HARMONIC MEAN CORDIAL LABELING OF SOME GRAPHS

J. PAREJIYA1*, D. JANI2, Y. HATHI3, §

Abstract. All the graphs considered in this article are simple and undirected. Let \( G = (V(G), E(G)) \) be a simple undirected Graph. A function \( f : V(G) \rightarrow \{1, 2\} \) is called Harmonic Mean Cordial if the induced function \( f^* : E(G) \rightarrow \{1, 2\} \) defined by
\[
f^*(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor
\]
satisfies the condition \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for any \( i, j \in \{1, 2\} \), where \( v_f(x) \) and \( e_f(x) \) denotes the number of vertices and number of edges with label \( x \) respectively and \( \lfloor x \rfloor \) denotes the greatest integer less than or equals to \( x \). A Graph \( G \) is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. In this article, we have provided some graphs which are not Harmonic Mean Cordial and also we have provided some graphs which are Harmonic Mean Cordial.

Keywords: Harmonic Mean Cordial, cycle, complete bipartite graph, join of two graphs.

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1. Introduction

We begin with simple, finite, connected and undirected graph \( G = (V(G), E(G)) \). The concept of cordial labeling was introduced by Cahit in [2] the year 1987. Recall from [5] that function \( f : V(G) \rightarrow \{1, 2\} \) is called Harmonic Mean Cordial if the induced function \( f^* : E(G) \rightarrow \{1, 2\} \) defined by \( f^*(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor \) satisfies the condition \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for any \( i, j \in \{1, 2\} \), where \( v_f(x) \) and \( e_f(x) \) denotes the number of vertices and number of edges with label \( x \) respectively and \( \lfloor x \rfloor \) denotes the greatest integer less than or equals to \( x \). A Graph \( G \) is called Harmonic Mean Cordial if it admits Harmonic Mean Cordial labeling. Let \( G = (V(G), E(G)) \) be a simple, undirected graph and \( \{v_1, v_2, \cdots, v_n\} \subseteq V(G) \), we call \( v_1, v_2, \cdots, v_n \) are in sequence if it forms a path. For the sake of convenience of the reader we use HMC for harmonic mean cordial labeling. Motivated by the Results proved in [5] and [4], in this article, we have provided some
examples of non \textit{HMC} graphs and also we have provided some \textit{HMC} graphs. It is useful to recall some useful definitions of graph theory to make this article self-contained.

**Definition 1.1.** \[1\] A simple graph \( G \) is said to be complete if every pair of distinct vertices of \( G \) are adjacent in \( G \). It is denoted by \( K_n \).

**Definition 1.2.** \[1\] A walk in a graph \( G \) is a finite alternating sequence of vertices and edges. A walk is called a trail if all the edges are distinct. Cycle is a closed trail in which all the vertices are distinct. It is denoted by \( C \).

**Definition 1.3.** \[1\] Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. Then union of \( G_1 \) and \( G_2 \) is denoted by \( G_1 \cup G_2 \) is the graphs whose vertex set is \( V_1 \cup V_2 \) and edge set is \( E_1 \cup E_2 \). When \( G_1 \) and \( G_2 \) are vertex disjoint \( G_1 \cup G_2 \) is called sum of \( G_1 \) and \( G_2 \) and it is denoted by \( G_1 + G_2 \).

**Definition 1.4.** \[1\] Let \( G_1 \) and \( G_2 \) be two vertex disjoint graphs. Then the join \( G_1 \lor G_2 \) of \( G_1 \) and \( G_2 \) is the supergraph of \( G_1 + G_2 \) in which each vertex of \( G_1 \) is also adjacent to every vertex of \( G_2 \).

**Definition 1.5.** \[1\] A graph is bipartite if its vertex set can be partitioned into two non empty subsets \( V_1 \) and \( V_2 \) such that each edge of \( G \) has one end in \( V_1 \) and other in \( V_2 \). The pair \( (V_1, V_2) \) is called bipartition of a bipartite graph. It is denoted by \( G(V_1, V_2) \). A simple bipartite graph \( G(V_1, V_2) \) is complete if each vertex of \( V_1 \) is adjacent to all the vertices of \( V_2 \). If \( G(V_1, V_2) \) is complete with \( |V_1| = m \) and \( |V_2| = n \) then \( G(V_1, V_2) \) is denoted by \( K_{m,n} \).

**Definition 1.6.** \[6\] Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. Then the Corona of \( G_1 \) and \( G_2 \) is denoted as \( G_1 \odot G_2 \) is a graph obtained by taking one copy of \( G_1 \) (which has \( p_1 \) vertices) and \( p_1 \) copies of \( G_2 \) and then joining the \( i^{th} \) vertex of \( G_1 \) to every point in the \( i^{th} \) copy of \( G_2 \).

**Definition 1.7.** \[1\] The Helm graph \( H_n \) is the graph obtained from a wheel \( W_n \) by attaching a pendant edge at each vertex of the cycle.

**Definition 1.8.** \[3\] A Closed helm is the graph obtained from a helm by joining each pendant vertex to form a cycle. It is denoted by \( CH_n \).

**Definition 1.9.** \[3\] The direct (tensor) product \( G \times H \) of two graphs \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) is a graph with the vertex set \( V(G \times H) = V(G) \times V(H) \) and edge set \( E(G \times H) = \{(x,y)(x',y')|xx' \in E(G) \text{ and } yy' \in E(H)\} \).

Here, we mention the results proved in the Section 2. In Theorem 2.1, we have proved that the \( CH_n \odot K_1 \) is HMC. We have shown in Theorem 2.2 that the tensor Product \( P_m \times P_n \) is HMC. The Complete bipartite graph \( K_{m,n} \) is not HMC is proved in theorem 2.3. We have discussed harmonic mean cordial labelling of \( K_n \lor C_m \) and we have proved that it is not HMC in Corollary 2.1 for any \( n \geq 2, m \geq 3 \) and \( n, m \in N \). In Corollary 2.2, we have proved that \( C_m \lor C_n \) is not HMC for any \( n, m \geq 3 \) and \( n, m \in N \).

2. Main Results

**Theorem 2.1.** \( CH_n \odot K_1 \) is HMC.

**Proof.** Note that \(|V(CH_n \odot K_1)| = 4n + 2 \) and \(|E(CH_n \odot K_1)| = 6n + 1 \). Let \( V(CH_n) = \{x_1, x_2, ..., x_{n+1}\} \) be the vertex set of \( CH_n \) with \( x_1 \) as an apex vertex and \( y_i \) be the pendant vertex, adjacent to \( x_i \) in \( CH_n \odot K_1 \) for \( 1 \leq i \leq 2n + 1 \) as shown in the following figure.
Let us define the labeling function $f : V(CH_n \odot K_1) \to \{1, 2\}$ as follows

- $f(x_i) = 2$, $1 \leq i \leq n+1$
- $f(x_i) = 1$, $n+1 \leq i \leq 2n+1$
- $f(y_i) = 2$, $1 \leq i \leq n$
- $f(x_i) = 1$, $n+1 \leq i \leq 2n+1$

Note that, $v_f(1) = 2n+1 = v_f(2)$ and $e_f(1) = 3n+1$, $e_f(2) = 3n$. Therefore, $CH_n \odot K_1$ is HMC.

