

APPLICATIONS OF THE GENERALIZED HOMOGENEOUS q -SHIFT OPERATOR IN q -POLYNOMIALS

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ABSTRACT. In this paper, we construct the generalized homogeneous q -shift operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right)$. Then, we apply this operator to derive some q -identities such as: the generating function and its extension, Rogers formula and its extension, Mehler's formula and its extension, Srivastava-Agarwal type bilinear generating functions for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$. Also, we obtain a transformation formula involving generating functions for $\phi_n^{(a,b)}(x, y, c|q)$. We provide some special values for $\phi_n^{(a,b)}(x, y, c|q)$ in order to establish identities for the polynomials $\phi_n^{(a)}(x)$ and $\phi_n^{(a,b)}(x, y|q)$.

Keywords: the homogeneous q -difference operator, the homogeneous q -shift operator, q -Hahn polynomials, the generalized Al-Salam-Carlitz q -polynomials, generating function, Rogers formula, Mehler's formula.

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1. INTRODUCTION

In this paper, we will follow common notations and definitions for the q -series that used in [8]. We assume that $|q| < 1$.

For $a \in \mathbb{C}$, the q -shifted factorial is defined by [8]

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),$$

and the multiple q -shifted factorials by:

$$(a_1, a_2, \dots, a_r; q)_m = (a_1; q)_m (a_2; q)_m \cdots (a_r; q)_m,$$

where $m \in \mathbb{Z}$ or ∞ .

The basic hypergeometric series ${}_r\phi_s$ is presented as follows [8]:

$${}_r\phi_s \left(\begin{matrix} \alpha_1, \dots, \alpha_r \\ \beta_1, \dots, \beta_s \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1, \dots, \alpha_r; q)_n}{(q, \beta_1, \dots, \beta_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n,$$

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where $q \neq 0$ when $r > s + 1$. Note that

$${}_{r+1}\phi_r \left(\begin{matrix} \alpha_1, \dots, \alpha_{r+1} \\ \beta_1, \dots, \beta_r \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1, \dots, \alpha_{r+1}; q)_n}{(\beta_1, \dots, \beta_r; q)_n} x^n, \quad |x| < 1.$$

The q -binomial coefficient is defined as [8]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \quad \text{for } 0 \leq k \leq n,$$

where $n, k \in \mathbb{N}$.

The Cauchy identity is given by:

$$\sum_{m=0}^{\infty} \frac{(a; q)_m}{(q; q)_m} x^m = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad |x| < 1. \quad (1)$$

For $a = 0$, Cauchy identity becomes Euler's identity [8]:

$$\sum_{m=0}^{\infty} \frac{x^m}{(q; q)_m} = \frac{1}{(x; q)_{\infty}}, \quad |x| < 1, \quad (2)$$

which has the following inverse relation:

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} x^m}{(q; q)_m} = (x; q)_{\infty}. \quad (3)$$

The q -Chu-Vandermonde's sum is given by [8]:

$${}_2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a; q)_n}{(c; q)_n} a^n. \quad (4)$$

Heine's transformation of ${}_2\phi_1$ series [8, Appendix III, equation (III.1)] is:

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(b, az; q)_{\infty}}{(c, z; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} c/b, z \\ az \end{matrix} ; q, b \right). \quad (5)$$

Heine's transformation of ${}_2\phi_1$ series [8, Appendix III, equation (III.3)] is:

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(abz/c; q)_{\infty}}{(z; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} c/a, c/b \\ c \end{matrix} ; q, abz/c \right). \quad (6)$$

Transformation of ${}_3\phi_2$ series [8, Appendix III, equation (III.12)] is:

$${}_3\phi_2 \left(\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix} ; q, q \right) = \frac{(e/c; q)_n}{(e; q)_n} c^n {}_3\phi_2 \left(\begin{matrix} q^{-n}, c, d/b \\ d, q^{1-n}c/e \end{matrix} ; q, bq/e \right). \quad (7)$$

Transformation of ${}_2\phi_1$ [2, equation (5.3)] is:

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(abz/c; q)_{\infty}}{(az/c; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} b, c/a, 0 \\ qc/az, c \end{matrix} ; q, q \right). \quad (8)$$

We will use the following identities in this paper:

$$(aq^{-n}; q)_n = (q/a; q)_n (-a)^n q^{-n - \binom{n}{2}}. \quad (9)$$

$$(q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk}. \quad (10)$$

In 1949, Hahn [10] defined the following polynomials:

$$\phi_n^{(\alpha)}(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (\alpha; q)_k x^k. \quad (11)$$

In 1965, Al-Salam and Carlitz [1] provided the generating function and Mehler's formula for $\phi_n^{(a)}(x)$ as follows:

The generating function for $\phi_n^{(a)}(x)$ is [1, 12, 14].

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x) \frac{\tau^n}{(q; q)_n} = \frac{(ax\tau; q)_{\infty}}{(\tau, x\tau; q)_{\infty}}, \quad \max\{|\tau|, |x\tau|\} < 1. \tag{12}$$

Mehler's formula for $\phi_n^{(a)}(x)$ is [1, 12, 14]:

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x) \phi_n^{(b)}(y) \frac{\tau^n}{(q; q)_n} = \frac{(ax\tau, by\tau; q)_{\infty}}{(\tau, x\tau, y\tau; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} a, b, \tau \\ ax\tau, by\tau \end{matrix}; q, xy\tau \right), \tag{13}$$

where $\max\{|\tau|, |x\tau|, |y\tau|, |xy\tau|\} < 1$.

