(j = 1,\cdots,a) = 1$ *in above corollaries, then we may obtain the linear and bilateral generating relations involving incomplete* $\overline{H}$*-functions defined by Srivastava et al.* [13].

Remark 3.2. *Again, if we set* $E_j\,(j =,\cdots,c) = 1$ *and* $F_j\,(j = 1,\cdots,d) = 1$ *in above theorems, then we obtain the linear and bilateral generating relations involving incomplete* H*-functions given by Jangid et al.* [5].

4. Conclusion

We have derived linear and bilateral generating functions involving incomplete I-functions and incomplete $\overline{I}$-functions. The results obtained are of a general nature, as special cases may result in generating functions involving special functions such as H-function, G-function, Fox-Wright function and hypergeometric functions.

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