Example 2.1. HMC labeling of $CH_4 \odot K_1$ is shown in the following figure.

![Diagram of CH_n ⊕ K_1 and CH_4 ⊕ K_1 with labeling]
Theorem 2.2. The $P_m \times P_n$ is HMC $\forall m, n \in \mathbb{N}$.

Proof. Let $G = (V, E)$ be the $P_m \times P_n$. Note that $|V| = mn$ and $|E| = 2mn - 2m - 2n + 2$. Let $V = \{x_{1,1}, x_{1,2}, ..., x_{1,n}, x_{2,1}, x_{2,2}, ..., x_{2,n}, ..., x_{m,1}, x_{m,2}, ..., x_{m,n}\}$ be a vertex set of $G$ as shown in the following figure.

Case 1: $mn$ is even
Define a labeling function $f : V(P_m \times P_n) \to \{1, 2\}$ as follows,

$$f(x_{i,j}) = \begin{cases} 
1 & \text{if } 1 \leq i \equiv 1 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 1 \pmod{2} \leq n \\
1 & \text{if } 1 \leq i \equiv 0 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 0 \pmod{2} \leq n \\
2 & \text{if } 1 \leq i \equiv 1 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 0 \pmod{2} \leq n \\
2 & \text{if } 1 \leq i \equiv 0 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 1 \pmod{2} \leq n 
\end{cases}$$

Then $v_f(1) = \frac{mn}{2}$ and $e_f(1) = e_f(2) = \frac{2mn - 2m - 2n + 2}{2}$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 0$.

Case 2: $mn$ is odd
Define a labeling function $f : V(P_m \times P_n) \to \{1, 2\}$ as follows,

$$f(x_{i,j}) = \begin{cases} 
1 & \text{if } 1 \leq i \equiv 1 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 1 \pmod{2} \leq n \\
1 & \text{if } 1 \leq i \equiv 0 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 0 \pmod{2} \leq n \\
2 & \text{if } 1 \leq i \equiv 1 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 0 \pmod{2} \leq n \\
2 & \text{if } 1 \leq i \equiv 0 \pmod{2} \leq m \text{ and } 1 \leq j \equiv 1 \pmod{2} \leq n 
\end{cases}$$

Then $v_f(1) = \frac{mn+1}{2}$, $v_f(2) = \frac{mn-1}{2}$ and $e_f(1) = e_f(2) = \frac{2mn - 2m - 2n + 2}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 0$.

Hence, The $P_m \times P_n$ is HMC. $\square$
Example 2.2. HMC labeling of $P_4 \times P_5$ and $P_5 \times P_5$ is shown in the following figure.

$P_4 \times P_5$

$P_5 \times P_5$

Theorem 2.3. Complete bipartite graph $K_{m,n}$ is not HMC, where $m, n \geq 2$.

Proof. Without loss of generality, we may assume $m \geq n$. Let $V(K_{m,n}) = V_1 \cup V_2$. Where, $|V_1| = n$ and $|V_2| = m$. Suppose that $K_{m,n}$ is HMC.

Case 1: $n + m$ is even
Since $K_{m,n}$ is HMC, we have $v_f(1) = \frac{n+m}{2} = v_f(2)$. Suppose that there exist $t$ vertices with label 1 in $V_1$. So, we have $n - t$ vertices with label 2 in $V_1$. Hence, there exists $(\frac{m+n}{2} - t)$ vertices with label 1 in $V_2$ and $m - (\frac{m+n}{2} - t) = (\frac{m+n}{2} + t)$ vertices with label 2 in $V_2$. Note that, $e_f(1) = mt + (n-t)(\frac{m+n}{2} - t)$ and $e_f(2) = (n-t)(m - \frac{m+n}{2} + t)$. Now, $e_f(1) - e_f(2) = mt + (n-t)(\frac{m+n}{2} - t) - (n-t)(m - \frac{m+n}{2} + t) = mt + (n-t)^2 - nt + t^2 > 2$.

Case 2: $n + m$ is odd
In this Case we have two possibilities

(i) $v_f(1) = \frac{n+m+1}{2}$ and $v_f(2) = \frac{n+m-1}{2}$
(ii) $v_f(1) = \frac{n+m-1}{2}$ and $v_f(2) = \frac{n+m+1}{2}$

So, we consider the following Cases.

Subcase 2.1: $v_f(1) = \frac{n+m+1}{2}$ and $v_f(2) = \frac{n+m-1}{2}$
Suppose that there exist $t$ vertices with label 1 in $V_1$. So, we have, $(n-t)$ vertices with label 2 in $V_1$. Hence, there exists $(\frac{m+n+1}{2} - t)$ vertices with label 1 in $V_2$ and $m - (\frac{m+n+1}{2} - t) = (\frac{m+n-1}{2} + t)$ vertices with label 2 in $V_2$. Note that, $e_f(1) = mt + (n-t)(\frac{m+n+1}{2} - t)$ and $e_f(2) = (n-t)(\frac{m+n-1}{2} + t)$. Now, $e_f(1) - e_f(2) = mt + (n-t)(\frac{m+n+1}{2} - t) - (n-t)(\frac{m+n-1}{2} + t) = mt + (n-t)^2 + n - nt + t^2 - t > 2$.

Subcase 2.2: $v_f(1) = \frac{n+m-1}{2}$ and $v_f(2) = \frac{n+m+1}{2}$
Suppose that there exist $t$ vertices with label 1 in $V_1$. So, we have, $(n-t)$ vertices with label 2 in $V_1$. Hence, there exists $(\frac{m+n-1}{2} - t)$ vertices with label 1 in $V_2$ and $m - (\frac{m+n-1}{2} - t) = (\frac{m+n+1}{2} + t)$ vertices with label 2 in $V_2$. Note that, $e_f(1) = mt + (n-t)(\frac{m+n-1}{2} - t)$ and $e_f(2) = (n-t)(\frac{m+n+1}{2} + t)$. Now, $e_f(1) - e_f(2) = mt + (n-t)(\frac{m+n+1}{2} - t) - (n-t)(\frac{m+n+1}{2} + t) = mt + (n-t)^2 + n - nt + t^2 > 2$. Hence, $K_{m,n}$ is not HMC, where $m, n \geq 2$. 

\[\blacksquare\]
Theorem 2.4. $K_n \lor C_n$ is not HMC, where $n \geq 3$.

Proof. Suppose that $K_n \lor C_n$ is HMC. Note that, $|V(K_n \lor C_n)| = 2n$ and $|E(K_n \lor C_n)| = n \frac{(n-1)}{2} + n + n^2$. Since, $|V(K_n \lor C_n)| = 2n$ and we have assume that $K_n \lor C_n$ is HMC. We have $v_f(1) = v_f(2) = n$.