In 1972, Carlitz [4] extended the generating function for $\phi_n^{(a)}(x)$ as follows:

$$\sum_{n=0}^{\infty} \phi_{n+k}^{(a)}(x) \frac{t^n}{(q; q)_n} = \frac{(\alpha xt; q)_{\infty}}{(t, xt; q)_{\infty}} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(t, \alpha; q)_j}{(\alpha xt; q)_j} x^j, \quad \max\{|t|, |xt|\} < 1. \tag{14}$$

In 1989, Srivastava and Agarwal [17] proposed the following generating function for $\phi_n^{(a)}(x)$:

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x) (\lambda; q)_n \frac{\tau^n}{(q; q)_n} = \frac{(\lambda\tau; q)_{\infty}}{(\tau; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} \lambda, a \\ \lambda\tau \end{matrix}; q, x\tau \right), \quad \max\{|\tau|, |x\tau|\} < 1. \tag{15}$$

The Cauchy polynomials are defined as follows [6, 13, 15, 16]:

$$P_n(x, y) = (x - y)(x - qy) \cdots (x - q^{n-1}y) = (y/x; q)_n x^n,$$

which has the following generating function:

$$\sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}, \quad |xt| < 1. \tag{16}$$

Goulden and Jackson in [9] gave the following identity:

$$P_n(x, y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k x^{n-k}. \tag{17}$$

In 2003, Chen et al. [5] presented the homogeneous q -difference operator D_{xy} , which performs on functions in two variables as follows:

$$D_{xy} \{f(x, y)\} = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}.$$

The homogeneous q -shift operator was built as the q -exponential of the homogeneous q -shift operator, as shown below:

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D_{xy}^k}{(q; q)_k}. \tag{18}$$

Proposition 1.1. [5]. *We have*

$$D_{xy}^k \{P_n(x, y)\} = \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y). \tag{19}$$

$$D_{xy}^k \left\{ \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} = t^k \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}. \tag{20}$$

In 2010, Chen et al. [7] constructed the following homogeneous q -shift operator:

$$\mathbb{F}(aD_{xy}) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (aD_{xy})^k}{(q; q)_k}. \quad (21)$$

In 2014, Cao [3] defined the homogeneous q -difference operator

$$T(a, zD_{xy}) = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} (zD_{xy})^k. \quad (22)$$

In 2014, Zhou and Luo [19] discovered some new generating functions for q -Hahn polynomials and proved them using the homogeneous q -difference operator $\mathbb{E}(D_{xy})$.

The Rogers formula for $\phi_n^{(a)}(x|q)$ is [19]:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \phi_{n+k}^{(a)}(x) \frac{t^n}{(q; q)_n} \frac{\ell^k}{(q; q)_k} = \frac{(ax\ell; q)_{\infty}}{(\ell, x\ell, xt; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} xa, x\ell \\ xal \end{matrix}; q, t \right), \quad (23)$$

where $\max\{|t|, |\ell|, |x\ell|, |xt|\} < 1$.

The extension of Mehler's formula for $\phi_n^{(a)}(x)$ is [19]:

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_{m+n}^{(\alpha)}(x) \phi_n^{(\beta)}(y) \frac{t^n}{(q; q)_n} \\ &= \frac{(\alpha xt; q)_{\infty}}{(t, xt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(\beta; q)_k (xyt)^k}{(q; q)_k} \sum_{j=0}^{m+k} \begin{bmatrix} m+k \\ j \end{bmatrix} \frac{(xt, x\alpha; q)_j}{(\alpha xt; q)_j} x^{m-j}, \end{aligned} \quad (24)$$

provided $\max\{|t|, |xt|\} < 1$.

In 2020, Srivastava and Arjika [18] established the generalized Al-Salam-Carlitz q -polynomials $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y|q)$ in the following form:

$$\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1, a_2, \dots, a_{s+1}; q)_k}{(b_1, b_2, \dots, b_s; q)_k} x^k y^{n-k}. \quad (25)$$

Srivastava and Arjika [18] gave the following results:

Theorem 1.1. [18]. *We have*

The generating function for $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y|q)$ is:

$$\sum_{n=0}^{\infty} \phi_n^{(\mathbf{a}, \mathbf{b})}(x, y|q) \frac{t^n}{(q; q)_n} = \frac{1}{(yt; q)_{\infty}} {}_{s+1}\phi_s \left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, xt \right), \quad (26)$$

where $\max\{|xt|, |yt|\} < 1$.

The Srivastava-Agarwal type generating function for $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y|q)$ is:

$$\sum_{n=0}^{\infty} \phi_n^{(\mathbf{a}, \mathbf{b})}(x, y|q) (\lambda; q)_n \frac{t^n}{(q; q)_n} = \frac{(\lambda yt; q)_{\infty}}{(yt; q)_{\infty}} {}_{s+2}\phi_{s+1} \left(\begin{matrix} a_1, \dots, a_{s+1}, \lambda \\ b_1, \dots, b_s, \lambda yt \end{matrix}; q, xt \right), \quad (27)$$

where $\max\{|xt|, |yt|\} < 1$.

In 2022, the generalised q -hypergeometric polynomials were defined by Reshem and Saad [11] as follows:

$$\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y, c|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} c^k P_{n-k}(x, y), \quad (28)$$

where $\mathbf{a} = (a_1, a_2, \dots, a_r)$, $\mathbf{b} = (b_1, b_2, \dots, b_s)$.

- Setting $r = 1, s = 0, a_1 = a, x = 1, y = 0, c = x$ in (28), we get the q -Hahn polynomials $\phi_n^{(a)}(x)$ defined in equation (11).
- Letting $r = s + 1, y = 0, x = y, c = x$ in (28), we get the generalized Al-Salam-Carlitz q -polynomials $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y)$ defined in (25).

This paper is arranged as follows: In Section 2, we'll present the generalized homogeneous q -shift operator ${}_r\Phi_s$ and then find some of its identities. The generating function and its extension for the polynomials $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y, c|q)$ are presented in Section 3 using the operator representation (30). In Section 4, the operator approach to Rogers' formula and its extension for the polynomials $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y, c|q)$ will be driven. For the polynomials $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y, c|q)$, we establish Mehler's formula and its extension in Section 5. In Section 6, for the polynomials $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y, c|q)$, we develop a Srivastava-Agarwal type bilinear generating function. We use the homogeneous q -difference operator ${}_r\Phi_s$ in Section 7 to obtain the transformation formula for the polynomials $\phi_n^{(\mathbf{a}, \mathbf{b})}(x, y, c|q)$.