Case 1: All the vertices of label 1 and label 2 are in sequence in $C_n$

Suppose that we have $t$ no. of vertices with label 1 in $K_n$. So, we have $(n-t)$ vertices of of label 1 in $C_n$. Hence, we have $(n-t)$ vertices of label 2 in $K_n$ and $t$ vertices of label 2 in $C_n$. Note that, $e_f(1) = (n-t)t + \frac{(t-1)}{2} + (n-t)^2 + (n-t+1)+nt$ and $e_f(2) = (n-t)(n-t-1) + t(n-t)+(t-1)$. Now, $e_f(1) - e_f(2) = n^2 + t^2 + \frac{3n}{2} - 3t + 2$. If $t \geq 3$ then as $n \geq 2$, we have $e_f(1) - e_f(2) > 1$.

If $t = 1$ then $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} > 1$.

If $t = 2$ then $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} > 1$.

So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in $C_n$

Suppose that we have $t$ no. of vertices with label 1 in $K_n$. So, we have $(n-t)$ vertices of label 2 in $C_n$. We have $t$ no. of vertices with label 2 in $K_n$ and $t$ vertices of label 2 in $C_n$. Note that, $e_f(1) = (n-t)t + \frac{(t-1)}{2} + (n-t)^2 + (n-t+1)+nt$ and $e_f(2) = (n-t)(n-t-1) + t(n-t)+(t-1)$. Now, $e_f(1) - e_f(2) = n^2 + t^2 + \frac{3n}{2} - 3t + 2$. If $t \geq 3$ then as $n \geq 2$, we have $e_f(1) - e_f(2) > 1$.

If $t = 1$ then $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} > 1$.

If $t = 2$ then $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} > 1$.

So, $e_f(1) - e_f(2) > 1$.

Case 3: We have $n$ no. of vertices with label 1 in $K_n$ and $n$ no. of vertices with label 2 in $C_n$

Then, we have $e_f(1) = \frac{n(n-1)}{2} + n^2$ and $e_f(2) = n$. Then, $e_f(1) - e_f(2) = \frac{n(n-1)}{2} + n^2 - n = \frac{3n^2}{2} - \frac{3n}{2} > 1$ as $n^2 > n$.

Case 4: We have $n$ no. of vertices with label 2 in $K_n$ and $n$ no. of vertices with label 1 in $C_n$

Then we have, $e_f(1) = n^2 + n$ and $e_f(2) = \frac{n(n-1)}{2}$. Then, $e_f(1) - e_f(2) = n^2 + n - \frac{n(n-1)}{2} = \frac{n^2}{2} + \frac{3n}{2} > 1$.

Hence, $K_n \lor C_n$ is not HMC where, $n \geq 3$.

□

Theorem 2.5. $K_n \lor C_m$ is not HMC, where $m + n$ is even and $n \geq 2$, $m \geq 3$.

Proof. Note that, $|V(K_n \lor C_m)| = m + n$. Suppose that $K_n \lor C_m$ is HMC. Then we have, $|v_f(1)| = \frac{m+n}{2} = |v_f(2)|$.

Case 1: All the vertices with label 1 and label 2 are in sequence in $C_m$

Suppose that we have $t$ no. of vertices with label 1 in $K_n$. So, we have $(m+n-t)$ vertices with label 1 in $C_n$. Hence, we have $(n-t)$ vertices with label 2 in $K_n$ and $m-(m+n-t) = (m-n)$ vertices with label 2 in $C_n$. Then we have, $e_f(1) = \frac{t(t-1)}{2} + tm + (n-t)(m-n)$ and $e_f(2) = \frac{(m-t)}{2} + tm + (n-t)(m-n)$ and $e_f(2) = \frac{(m-t)}{2} + tm + (m-n+t+1)+t(n-t)+(n-t)$.

Then, $e_f(1) - e_f(2) = m^2 + \frac{n^2}{2} - nt + \frac{3n}{2} + t^2 - 3t + 2 = (t-n)^2(\frac{1}{2}) + \frac{t^2}{2} + \frac{3n}{2} + 2 + tm - 3$. If $m \geq 3$, then $e_f(1) - e_f(2) > 1$.

If $m = 2$, then $e_f(1) - e_f(2) = (t-n)^2(\frac{1}{2}) + \frac{t^2}{2} + \frac{n}{2} + (n-t) + 2$.

Now, $n > t$. So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices with label 2 are not in sequence in $C_m$

Suppose that we have $t$ no. of vertices with label 1 in $K_n$. So, we have $(m+n-t)$
vertices with label 1 in $C_m$. Hence, we have $(n - t)$ vertices with label 2 in $K_n$ and $m - \left(\frac{m + n}{2} - t\right) = \left(\frac{m - n}{2} + t\right)$ vertices with label 2 in $C_m$. Suppose that there exist i.
o. of vertices from $\left(\frac{m - n}{2} + t\right)$ with label 2 are not in sequence in $C_m$. Then we have, 
\[ e_f(1) = \frac{t(n - t)}{2} + t(n - t) + tm + (n - t)(\frac{m + n}{2} - t) + (\frac{m + n}{2} - t + i + 1) \]
and $e_f(2) = \left(\frac{n - t}{2}\right) + (n - t)(\frac{m - n}{2} + t) + (\frac{m - n}{2} + t - i - 1)$. Now, $e_f(2)$ in Case 2 $\leq e_f(2)$ in Case 1 and $e_f(1)$ in Case 2 $\geq e_f(1)$ in Case 1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Case 1. Now, we have already proved in Case 1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

**Case 3: $m < n$**

**Subcase 3.1:** All the vertices in $C_m$ are with label 1.

Suppose that we have t.
o. of vertices with label 1 in $K_n$. So, we have $(n - t)$ vertices with label 2 in $K_n$. Then we have, $e_f(1) = \frac{t(n - t)}{2} + mn + m + t(n - t)$ and $e_f(2) = \left(\frac{n - t}{2}\right).$ We know that, $t = \frac{m + n}{2}$. Then, $e_f(1) - e_f(2) = \frac{3mn}{t} + \frac{m^2}{2} + \frac{n^2}{1} - \frac{m^2}{2}$. We know that $n > m$. So, $e_f(1) - e_f(2) > 1$.

**Subcase 3.2:** All the vertices in $C_m$ are with label 2.

Suppose that we have t.
o. of vertices with label 1 in $K_n$. So, we have $(n - t)$ vertices with label 2 in $K_n$. Then we have, $e_f(1) = \frac{t(n - t)}{2} + tm + t(n - t)$ and $e_f(2) = \left(\frac{n - t}{2}\right).$ We know that, $t = \frac{m + n}{2}$. Then, $e_f(1) - e_f(2) = \frac{3mn}{t} + \frac{m^2}{2} + \frac{n^2}{1} - \frac{m^2}{2}$. As $m \geq 2$. So, $e_f(1) - e_f(2) > 1$.