2. GENERALIZED HOMOGENEOUS q -SHIFT OPERATOR AND SOME OF ITS OPERATOR IDENTITIES

In this section, we will begin by introducing the generalised homogeneous q -shift operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right)$ and then we will proceed to find some of its identities.

Definition 2.1. We define the generalized homogeneous q -shift operator as follows:

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) = \sum_{k=0}^{\infty} \frac{W_k}{(q; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} (cD_{xy})^k, \tag{29}$$

where $W_k = \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k}$.

By providing specific values to the generalized homogeneous q -shift operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right)$, several previously stated operators can be obtained, for more information, see [3, 5, 7].

The following lemma is straightforward to prove:

Lemma 2.1. Let the operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right)$ be defined as in (29), then

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) \{P_n(x, y)\} = \phi_n^{(\mathbf{a}, \mathbf{b})}(x, y, c|q). \tag{30}$$

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) \left\{ \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ct \right), \tag{31}$$

where $|xt| < 1$.

Lemma 2.2. Let the operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right)$ be defined as in (29), then

$${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) \left\{ \frac{P_k(x, y) (yt; q)_{\infty}}{(yt; q)_k (xt; q)_{\infty}} \right\}$$

$$= \frac{(yt; q)_\infty}{(xt; q)_\infty} t^{-k} \sum_{j=0}^k \frac{(q^{-k}, xt; q)_j}{(q, yt; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right), \quad |xt| < 1. \tag{32}$$

Proof. By using q -Chu-Vandermonde sum (4), we find that

$$\begin{aligned} \frac{P_k(x, y)}{(yt; q)_k} \frac{(yt; q)_\infty}{(xt; q)_\infty} &= \frac{(yt; q)_\infty}{(xt; q)_\infty} \frac{(y/x; q)_k x^k}{(yt; q)_k} \\ &= \frac{(yt; q)_\infty}{(xt; q)_\infty} t^{-k} {}_2\phi_1(q^{-k}, xt; yt; q, q) \\ &= t^{-k} \sum_{j=0}^k \frac{(q^{-k}; q)_j}{(q; q)_j} \frac{(q^j yt; q)_\infty}{(q^j xt; q)_\infty} q^j. \end{aligned} \tag{33}$$

By using (33) and (31), we obtain

$$\begin{aligned} &{}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{P_k(x, y)}{(yt; q)_k} \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\} \\ &= t^{-k} \sum_{j=0}^k \frac{(q^{-k}; q)_j}{(q; q)_j} q^j {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{(q^j yt; q)_\infty}{(q^j xt; q)_\infty} \right\} \\ &= t^{-k} \sum_{j=0}^k \frac{(q^{-k}; q)_j}{(q; q)_j} q^j \frac{(ytq^j; q)_\infty}{(xtq^j; q)_\infty} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right). \end{aligned}$$

□

3. THE GENERATING FUNCTION FOR $\phi_n^{(a,b)}(x, y, c|q)$

In this section, we use the operator representation (30) to drive the generating function and its extension for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$. We give some special values for the parameters in the generating function as well as its extension for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$ to get the generating function and its extension for both polynomials $\phi_n^{(a)}(x)$ and $\phi_n^{(a,b)}(x, y|q)$.

Theorem 3.1. (Generating function for $\phi_n^{(a,b)}(x, y, c|q)$). *We have*

$$\sum_{n=0}^{\infty} \phi_n^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ct \right), \quad |xt| < 1. \tag{34}$$

Proof. By using (30), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \phi_n^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \{P_n(x, y)\} \frac{t^n}{(q; q)_n} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\}, \quad |xt| < 1 \\ &= \frac{(yt; q)_\infty}{(xt; q)_\infty} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ct \right). \end{aligned}$$

□

- Setting $r = 1, s = 0, a_1 = a, x = 1, y = 0, c = x$ in (3.1), we recover the generating function for the polynomials $\phi_n^{(a)}(x)$ (equation (12)).
- Letting $r = s + 1, y = 0, x = y, c = x$ in (3.1), we regain the generating function for the polynomials $\phi_n^{(a,b)}(x, y|q)$ (equation (26)).

Theorem 3.2. (An extension to generating function for $\phi_n^{(a,b)}(x, y, c|q)$). *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_{n+k}^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_n} \\ &= \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} t^{-k} \sum_{j=0}^k \frac{(q^{-k}, xt; q)_j}{(q, yt; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right), \quad |xt| < 1. \end{aligned} \tag{35}$$

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_{n+k}^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \{P_{n+k}(x, y)\} \frac{t^n}{(q; q)_n} \quad (\text{by using (30)}) \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_{n+k}(x, y) \frac{t^n}{(q; q)_n} \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ P_k(x, y) \sum_{n=0}^{\infty} P_n(x, q^k y) \frac{t^n}{(q; q)_n} \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ P_k(x, y) \frac{(q^k yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} \quad (\text{by using (16)}) \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{P_k(x, y) (yt; q)_{\infty}}{(yt; q)_k (xt; q)_{\infty}} \right\} \\ &= \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} t^{-k} \sum_{j=0}^k \frac{(q^{-k}, xt; q)_j}{(q, yt; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right). \quad (\text{by using (32)}) \end{aligned}$$

□

- Setting $k = 0$ in equation (35), we obtain the generating function for $\phi_n^{(a,b)}(x, y, c|q)$ (equation (34)).

Lemma 3.1. *We have*

$${}_3\phi_2 \left(\begin{matrix} q^{-n}, b, c \\ d, 0 \end{matrix}; q, q \right) = c^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(c, d/b; q)_k}{(d; q)_k} (b/c)^k. \tag{36}$$

Proof. Setting $e = 0$ in (7), we obtain

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} q^{-n}, b, c \\ d, 0 \end{matrix}; q, q \right) &= c^n \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(c, d/b; q)_k}{(d; q)_k} \frac{(bq)^k}{(-1)^k q^{\binom{k}{2}} q^{-nk} (qc)^k} \\ &= c^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(c, d/b; q)_k}{(d; q)_k} (b/c)^k. \quad (\text{by using (10)}) \end{aligned}$$

□

- By giving special values to the parameters in equation (35), we retrieve an extension to the generating function for the polynomials $\phi_n^{(a)}(x)$ (equation (14)).