**Case 4: $m > n$**

**Subcase 4.1:** All the vertices in $K_n$ are with label 1.

Suppose that we have t.
o. of vertices with label 1 in $C_m$. So, we have $(m - t)$ vertices with label 2 in $C_m$.

**Subsubcase 4.1.1:** All the vertices with label 2 are in sequence in $C_m$.

Then we have, $e_f(1) = \frac{n(n - t)}{2} + (t + 1) + nm + e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = \frac{n(n - t)}{2} + (t + 1) + nm - m + t + 1$. We know that, $nm > m$. So, $e_f(1) - e_f(2) > 1$.

**Subsubcase 4.1.2:** Some of the vertices with label 2 are not in sequence in $C_m$.

Suppose that we have i.
o. of vertices with label 2 are not in sequence in $C_m$. Suppose that i.
o. of vertices are not i.
o. sequence. Then we have, $e_f(1) = \frac{n(n - t)}{2} + (m + i + 1)$ and $e_f(2) = m - t - i - 1$. Now, $e_f(2)$ in Subsubcase 4.1.2 $\leq e_f(2)$ in Subsubcase 4.1.1 and $e_f(1)$ in Subsubcase 4.1.2 $\geq e_f(1)$ in Subsubcase 4.1.1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subsubcase 4.1.1. Now, we have already proved in Subsubcase 4.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

**Subcase 4.2:** All the vertices in $K_n$ are with label 2.

Suppose that we have t.
o. of vertices with label 1 in $C_m$. So, we have $(m - t)$ vertices with label 2 in $C_m$.

**Subsubcase 4.2.1:** All the vertices with label 2 are in sequence in $C_m$.

Then we have, $e_f(1) = (t + 1) + nt$ and $e_f(2) = \frac{n(n - t)}{2} + (m - t - 1) + n(m - t)$. Then, $e_f(1) - e_f(2) = (t + 1) + tn - \frac{n(n - t)}{2} - m + t + 1 - mn + tn$. We know that, $t = \frac{m + n}{2}$. Then, $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{1} + 2 > 1$.

**Subsubcase 4.2.2:** Some of the vertices with label 2 are not in sequence in $C_m$.

Suppose that i.
o. of vertices are not i.
o. sequence. Then we have, $e_f(1) = tn$ and $e_f(1) = \frac{n(n - t)}{2} + (m - t - i - 1) + n(m - t)$. Now, $e_f(2)$ in Subsubcase 4.2.2 $\leq e_f(2)$ in Subsubcase 4.2.1 and $e_f(1)$ in Subsubcase 4.2.2 $\geq e_f(1)$ in Subsubcase 4.2.1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Subsubcase 4.2.1. Now, we have already proved in Subsubcase
4.2.1 that \( e_f(2) - e_f(1) > 1 \). Hence, \( e_f(2) - e_f(1) > 1 \) in this Case.
Hence, \( K_n \lor C_m \) is not HMC, where \( m + n \) is even and \( n \geq 2 \), \( m \geq 3 \). □

**Theorem 2.6.** \( K_n \lor C_m \) is not HMC, where \( m + n \) is odd and \( n \geq 2 \), \( m \geq 3 \).

**Proof.** Note that, \(|V(K_n \lor C_m)| = m + n\). Suppose that \( K_n \lor C_m \) is HMC.

**Case 1: All the vertices with label 1 and label 2 in \( C_m \) are in sequence in \( C_m \)**

In this Case we have two possibilities
(i) \( v_f(1) = \frac{m+n+1}{2} \) and \( v_f(2) = \frac{m+n-1}{2} \)
(ii) \( v_f(1) = \frac{m+n-1}{2} \) and \( v_f(2) = \frac{m+n+1}{2} \)

So, we consider the following cases.

**Subcase 1.1:** \( v_f(1) = \frac{m+n+1}{2} \) and \( v_f(2) = \frac{m+n-1}{2} \)

Suppose that we have \( t \) no. of vertices with label 1 in \( K_n \). So, we have \( \frac{(m+n+1) - t}{2} \) vertices of label 1 in \( C_m \). Hence, we have \( (n-t) \) vertices with label 2 in \( K_n \) and \( m - \frac{(m+n+1) - t}{2} = \left(\frac{m-n-1}{2} + t\right) \) vertices with label 2 in \( C_m \). Then we have, \( e_f(1) = \frac{t(t-1)}{2} + tn + t(n-t) + \frac{m+n+1-t+1}{2} + (n-t)(\frac{m+n+1}{2} - t) \) and \( e_f(2) = \frac{\frac{n-n(t-1)}{2} + (n-t)(\frac{m-n-1}{2} + t)}{2} + \frac{m-n-1}{2} + 1 \).

Then, \( e_f(1) - e_f(2) = t^2 + \frac{n^2}{2} + \frac{n}{2} + t^2 - 3t + 3 = (t-1)^2 + \frac{n^2}{2} + 2 + \frac{n}{2} + 2(n-t) > 1 \) as \( n > t \).

**Subcase 1.2:** \( v_f(1) = \frac{m+n-1}{2} \) and \( v_f(2) = \frac{m+n+1}{2} \)

Suppose that we have \( t \) no. of vertices with label 1 in \( K_n \). So, we have \( \frac{(m+n-1) - t}{2} \) vertices of label 1 in \( C_m \). Hence, we have \( (n-t) \) vertices with label 2 in \( K_n \) and \( m - \frac{(m+n-1) - t}{2} = \left(\frac{m-n-1}{2} + t\right) \) vertices with label 2 in \( C_m \). Then we have, \( e_f(1) = \frac{t(t-1)}{2} + tm + t(n-t) + \frac{m+n-1-t+1}{2} + (n-t)(\frac{m+n-1}{2} - t) \) and \( e_f(2) = \frac{\frac{n-n(t-1)}{2} + (n-t)(\frac{m-n-1}{2} + t)}{2} + \frac{m-n-1}{2} + 1 \).

Then, \( e_f(1) - e_f(2) = t^2 + \frac{n^2}{2} + \frac{n}{2} + mt + 1 - nt = \frac{(n-1)^2}{2} + (t-1)^2 + mt + \frac{n}{2} - \frac{1}{2} > 1 \) as \( n \geq 2 \).