Proof. Setting $r = 1, s = 0, a_1 = a, x = 1, y = 0, c = x$ in (35), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \phi_{n+k}^{(a)}(x|q) \frac{t^n}{(q; q)_n} &= \frac{t^{-k}}{(t; q)_{\infty}} \sum_{j=0}^k \frac{(q^{-k}, t; q)_j}{(q; q)_j} q^j {}_1\phi_0 \left(\begin{matrix} a \\ 0 \end{matrix}; q, xtq^j \right) \\ &= \frac{t^{-k}}{(t; q)_{\infty}} \sum_{j=0}^k \frac{(q^{-k}, t; q)_j}{(q; q)_j} q^j \frac{(axtq^j; q)_{\infty}}{(xtq^j; q)_{\infty}} \quad (\text{by using (1)}) \\ &= \frac{(axt; q)_{\infty}}{(t, xt; q)_{\infty}} t^{-k} \sum_{j=0}^k \frac{(q^{-k}, t, xt; q)_j}{(q, axt; q)_j} q^j \\ &= \frac{(axt; q)_{\infty}}{(t, xt; q)_{\infty}} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(t, a; q)_j}{(axt; q)_j} x^j, \quad (\text{by using (36)}) \end{aligned}$$

where $\max\{|t|, |xt|\} < 1$. □

- If $r = s + 1, y = 0, x = y, c = x$ in equation (35), we obtain an extension to the generating function for the polynomials $\phi_n^{(a,b)}(x, y|q)$ as follows:

Corollary 3.1. (An extension to generating function for $\phi_n^{(a,b)}(x, y|q)$). *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} \phi_{n+k}^{(a,b)}(x, y|q) \frac{t^n}{(q; q)_n} &= \frac{t^{-k}}{(yt; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{s+1}; q)_n}{(b_1, \dots, b_s; q)_n} (xt)^n \\ &\quad \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (yt; q)_j (-1)^j q^{\binom{j}{2} + (n-k+1)j}, \quad |yt| < 1. \end{aligned}$$

4. ROGERS FORMULA FOR $\phi_n^{(a,b)}(x, y, c|q)$

In this section, we will present an operator approach to Rogers formula and its extension for the generalized q -hypergeometric polynomials $\phi_n^{(a,b)}(x, y, c|q)$. The Rogers formula and its extension for the q -Hahn polynomials $\phi_n^{(a)}(x)$ and the extension of the Rogers formula for the generalized Al-Salam-Carlitz q -polynomials $\phi_n^{(a,b)}(x, y|q)$ are obtained by incorporating special values for the parameters in the Rogers formula and its extension for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$.

Theorem 4.1. (Rogers formula for $\phi_n^{(a,b)}(x, y, c|q)$). *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \phi_{n+k}^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_n} \frac{\ell^k}{(q; q)_k} &= \frac{(y\ell; q)_{\infty}}{(t/\ell, x\ell; q)_{\infty}} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(x\ell; q)_k}{(q, y\ell, q\ell/t; q)_k} q^k {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, c\ell q^k \right), \quad \max\{|t/\ell|, |x\ell|\} < 1. \end{aligned} \quad (37)$$

Proof.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \phi_{n+k}^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_n} \frac{\ell^k}{(q; q)_k}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) \{ P_{n+k}(x, y) \} \frac{t^n}{(q; q)_n} \frac{\ell^k}{(q; q)_k} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \sum_{k=0}^{\infty} P_k(x, q^n y) \frac{\ell^k}{(q; q)_k} \right\} \\
 &\qquad\qquad\qquad \text{(by using (16))} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{(q^n y \ell; q)_{\infty}}{(x \ell; q)_{\infty}} \frac{t^n}{(q; q)_n} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{xy} \right) \left\{ \frac{P_n(x, y) (y \ell; q)_{\infty}}{(y \ell; q)_n (x \ell; q)_{\infty}} \right\} \quad \text{(by using (32))} \\
 &= \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} \ell^{-n} \frac{(y \ell; q)_{\infty}}{(x \ell; q)_{\infty}} \sum_{k=0}^n \frac{(q^{-n}, x \ell; q)_k}{(q, y \ell; q)_k} q^k {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right) \\
 &= \frac{(y \ell; q)_{\infty}}{(x \ell; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(t/\ell)^n}{(q; q)_n} \sum_{k=0}^n \frac{(q^{-n}, x \ell; q)_k}{(q, y \ell; q)_k} q^k {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right) \\
 &= \frac{(y \ell; q)_{\infty}}{(x \ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x \ell; q)_k}{(q, y \ell; q)_k} q^k {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right) \sum_{n=k}^{\infty} \frac{(t/\ell)^n (-1)^k q^{\binom{k}{2} - nk}}{(q; q)_{n-k}} \\
 &\qquad\qquad\qquad \text{(by using (10))} \\
 &= \frac{(y \ell; q)_{\infty}}{(x \ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x \ell; q)_k}{(q, y \ell; q)_k} q^k {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right) \sum_{n=0}^{\infty} \frac{(t/\ell)^{n+k} (-1)^k q^{\binom{k}{2} - (n+k)k}}{(q; q)_n} \\
 &= \frac{(y \ell; q)_{\infty}}{(x \ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x \ell; q)_k (t/\ell)^k (-1)^k q^{-\binom{k}{2}}}{(q, y \ell; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right) \sum_{n=0}^{\infty} \frac{(q^{-k} t/\ell)^n}{(q; q)_n} \\
 &= \frac{(y \ell; q)_{\infty}}{(x \ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x \ell; q)_k (t/\ell)^k (-1)^k q^{-\binom{k}{2}}}{(q, y \ell; q)_k (q^{-k} t/\ell; q)_{\infty}} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right) \\
 &= \frac{(y \ell; q)_{\infty}}{(x \ell, t/\ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x \ell; q)_k (t/\ell)^k (-1)^k q^{-\binom{k}{2}}}{(q, y \ell; q)_k (q^{-k} t/\ell; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right) \\
 &= \frac{(y \ell; q)_{\infty}}{(x \ell, t/\ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x \ell; q)_k q^k}{(q, y \ell, q \ell/t; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, c \ell q^k \right). \quad \text{(by using (9))}
 \end{aligned}$$

□

- Rogers formula for polynomials $\phi_n^{(a)}(x)$ (equation (23)) can be preserved by using special values for the parameters in equation (37).