**Case 2:** Some of the vertices with label 2 are not in sequence in \( C_m \)

**Subcase 2.1:** Suppose that \( v_f(1) = \frac{m+n+1}{2} \) and \( v_f(2) = \frac{m+n-1}{2} \)

Suppose that we have \( t \) no. of vertices with label 1 in \( K_n \). So, we have \( \frac{(m+n+1) - t}{2} \) vertices of label 1 in \( C_m \). Hence, we have \( (n-t) \) vertices with label 2 in \( K_n \) and \( m - \frac{(m+n+1) - t}{2} = \left(\frac{m-n-1}{2} + t\right) \) vertices with label 2 in \( C_m \). Suppose that there exist \( n \) no. of vertices from \( \frac{m-n-1}{2} + t \) with label 2 are not in sequence in \( C_m \). Then we have, \( e_f(1) = \frac{t(t-1)}{2} + tm + t(n-t) + \frac{(m+n+1) - t + i + 1}{2} + (n-t)(\frac{m-n-1}{2} - t) \) and \( e_f(2) = \frac{(m-n-1) - t - i - 1}{2} + (n-t)(\frac{m+n-1}{2} - t) \).

Now, \( e_f(2) \) in Subcase 2.1 \( \leq e_f(2) \) in Subcase 1.1 and \( e_f(1) \) Subcase 2.1 \( \geq e_f(1) \) in Subcase 1.1. So, \( e_f(1) - e_f(2) \) in this Case \( \geq e_f(1) - e_f(2) \) in Subcase 1.1. Now, we have already proved in Subcase 1.1 that \( e_f(1) - e_f(2) > 1 \). Hence, \( e_f(1) - e_f(2) > 1 \) in this Case.

**Subcase 2.2:** \( v_f(1) = \frac{m+n-1}{2} \) and \( v_f(2) = \frac{m+n+1}{2} \)

Suppose that we have \( t \) no. of vertices with label 1 in \( K_n \). So, we have \( \frac{(m+n-1) - t}{2} \) vertices of label 1 in \( C_m \). Hence, we have \( (n-t) \) vertices with label 2 in \( K_n \) and \( m - \frac{(m+n-1) - t}{2} = \left(\frac{m-n+1}{2} + t\right) \) vertices with label 2 in \( C_m \). Suppose that there exist \( n \) no. of vertices from \( \frac{m-n+1}{2} + t \) with label 2 are not in sequence in \( C_m \). Then we have, \( e_f(1) = \frac{t(t-1)}{2} + tm + t(n-t) + \frac{(m+n-1) - t + i + 1}{2} + (n-t)(\frac{m+n-1}{2} - t) \) and \( e_f(2) = \frac{(m-n+1) - t - i - 1}{2} + (n-t)(\frac{m+n-1}{2} + t) \).

Now, \( e_f(2) \) in Subcase 2.2 \( \leq e_f(2) \) in Subcase 1.2 and \( e_f(1) \) Subcase 2.2 \( \geq e_f(1) \) in Subcase 1.2. So, \( e_f(1) - e_f(2) \) in this Case \( \geq e_f(1) - e_f(2) \) in Subcase 2.1. Now, we have already proved in Subcase 2.1 that \( e_f(1) - e_f(2) > 1 \). Hence, \( e_f(1) - e_f(2) > 1 \) in this Case.

**Case 3:** \( m < n \)

**Subcase 3.1:** All the vertices in \( C_m \) are with label 1 and some vertices with label 1 are
in $K_n$.

Suppose that there exist $t$ no. of vertices with label 1 in $K_n$. So, there exists $(n - t)$ vertices with label 2 in $K_n$. Suppose that we have $m$ no. of vertices with label 1 in $C_m$. Then we have, $e_f(1) = \frac{(t-1)}{2} + t(n-t) + mn + m$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2}$. Then, $e_f(1) - e_f(2) = mn + m + 2nt + \frac{n}{2} - t - t^2 - \frac{n^2}{2}$.

In this case we have two possibilities

(i) $m + t = \frac{m+n+1}{2}$

(ii) $m + t = \frac{m+n-1}{2}$

So, we consider the following cases.

**Subsubcase 3.1.1:** $m + t = \frac{m+n+1}{2}$.

Therefore, $t = \frac{n-m+1}{2}$. Then, $e_f(1) - e_f(2) = \frac{mn}{2} + (2m - \frac{3}{4}) + (\frac{n^2}{4} - \frac{m^2}{4}) + \frac{n}{2} > 1$ as $m < n$ and $2m > \frac{n}{2}$ as $m \geq 2$.

**Subsubcase 3.1.2:** $m + t = \frac{m+n-1}{2}$.

Therefore, $t = \frac{n-m-1}{2}$. Then, $e_f(1) - e_f(2) = \frac{mn}{2} - \frac{n}{2} + m + (\frac{n^2}{4} - \frac{m^2}{4}) + \frac{1}{4} > 1$ as $n > m$.

**Subcase 3.2:** All the vertices in $C_m$ are with label 2 and some vertices with label 2 are in $K_n$.

Suppose that there exist $t$ no. of vertices with label 1 in $K_n$. So, there exists $(n - t)$ vertices with label 2 in $K_n$. Suppose that we have $m$ no. of vertices with label 2 in $C_m$. Then we have, $e_f(1) = \frac{(t(t-1)}{2} + t(n-t) + tm$ and $e_f(2) = \frac{(n-t)(n-t-1)}{2} + m(n-t) + m$.

Then, $e_f(1) - e_f(2) = 2mt - mn - \frac{n^2}{2} + 2nt + \frac{n}{2} - t^2 - t - m$.

**Subsubcase 3.2.1:** $t = \frac{m+n+1}{2}$.

Then, $e_f(1) - e_f(2) = \frac{3n^2}{4} + (\frac{mn}{2} - m) + (\frac{n^2}{4} - \frac{3}{4}) + \frac{n}{2} > 1$ as $m, n \geq 2$.

**Subsubcase 3.2.2:** $t = \frac{m+n-1}{2}$.

Then, $e_f(1) - e_f(2) = \frac{3n^2}{4} + \frac{mn}{2} - 2m + \frac{n^2}{2} + \frac{1}{2} - \frac{n}{2} = \frac{n^2}{2} - \frac{n}{2} + m(\frac{3m}{4} + \frac{n}{2} - 2) + \frac{1}{4} > 1$ as $m, n \geq 2$.

**Case 4:** $m > n$ and all the vertices with label 2 are in sequence in $C_m$.

**Subcase 4.1:** All the vertices in $K_n$ are with label 1 and some vertices with label 1 are in $C_m$.

Suppose that there exist $t$ no. of vertices with label 1 in $K_n$. So, there exists $(m - t)$ vertices with label 2 in $C_m$. Suppose that we have $n$ no. of vertices with label 1 in $K_n$. Then we have, $e_f(1) = mn + (t + 1) + n(\frac{n-1}{2})$ and $e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = (mn - m) + 2t + (\frac{n^2}{2} - \frac{n}{2}) > 1$ as $mn > m$ and $\frac{n^2}{2} > \frac{n}{2}$, where, $m, n \geq 2$.