Proof. Substitute $r = 1, s = 0, a_1 = a, x = 1, y = 0, c = x$ in equation (37), we get

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \phi_{n+k}^{(a)}(x|q) \frac{t^n}{(q; q)_n} \frac{\ell^k}{(q; q)_k} \\
 &= \frac{1}{(\ell, t/\ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(\ell; q)_k}{(q, q \ell/t; q)_k} q^k {}_1\phi_0 \left(\begin{matrix} a \\ 0 \end{matrix} ; q, x \ell q^k \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(\ell, t/\ell; q)_\infty} \sum_{k=0}^\infty \frac{(\ell; q)_k}{(q, q\ell/t; q)_k} q^k \sum_{n=0}^\infty \frac{(a; q)_n}{(q; q)_n} (x\ell q^k)^n \\
 &= \frac{1}{(\ell, t/\ell; q)_\infty} \sum_{k=0}^\infty \frac{(\ell; q)_k}{(q, q\ell/t; q)_k} q^k \frac{(q^k a x \ell)_\infty}{(q^k x \ell)_\infty} \\
 &= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, t/\ell; q)_\infty} \sum_{k=0}^\infty \frac{(\ell, x\ell; q)_k}{(q, ax\ell, q\ell/t; q)_k} q^k \\
 &= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, t/\ell; q)_\infty} {}_3\phi_2 \left(\begin{matrix} \ell, x\ell, 0 \\ ax\ell, q\ell/t \end{matrix}; q, q \right) \\
 &= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, t; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, \ell \\ ax\ell \end{matrix}; q, xt \right) \quad (\text{by using (8)}) \\
 &= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, xt; q)_\infty} {}_2\phi_1 \left(\begin{matrix} ax, x\ell \\ ax\ell \end{matrix}; q, t \right), \quad (\text{by using (6)})
 \end{aligned}$$

where $\max\{|\ell|, |x\ell|, |xt|, |t|\} < 1$. □

Theorem 4.2. (An extension to Rogers formula for $\phi_n^{(a,b)}(x, y, c|q)$). *we have*

$$\begin{aligned}
 &\sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty \phi_{n+m+k}^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\
 &= \frac{(y\ell; q)_\infty}{(x\ell, \tau/t, t/\ell; q)_\infty} \sum_{k=0}^\infty \frac{(x\ell; q)_k q^k}{(q, y\ell, q\ell/t; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, c\ell q^k \right), \quad (38)
 \end{aligned}$$

where $\max\{|x\ell|, |\tau/t|, |t/\ell|\} < 1$.

Proof.

$$\begin{aligned}
 &\sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty \phi_{n+m+k}^{(a,b)}(x, y, c|q) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\
 &= \sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \{P_{n+m+k}(x, y)\} \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^\infty \sum_{m=0}^\infty P_{n+m}(x, y) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \right. \\
 &\quad \left. \times \sum_{k=0}^\infty P_k(x, q^{n+m}y) \frac{\ell^k}{(q; q)_k} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^\infty \sum_{m=0}^\infty P_{n+m}(x, y) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{(q^{n+m}y\ell, q)_\infty}{(x\ell; q)_\infty} \right\} \\
 &\quad (\text{by using (16)}) \\
 &= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{P_{n+m}(x, y)}{(y\ell; q)_{n+m}} \frac{(y\ell; q)_\infty}{(x\ell; q)_\infty} \right\} \\
 &= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \ell^{-(n+m)} \frac{(y\ell; q)_\infty}{(x\ell; q)_\infty} \sum_{k=0}^{n+m} \frac{(q^{-(n+m)}, x\ell, q)_k q^k}{(q, y\ell; q)_k}
 \end{aligned}$$

$$\begin{aligned}
 & \times {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \quad (\text{by using (32)}) \\
 &= \frac{(yl; q)_\infty}{(xl; q)_\infty} \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(t/\ell)^n (\tau/\ell)^m}{(q; q)_{n+m} (q; q)_m} \sum_{k=0}^{n+m} \frac{(q^{-(n+m)}, xl, q)_k q^k}{(q, yl; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \\
 &= \frac{(yl; q)_\infty}{(xl; q)_\infty} \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(t/\ell)^n (\tau/\ell)^m}{(q; q)_m} \sum_{k=0}^{n+m} \frac{(xl, q)_k q^k (-1)^k q^{\binom{k}{2}} q^{-(n+m)k}}{(yl; q)_k (q; q)_{n+m-k} (q; q)_k} \\
 & \times {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \quad (\text{by using (10)}) \\
 &= \frac{(yl; q)_\infty}{(xl; q)_\infty} \sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{n+m=k}^\infty \frac{(t/\ell)^n (\tau/\ell)^m (xl, q)_k q^k (-1)^k q^{\binom{k}{2}} q^{-(n+m)k}}{(q; q)_m (yl; q)_k (q; q)_{n+m-k} (q; q)_k} \\
 & \times {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \\
 &= \frac{(yl; q)_\infty}{(xl; q)_\infty} \sum_{m=0}^\infty \sum_{i=0}^\infty \sum_{k=0}^\infty \frac{(\tau/t)^m (t/\ell)^{k+i} (-1)^k q^{\binom{k}{2}} q^{-(k+i)k+k} (xl, q)_k}{(q; q)_m (q; q)_i (q; q)_k (yl; q)_k} \\
 & \times {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \\
 &= \frac{(yl; q)_\infty}{(xl; q)_\infty} \sum_{m=0}^\infty \frac{(\tau/t)^m}{(q; q)_m} \sum_{k=0}^\infty \frac{(-1)^k q^{-\binom{k}{2}} (t/\ell)^k (xl, q)_k}{(q; q)_k (yl; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \\
 & \times \sum_{i=0}^\infty \frac{(q^{-k}t/\ell)^i}{(q; q)_i} \\
 &= \frac{(yl; q)_\infty}{(xl, \tau/t; q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{-\binom{k}{2}} (t/\ell)^k (xl, q)_k}{(q; q)_k (yl; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \frac{1}{(q^{-k}t/\ell; q)_\infty} \\
 &= \frac{(yl; q)_\infty}{(xl, \tau/t, t/\ell; q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{-\binom{k}{2}} (t/\ell)^k (xl, q)_k}{(q; q)_k (yl, q^{-k}t/\ell; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right) \\
 &= \frac{(yl; q)_\infty}{(xl, \tau/t, t/\ell; q)_\infty} \sum_{k=0}^\infty \frac{(xl; q)_k q^k}{(q, yl, ql/t; q)_k} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, clq^k \right). \quad (\text{by using (9)})
 \end{aligned}$$