**Subcase 4.2:** All the vertices in $K_n$ are with label 2 and some vertices with label 2 are in $C_m$.

Suppose that there exist $t$ no. of vertices with label 1 in $C_m$. So, there exists $(m - t)$ vertices with label 2 in $C_m$. Suppose that we have $n$ no. of vertices with label 2 in $K_n$. Then we have, $e_f(1) = tn + (t + 1)$ and $e_f(2) = \frac{n(n-1)}{2} + n(m-t) + (m - t - 1)$. Then, $e_f(1) - e_f(2) = 2t + 2nt - \frac{n^2}{2} + \frac{n}{2} - mn - m$.

**Subsubcase 4.2.1:** $t = \frac{m+n+1}{2}$.

Then, $e_f(1) - e_f(2) = \frac{3n^2}{4} + \frac{n^2}{2} + 3 > 1$.

**Subsubcase 4.2.2:** $t = \frac{m+n-1}{2}$.

Then, $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{n}{2} + 1 > 1$.

**Case 5:** $m > n$ and Suppose that some of the vertices with label 2 are not in sequence in $C_m$.

**Subcase 5.1:** All the vertices in $K_n$ are with label 1 and some vertices with label 1 are in $C_m$. 
Suppose that there exist $t$ no. of vertices with label 1 in $C_m$. So, there exists $(m - t)$ vertices with label 2 in $C_m$. Suppose that we have $n$ no. of vertices with label 1 in $K_n$. Suppose that we have $i$ no. of vertices with label 2 are not in sequence in $C_m$. Then, $e_f(1) = \frac{n(n-1)}{2} + mn + (t + i + 1)$ and $e_f(2) = m - t - i - 1$. Now, $e_f(2)$ in Subcase 5.1 $\leq e_f(2)$ in Subcase 4.1 and $e_f(1)$ in Subsubcase 5.1 $\geq e_f(1)$ in Subcase 4.1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Subcase 4.1. Now, we have already proved in Subcase 4.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

**Subcase 5.2:** All the vertices in $K_n$ are with label 2 and some vertices with label 2 are in $C_m$.

Suppose that there exist $t$ no. of vertices with label 1 in $C_m$. So, there exists $(m - t)$ vertices with label 2 in $C_m$. Suppose that we have $i$ no. of vertices with label 2 in $K_n$. Suppose that we have $j$ no. of vertices with label 2 are not in sequence in $C_m$. Then, $e_f(1) = nt + (t + i + 1)$ and $e_f(2) = \frac{n(n-1)}{2} + (m - t - i - 1)$. Now, $e_f(2)$ in Subcase 5.2 $\leq e_f(2)$ in Subcase 4.2 and $e_f(1)$ in Subcase 5.2 $\geq e_f(1)$ in Subcase 4.2. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Subcase 4.2. Now, we have already proved in Subcase 4.2 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case. Hence, $K_n \cap C_m$ is not HMC, where $m + n$ is odd and $n \geq 2, m \geq 3$.

**Corollary 2.1.** $K_n \cap C_m$ is not HMC, where $n \geq 2, m \geq 3, n, m \in \mathbb{N}$

**Proof.** Proof follows from Theorems 2.4, 2.5 and 2.6.

**Theorem 2.7.** $C_m \cap C_n$ is not HMC, where $m = n$ and $m \geq 3$.

**Proof.** Suppose that $C_m \cap C_n$ is HMC for $m = n$. Note that, $|V(C_m \cap C_n)| = 2n$ and $|E(C_m + C_n)| = n + m + mn = 2n + n^2$ as $n = m$. Since, $|V(C_m \cap C_n)| = m + n = 2n$ as $n = m$. We have assume that $C_m \cap C_n$ is HMC for $n = m$. We have $v_f(1) = v_f(2) = n$.

**Case 1:** All the vertices of label 1 are in sequence in $C_m$ and $C_n$.

Then, it is clear that all the vertices of label 2 are in sequence in $C_m$ and $C_n$. Suppose that we have $t$ no. of vertices with label 1 in $C_m$. So, we have $(n - t)$ vertices of label 1 in $C_n$. Hence, we have $(m - t)$ vertices of label 2 in $C_m$ and $t$ vertices of label 2 in $C_n$. Note that, $e_f(1) = (t + 1) + (n - t + 1) + (n - t)n$ and $e_f(2) = (n - t - 1) + (t - 1) + (t - n - t)$. Then, $e_f(1) - e_f(2) = 2n + 2 + tn + n^2 - t^2$. We know that, $n > t$. So, $e_f(1) - e_f(2) > 1$.

**Case 2:** Some of the vertices of label 2 are not in sequence in $C_m$ and $C_n$.

Suppose that we have $t$ no. of vertices with label 1 in $C_m$. So, we have $(n - t)$ vertices of label 1 in $C_n$. Hence, we have $(m - t)$ vertices of label 2 in $C_m$ and $t$ vertices of label 2 in $C_n$. Suppose that there exist $i$ no. of vertices with label 2 are not in sequence in $C_m$ and $j$ no. of vertices with label 2 are not in sequence in $C_n$. Note that, $e_f(1) = (t+i+1) + (n-t+j+1) + (n-t)m$ and $e_f(2) = (n-t-i-1) + (t-j-1) + (t-n-t)$. Now, $e_f(2)$ in Case 2 $\leq e_f(2)$ in Case 1 and $e_f(1)$ in Case 2 $\geq e_f(1)$ in Case 1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$. We have already proved in Case 1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

**Case 3:** We have $m$ no. of vertices with label 1 in $C_m$ and $n$ no. of vertices with label 2 in $C_n$.

Note that, $e_f(1) = mn + m$ and $e_f(2) = n$. Then, $e_f(1) - e_f(2) = mn + m - n = mn > 1$ as $n = m$.

**Case 4:** We have $m$ no. of vertices with label 2 in $C_m$ and $n$ no. of vertices with label 1 in $C_n$.

Note that, $e_f(1) = mn + n$ and $e_f(2) = m$. Then, $e_f(1) - e_f(2) = mn + n - m > 1$ as $n = m$. Hence, $C_m \cap C_n$ is not HMC, where $m = n$ and $m \geq 3$.

□
Theorem 2.8. $C_m \lor C_n$ is not HMC, where $m + n$ is even and $m, n \geq 3$.

Proof. Note that, $|V(C_m \lor C_n)| = n + m$. Suppose that $C_m \lor C_n$ is HMC. Since we have $v_f(1) = \frac{n+m}{2} = v_f(2)$.