□

- Letting $\tau = 0$ in equation (38), we get Rogers formula for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$ (equation (4.1)).
- When $r = 1, s = 0, a_1 = a, x = 1, y = 0, c = x$ in equation (38), we get an extension to the Rogers formula for the polynomials $\phi_n^{(a)}(x|q)$.

Corollary 4.1 (An extension to Rogers formula for $\phi_n^{(a)}(x|q)$). *We have*

$$\begin{aligned}
 & \sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty \phi_{n+m+k}^{(a)}(x|q) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\
 &= \frac{(ax\ell; q)_\infty}{(\ell, xl, xt, \tau/\ell; q)_\infty} {}_2\phi_1 \left(\begin{matrix} ax, xl \\ ax\ell \end{matrix} ; q, t \right),
 \end{aligned}$$

where $\max\{|\ell|, |x\ell|, |xt|, \tau/\ell\} < 1$.

- Setting $r = s + 1, y = 0, x = y, c = x$ in equation (38), we get an extension to the Rogers formula for the polynomials $\phi_n^{(a,b)}(x, y|q)$.

Corollary 4.2 (An extension to Rogers formulaa for $\phi_n^{(a,b)}(x, y|q)$).

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \phi_{n+m+k}^{(a,b)}(x, y|q) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\ &= \frac{1}{(yt, \tau/t, t/\ell; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(y\ell; q)_k}{(q, \ell q/t; q)_k} q^k {}_{s+1}\phi_s \left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, x\ell q^k \right), \end{aligned}$$

where $\max\{|y\ell|, |\tau/t|, |t/\ell|\} < 1$.

5. MEHLER'S FORMULA FOR $\phi_n^{(a,b)}(x, y, c|q)$

In this section, we construct Mehler's formula and its extension for the generalized q -hypergeometric polynomials $\phi_n^{(a,b)}(x, y, c|q)$ using the operator representation (30). Miller's formula and its extension for both q -Hahn polynomials $\phi_n^{(a)}(x|q)$ and generalized Al-Salam-Carlitz polynomials $\phi_n^{(a,b)}(x, y|q)$ are derived by providing special values for the variables in the Mehler's formula and its extension for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$.

Theorem 5.1. (Mehler's formula for $\phi_n^{(a,b)}(x, y, c|q)$). *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_n^{(a,b)}(x, y, c|q) \phi_n^{(d,e)}(u, v, f|q) \frac{t^n}{(q; q)_n} \\ &= \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_k(u, v) \frac{f^k}{(q; q)_k} \\ & \quad \times \sum_{j=0}^k \frac{(q^{-k}, xt; q)_j}{(q, yt; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right), \quad |xt| < 1. \end{aligned} \tag{39}$$

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_n^{(a,b)}(x, y, c|q) \phi_n^{(d,e)}(u, v, f|q) \frac{t^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \{P_n(x, y)\} \phi_n^{(d,e)}(u, v, f|q) \frac{t^n}{(q; q)_n} \quad (\text{by using (30)}) \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \right\} \\ & \quad \times \sum_{k=0}^n \binom{n}{k} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} f^k P_{n-k}(u, v) \quad (\text{by using (28)}) \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \right. \\ & \quad \left. \times \frac{f^k t^n}{(q; q)_k (q; q)_{n-k}} P_{n-k}(u, v) P_n(x, y) \right\} \end{aligned}$$

$$\begin{aligned}
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} [(-1)^k q^{\binom{k}{2}}]^{1+s-r} \frac{f^k t^{n+k}}{(q; q)_k (q; q)_n} \right. \\
 &\quad \left. \times P_k(u, v) P_{n+k}(x, y) \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} [(-1)^k q^{\binom{k}{2}}]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} \right. \\
 &\quad \left. \times P_k(x, y) \sum_{n=0}^{\infty} P_n(x, q^k y) \frac{t^n}{(q; q)_n} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} [(-1)^k q^{\binom{k}{2}}]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} \right. \\
 &\quad \left. \times P_k(x, y) \frac{(q^k y t; q)_{\infty}}{(x t; q)_{\infty}} \right\} \quad (\text{by using (16)}) \\
 &= \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} [(-1)^k q^{\binom{k}{2}}]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} \\
 &\quad \times {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{P_k(x, y) (y t; q)_{\infty}}{(y t; q)_k (x t; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} [(-1)^k q^{\binom{k}{2}}]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} t^{-k} \frac{(y t; q)_{\infty}}{(x t; q)_{\infty}} \\
 &\quad \times \sum_{j=0}^k \frac{(q^{-k}, x t; q)_j}{(q, y t; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, c t q^j \right) \quad (\text{by using (32)}) \\
 &= \frac{(y t; q)_{\infty}}{(x t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} [(-1)^k q^{\binom{k}{2}}]^{1+s-r} P_k(u, v) \frac{f^k}{(q; q)_k} \\
 &\quad \times \sum_{j=0}^k \frac{(q^{-k}, x t; q)_j}{(q, y t; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, c t q^j \right).
 \end{aligned}$$

□

- By using special values to the parameters in equation (39), we revive Mehler’s formula for the polynomials $\phi_n^{(a)}(x|q)$ (equation (13)).

Proof. Setting $r = 1, s = 0, a_1 = a, x = 1, y = 0, c = x, d_1 = b, u = 1, v = 0, f = y,$ and $t = \tau$ in equation (39), we get

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \phi_n^{(a)}(x|q) \phi_n^{(b)}(y|q) \frac{\tau^n}{(q; q)_n} \\
 &= \frac{1}{(\tau; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(b; q)_k}{(q; q)_k} y^k \sum_{j=0}^k \frac{(q^{-k}, \tau; q)_j}{(q; q)_j} q^j {}_1\phi_0 \left(\begin{matrix} a \\ 0 \end{matrix}; q, x \tau q^j \right) \\
 &= \frac{1}{(\tau; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(b; q)_k}{(q; q)_k} y^k \sum_{j=0}^k \frac{(q^{-k}, \tau; q)_j}{(q; q)_j} q^j \frac{(a x \tau q^j; q)_{\infty}}{(x \tau q^j; q)_{\infty}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(ax\tau; q)_\infty}{(\tau, x\tau; q)_\infty} \sum_{k=0}^\infty \frac{(b; q)_k}{(q; q)_k} y^k \sum_{j=0}^k \frac{(q^{-k}, \tau, x\tau; q)_j}{(q, ax\tau; q)_j} q^j \\
 &= \frac{(ax\tau; q)_\infty}{(\tau, x\tau; q)_\infty} \sum_{k=0}^\infty \frac{(b; q)_k}{(q; q)_k} (y\tau)^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(\tau, a; q)_j}{(ax\tau; q)_j} x^j \quad (\text{by using (36)}) \\
 &= \frac{(ax\tau; q)_\infty}{(\tau, x\tau; q)_\infty} \sum_{j=0}^\infty \frac{(\tau, a; q)_j}{(q, ax\tau; q)_j} x^j \sum_{k=j}^\infty \frac{(b; q)_k}{(q; q)_{k-j}} (y\tau)^k \\
 &= \frac{(ax\tau; q)_\infty}{(\tau, x\tau; q)_\infty} \sum_{j=0}^\infty \frac{(\tau, a, b; q)_j}{(q, ax\tau; q)_j} (xy\tau)^j \sum_{k=0}^\infty \frac{(bq^j; q)_k}{(q; q)_k} (y\tau)^k \\
 &= \frac{(ax\tau; q)_\infty}{(\tau, x\tau; q)_\infty} \sum_{j=0}^\infty \frac{(\tau, a, b; q)_j}{(q, ax\tau; q)_j} (xy\tau)^j \frac{(q^j by\tau; q)_\infty}{(y\tau; q)_\infty} \\
 &= \frac{(ax\tau, by\tau; q)_\infty}{(\tau, x\tau, y\tau; q)_\infty} \sum_{j=0}^\infty \frac{(\tau, a, b; q)_j}{(q, ax\tau, by\tau; q)_j} (xy\tau)^j \\
 &= \frac{(ax\tau, by\tau; q)_\infty}{(\tau, x\tau, y\tau; q)_\infty} {}_3\phi_2 \left(\begin{matrix} \tau, a, b \\ ax\tau, by\tau \end{matrix}; q, xy\tau \right).
 \end{aligned}$$

□

- Setting $r = s + 1, y = 0, x = y, c = x, v = 0, u = v, f = u$ in equation (39), we get Mehler’s formula for the polynomials $\phi_n^{(a,b)}(x, y|q)$.

Corollary 5.1.

$$\begin{aligned}
 &\sum_{n=0}^\infty \phi_n^{(a,b)}(x, y|q) \phi_n^{(d,e)}(u, v|q) \frac{t^n}{(q; q)_n} \\
 &= \frac{1}{(yt; q)_\infty} \sum_{k=0}^\infty \frac{(d_1, \dots, d_{s+1}; q)_k}{(e_1, \dots, e_s; q)_k} (uv)^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (yt; q)_j (-1)^j q^{\binom{j}{2} - j(k-1)} \\
 &\quad \times {}_{s+1}\phi_s \left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, xtq^j \right), \quad |yt| < 1.
 \end{aligned} \tag{40}$$

Theorem 5.2. (An extension to Mehler’s formula for $\phi_n^{(a,b)}(x, y, c|q)$). *We have*

$$\begin{aligned}
 &\sum_{n=0}^\infty \phi_{n+m}^{(a,b)}(x, y, c|q) \phi_n^{(d,e)}(u, v, f|q) \frac{t^n}{(q; q)_n} \\
 &= \frac{(yt; q)_\infty}{(xt; q)_\infty} \sum_{k=0}^\infty \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_k(u, v) \frac{f^k}{(q; q)_k} \\
 &\quad \times t^{-m} \sum_{j=0}^{m+k} \frac{(q^{-(m+k)}, xt; q)_j}{(q, yt; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right), \quad |xt| < 1.
 \end{aligned} \tag{41}$$