Case 1: All the vertices of label 1 and 2 are in sequence in $C_m$ and $C_n$
Suppose that we have $t$ no. of vertices with label 1 in $C_m$. So, we have $(n-t)$ vertices of of label 2 in $C_m$. Hence, we have $\frac{n+m}{2} - t$ vertices of label 1 in $C_m$ and $m - \frac{n+m}{2} + t$ vertices of label 2 in $C_m$. Note that, $e_f(1) = (t+1) + (\frac{n+m}{2} - t + 1) + tm + (\frac{n+m}{2} - t)(n-t)$ and $e_f(2) = (n-t-1) + (m - \frac{n+m}{2} + t - 1) + (n-t)(m - \frac{n+m}{2} + t)$. Then, $e_f(1) - e_f(2) = mt + 4 + n^2 - 3nt + 2t^2$. We know that $t = \frac{n+m}{2}$, So, we have $e_f(1) - e_f(2) = m^2 + 4 > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in $C_m$ and $C_n$
Suppose that we have $t$ no. of vertices with label 1 in $C_m$. So, we have $\frac{n+m}{2} - t$ vertices of label 1 in $C_m$. Hence, we have $(n-t)$ vertices of label 2 in $C_m$ and $(m - \frac{n+m}{2} + t)$ vertices of label 2 in $C_m$. Suppose that there exist $i$ no. of vertices with label 2 are not in sequence in $C_m$ and $j$ no. of vertices with label 2 are not in sequence in $C_m$. Note that, $e_f(1) = (t+i+1) + (\frac{n+m}{2} - j + 1) + tm + (n-t)(\frac{n+m}{2} - t)$ and $e_f(2) = (n-t-i-1) + (m - \frac{n+m}{2} + j - 1) + (n-t)(m - \frac{n+m}{2} + t)$. Now, $e_f(2)$ in Case 2 $\leq e_f(2)$ in Case 1 and $e_f(1)$ in Case 2 $\geq e_f(1)$ in Case 1. So, $e_f(1) - e_f(2)$ in this Case is $\geq e_f(1) - e_f(2)$ in Case 1. Now, we have already proved in Case 1 that $e_f(1) - e_f(2) > 1$. Hence, in this Case $e_f(1) - e_f(2) > 1$.

Case 3: $m > n$

Subcase 3.1: All the vertices in $C_n$ are with label 1.
So, we have $n$ no. of vertices with label 1 in $C_n$. Suppose that we have $t$ no. of vertices with label 1 in $C_m$. So, there exist $m - t$ no. of vertices with label 2 in $C_m$.

Subcase 3.1.1: All the vertices in $c_m$ are in sequence.
Then, $e_f(1) = n + (t+1) + mn + tn$ and $e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = mn - m + n + 2t + n + 2 > 1$ as $mn > m$.

Subcase 3.1.2: All the vertices with label 2 are not in sequence in $c_m$.
Suppose that we have $i$ no. of vertices from $(m-t)$ no. of vertices are not in sequence in $c_m$. Then, $e_f(1) = n + (t+i+1) + mn$ and $e_f(2) = m - t - i - 1$. Now, $e_f(2)$ in Subcase 3.1.2 $\leq e_f(2)$ in Subcase 3.1.1 and $e_f(1)$ in Subcase 3.1.2 $\geq e_f(1)$ in Subcase 3.1.1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subcase 3.1.1. Now, we have already proved in Subcase 3.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

Subcase 3.2: All the vertices in $C_n$ are with label 2.
So, we have $n$ no. of vertices with label 2 in $C_n$. Suppose that we have $t$ no. of vertices with label 1 in $C_m$. So, there exist $m - t$ no. of vertices with label 2 in $C_m$.

Subcase 3.2.1: All the vertices in $c_m$ are in sequence.
Then, $e_f(1) = t + 1 + tn$ and $e_f(2) = n + m - t - 1$. Then, $e_f(1) - e_f(2) = nt - n - m + 2t + 2$.
We know that $t = \frac{n+m}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + \frac{n^2}{2} + 2 > 1$.

Subcase 3.2.2: All the vertices in $c_m$ are not in sequence.
Suppose that we have $i$ no. of vertices from $(n-t)$ no. of vertices are not in sequence in $c_m$. Then, $e_f(1) = t + i + 1 + tn$ and $e_f(2) = m - t - i - 1 + n + (m - t)$. Now, $e_f(2)$ in Subcase 3.2.2 $\leq e_f(2)$ in Subcase 3.2.1 and $e_f(1)$ in Subcase 3.2.2 $\geq e_f(1)$ in Subcase 3.2.1. So, $e_f(1) - e_f(2)$ in this Case $\geq e_f(1) - e_f(2)$ in Subcase 3.2.1. Now, we have already proved in Subcase 3.2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case. Hence, $C_m \lor C_n$ is not HMC, where $n + m$ is even and $m, n \geq 3$. □
Theorem 2.9. \( C_m \lor C_n \) is not HMC, where \( m + n \) is odd and \( m, n \geq 3 \).

Proof. Note that, \( |V(C_m \lor C_n)| = n + m = 2k + 1 \). Suppose that \( C_n \lor C_m \) is HMC. Without loss of generality we may assume that \( m > n \).

In this Case we have two possibilities.

(i) \( v_f(1) = \frac{m+n+1}{2} \) and \( v_f(2) = \frac{m+n-1}{2} \)

(ii) \( v_f(1) = \frac{m+n-1}{2} \) and \( v_f(2) = \frac{m+n+1}{2} \)

So, we consider the following cases.

Case 1: \( v_f(1) = \frac{n+m-1}{2} = k + 1 \) and \( v_f(2) = \frac{n+m+1}{2} = k \)

Subcase 1.1: All the vertices of label 1 are in sequence in \( C_n \) and \( C_m \)

Then, it is clear that all the vertices of label 2 are in sequence in \( C_n \) and \( C_m \). Suppose that we have \( t \) no. of vertices with label 1 in \( C_n \). So, we have \( (n-t) \) vertices of label 2 in \( C_n \). Hence, we have \((k+1-t)\) vertices of label 1 in \( C_m \) and \((k-n+t)\) vertices of label 2 in \( C_m \). Note that, \( e_f(1) = (t+1)+(k+2-t)+tm+(k+1-t)(n-t) \) and \( e_f(2) = (n-t-1)+(k-n+t-1)+(n-t)(k-n+t) \). Then, \( e_f(1) - e_f(2) = (n-t)^2 + 5tm + (n-t)(1-t) = (n-t)(n+1-2t)+tm+5 \). Now, \( e_f(1) - e_f(2) > 1 \) if \( n+1 \geq 2t \). If \( n+1 < 2t \), then \( \frac{n+1}{2} > \frac{n+1}{2} \). Therefore, \( m > k \).

Suppose that \( t = \frac{n+1}{2} + l \). Then, \( e_f(1) - e_f(2) = 2l + \frac{n+1}{2} + lm + 5 > 1 \).

Subcase 1.2: Some of the vertices of label 2 are not in sequence in \( C_n \) and \( C_m \)

Suppose that we have \( t \) no. of vertices with label 1 in \( C_n \). So, we have \( (n-t) \) vertices of label 2 in \( C_n \). Hence, we have \((k+1-t)\) vertices of label 1 in \( C_m \) and \((k-n+t)\) vertices of label 2 in \( C_m \). Suppose that there exist \( i \) no. of vertices with label 2 are not in sequence in \( C_n \) and \( j \) no. of vertices with label 2 are not in sequence in \( C_m \).

Note that, \( e_f(1) = (t+i+1)+(k-t+j+2)+tm+(k+1-t)(n-t) \) and \( e_f(2) = (n-t-i-1)+(k-n+t-j-1)+(n-t)(k-n+t) \). Then, \( e_f(1) - e_f(2) = (n-t)^2 + 3tm + (t-n)(1-t) = (n-t)(n-1+2t)+tm+3 \). Now, \( e_f(1) - e_f(2) > 1 \) if \( n+1 \geq 2t \). If \( n+1 < 2t \), then \( \frac{n+1}{2} < t \). Therefore, \( m > k \).

Suppose that \( t = \frac{n+1}{2} + l \). Then, \( e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{m}{2}) + 3l(m-n-1) > 1 \), if \( m \geq n+1 \). Suppose that \( m \leq n+1 \). Then since, \( m \geq n \), we have \( m = n+1 \). So, we have \( e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{m}{2}) + 3l(m-n-1) > 1 \).

Subcase 2.2: Some of the vertices of label 2 are not in sequence in \( C_n \) and \( C_m \)

Suppose that we have \( t \) no. of vertices with label 1 in \( C_n \). So, we have \( (n-t) \) vertices of label 2 in \( C_n \). Hence, we have \((k+1-t)\) vertices of label 1 in \( C_m \) and \((k-n+t)\) vertices of label 2 in \( C_m \). Suppose that there exist \( i \) no. of vertices with label 2 are not in sequence in \( C_n \) and \( j \) no. of vertices with label 2 are not in sequence in \( C_m \).

Note that, \( e_f(1) = (t+i+1)+(k-t+j+1)+tm+(n-t)(k-t) \) and \( e_f(2) = (n-t-i-1)+(k-n+t-j)+(n-t)(k-n+t+1) \). Now, \( e_f(2) \) in Subcase 2.2 \( \leq e_f(2) \) in Subcase 2.1 and \( e_f(1) \) in Subcase 2.2 \( \geq e_f(1) \) in Subcase 2.1. So, \( e_f(1) - e_f(2) \) in this
Case is $\geq e_f(1) - e_f(2)$ in Subcase 2.1. Now, we have already proved in Subcase 2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

**Case 3:** $m > n$

**Subcase 3.1:** All the vertices in $C_n$ are with label 1.

So, we have n no. of vertices with label 1 in $C_n$. Suppose that we have t no. of vertices with label 1 in $C_m$. So, there exist $m - t$ no. of vertices with label 2 in $C_m$.

**Subsubcase 3.1.1:** All the vertices in $C_m$ are in sequence.

Then, $e_f(1) = n + (t + 1) + mn + tn$ and $e_f(2) = m - t - 1$. Then, $e_f(1) - e_f(2) = (mn - m) + n + 2t + tn + 2 > 1$ as $mn > m$.

**Subsubcase 3.1.2:** All the vertices with label 2 are not in sequence in $C_m$.

Suppose that we have i no. of vertices from $(m - t)$ no. of vertices are not in sequence in $C_m$. Then, $e_f(1) = n + (t + i + 1) + mn$ and $e_f(2) = m - t - i - 1$. Now, $e_f(2)$ in Subcase 3.1.2 $\leq e_f(2)$ in subsubcase 3.1.1 and $e_f(1)$ in Subcase 3.1.2 $\geq e_f(1)$ in Subcase 3.1.1. So, $e_f(1) - e_f(2)$ in this case is $\geq e_f(1) - e_f(2)$ in Subcase 3.1.1. Now, we have already proved in Subcase 3.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this Case.

**Subcase 3.2:** All the vertices in $C_n$ are with label 2.

So, we have n no. of vertices with label 2 in $C_n$. Suppose that we have t no. of vertices with label 1 in $C_m$. So, there exist $m - t$ no. of vertices with label 2 in $C_m$.

**Subsubcase 3.2.1:** All the vertices in $C_n$ are in sequence.

Then, $e_f(1) = t + 1 + tn$ and $e_f(2) = n + m - t - 1$. Then, $e_f(1) - e_f(2) = nt - n - m + 2t + 2$.

In this Case we have two possibilities.

(i) $t = \frac{n + m + 1}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + \frac{n^2}{2} + \frac{n}{2} + 3 > 1$

(ii) $t = \frac{n + m - 1}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + \frac{n^2}{2} - \frac{n}{2} + 3 > 1$.

**Subsubcase 3.2.2:** All the vertices in $C_n$ are not in sequence.

Suppose that we have i no. of vertices from $(n - t)$ no. of vertices are not in sequence in $C_m$. Then, $e_f(1) = t + i + 1 + tn$ and $e_f(2) = m - t - i - 1 + n + n(m - t)$. Now, $e_f(2)$ in Subcase 3.2.2 $\leq e_f(2)$ in Subcase 3.2.1 and $e_f(1)$ in Subcase 3.2.2 $\geq e_f(1)$ in Subcase 3.2.1. So, $e_f(1) - e_f(2)$ in this case is $\geq e_f(1) - e_f(2)$ in Subcase 3.2.1. Now, we have already proved in Subcase 3.2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case. Hence, $C_m \lor C_n$ is not HMC, where $n + m$ is odd and $m, n \geq 3$.

**Corollary 2.2.** $C_m \lor C_n$ is not HMC, where $n, m \in \mathbb{N}$, $m, n \geq 3$.

**Proof.** Proof follows from Theorems 2.7, 2.8 and 2.9.

**3. Conclusion**

In this article we have proved that $CH_n \odot K_1$ and the tensor product $P_m \times P_n$ are HMC. Also we have proved that Complete bipartite graphs $K_{m,n}$, $K_n \lor C_m$ and $C_n \lor C_m$ are not HMC graphs.

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References


Jaydeep Parejiya is presently working as a lecturer in Mathematics in Government Polytechnic, Rajkot, Gujarat, India. He has completed his Ph.D. at Saurashtra University, Rajkot under the guidance of Dr. S. Visweswaran on Commutative Ring Theory.

Daxa B. Jani is presently working as an Assistant Professor at Darshan University, Rajkot, Gujarat, India. She obtained her B.Sc. Degree in Mathematics from Shree M. and N. Virani Science College, Saurashtra University, Rajkot, Gujarat in 2010 and her M. Sc. Degree in Applied Mathematics and in pure mathematics from the Faculty of Technology and Engineering, The Maharaja Sayajirao University of Baroda, Gujarat in 2015. Her fields of interest include Graph Theory, Numerical Analysis and Complex Analysis.

Yesha M. Hathi is a research scholar at Children’s University, Gandhinagar, Gujarat, doing her research work under the guidance of Dr. M.M. Jariya. She completed her B.Sc. at M. N. Virani Science College, Saurashtra University, Rajkot, Gujarat with distinction in 2016 and her M.Sc. from Shree M. and N. Virani Science College, Saurashtra University, Rajkot, Gujarat in 2018.