Proof. From (30), we have

$$\sum_{n=0}^\infty \phi_{m+n}^{(a,b)}(x, y, c|q) \phi_n^{(d,e)}(u, v, f|q) \frac{t^n}{(q; q)_n}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \{P_{m+n}(x, y)\} \phi_n^{(\mathbf{d}, \mathbf{e})}(u, v, f|q) \frac{t^n}{(q; q)_n} \\
&= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_{m+n}(x, y) \frac{(t)^n}{(q; q)_n} \right\} \\
&\quad \times \sum_{k=0}^n \frac{\begin{bmatrix} n \\ k \end{bmatrix} (d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_{n-k}(u, v) f^k \quad (\text{by using (28)}) \\
&= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \right. \\
&\quad \left. \times \frac{f^k t^n}{(q; q)_k (q; q)_{n-k}} P_{n-k}(u, v) P_{m+n}(x, y) \right\} \\
&= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \frac{f^k t^{n+k}}{(q; q)_k (q; q)_n} \right. \\
&\quad \left. \times P_k(u, v) P_{m+n+k}(x, y) \right\} \\
&= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} \right. \\
&\quad \left. \times P_{m+k}(x, y) \sum_{n=0}^{\infty} P_n(x, q^{m+k}y) \frac{t^n}{(q; q)_n} \right\} \\
&= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} \right. \\
&\quad \left. \times P_{m+k}(x, y) \frac{(q^{m+k}yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} \quad (\text{by using (16)}) \\
&= \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} \\
&\quad \times {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{P_{m+k}(x, y) (yt; q)_{\infty}}{(yt; q)_{m+k} (xt; q)_{\infty}} \right\} \\
&= \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_k(u, v) \frac{(ft)^k}{(q; q)_k} \\
&\quad \times t^{-(m+k)} \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \sum_{j=0}^{m+k} \frac{(q^{-(m+k)}, xt; q)_j}{(q, yt; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right) \\
&\quad \quad \quad (\text{by using (32)}) \\
&= \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_r; q)_k}{(e_1, \dots, e_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} P_k(u, v) \frac{f^k}{(q; q)_k} \\
&\quad \times t^{-m} \sum_{j=0}^{m+k} \frac{(q^{-(m+k)}, xt; q)_j}{(q, yt; q)_j} q^j {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right).
\end{aligned}$$

□

- Setting $m = 0$ in equation (41), we recover Mehler’s formula for $\phi_n^{(a,b)}(x, y, c|q)$ (equation (40)).
- Letting $r = 1, s = 0, a_1 = a, x = 1, y = 0, c = x, d_1 = b, u = 1, v = 0, f = y$ in equation (41), we reobtain an extension of Mehler’s formula for $\phi_n^{(a)}(x|q)$ (equation (24)).
- Setting $r = s + 1, y = 0, x = y, c = x, v = 0, u = v, f = u$ in (41), we obtain an extension of Mehler’s formula for $\phi_n^{(a,b)}(x, y|q)$.

Corollary 5.2. *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_{m+n}^{(a,b)}(x, y|q) \phi_n^{(d,e)}(u, v|q) \frac{t^n}{(q; q)_n} \\ &= \frac{1}{(yt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(d_1, \dots, d_{s+1}; q)_k}{(e_1, \dots, e_s; q)_k} (uv)^k t^{-m} \sum_{j=0}^{m+k} \begin{bmatrix} m+k \\ j \end{bmatrix} (yt; q)_j \\ & \quad \times (-1)^j q^{\binom{j}{2} - j(m+k-1)} {}_{s+1}\phi_s \left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, xtq^j \right), \quad |yt| < 1. \end{aligned}$$

6. THE SRIVASTAVA-AGARWAL TYPE BILINEAR GENERATING FUNCTIONS FOR $\phi_n^{(a,b)}(x, y, c|q)$

In this section, we create the Srivastava-Agarwal type bilinear generating function for the generalized q -hypergeometric polynomials $\phi_n^{(a,b)}(x, y, c|q)$. We use specific parameter values to reobtain Mehler’s formula for q -Hahn polynomials $\phi_n^{(a)}(x|q)$. In addition, we build the Srivastava-Agarwal type generating function for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$. We use given parameter values to reconstruct the Srivastava-Agarwal type generating function for both $\phi_n^{(a)}(x|q)$ and $\phi_n^{(a,b)}(x, y|q)$.

Theorem 6.1. (Srivastava-Agarwal type bilinear generating function for $\phi_n^{(a,b)}(x, y, c|q)$). *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q) \phi_n^{(a,b)}(u, v, c|q) \frac{t^n}{(q; q)_n} \\ &= \frac{(vt, \alpha x; q)_{\infty}}{(ut, x; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(\alpha, ut; q)_n}{(q, vt, q/x; q)_n} q^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^n \right), \end{aligned} \tag{42}$$

provided $\max\{|ut|, |x|\} < 1$.

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q) \phi_n^{(a,b)}(u, v, c|q) \frac{t^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (\alpha; q)_k x^k \phi_n^{(a,b)}(u, v, f|q) \frac{t^n}{(q; q)_n} \quad (\text{by using (11)}) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (\alpha; q)_k x^k {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{uv} \right) \{P_n(u, v)\} \frac{t^n}{(q; q)_n} \quad (\text{by using (30)}) \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{uv} \right) \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (\alpha; q)_k x^k P_n(u, v) \frac{t^n}{(q; q)_n} \right\} \end{aligned}$$

$$\begin{aligned}
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{uv} \right) \left\{ \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(\alpha; q)_k x^k P_n(u, v) t^n}{(q; q)_k (q; q)_{n-k}} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{uv} \right) \left\{ \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha; q)_k x^k P_{n+k}(u, v) t^{n+k}}{(q; q)_k (q; q)_n} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{uv} \right) \left\{ \sum_{k=0}^{\infty} P_k(u, v) \frac{(\alpha; q)_k (xt)^k}{(q; q)_k} \sum_{n=0}^{\infty} P_n(u, q^k v) \frac{t^n}{(q; q)_n} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{uv} \right) \left\{ \sum_{k=0}^{\infty} P_k(u, v) \frac{(\alpha; q)_k (xt)^k (q^k vt; q)_{\infty}}{(q; q)_k (ut; q)_{\infty}} \right\} \\
 &\hspace{15em} \text{(by using (16))} \\
 &= \sum_{k=0}^{\infty} \frac{(\alpha; q)_k (xt)^k}{(q; q)_k} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, cD_{uv} \right) \left\{ \frac{P_k(u, v) (vt; q)_{\infty}}{(vt; q)_k (ut; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(\alpha; q)_k (xt)^k}{(q; q)_k} \frac{(vt; q)_{\infty}}{(ut; q)_{\infty}} t^{-k} \sum_{n=0}^k \frac{(q^{-k}, ut; q)_n}{(q, vt; q)_n} q^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ctq^n \right) \\
 &\hspace{15em} \text{(by using (32))} \\
 &= \frac{(vt; q)_{\infty}}{(ut; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(\alpha; q)_k}{(q; q)_k} x^k \sum_{n=0}^k \frac{(q^{-k}, ut; q)_n}{(q, vt; q)_n} q^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ctq^n \right) \\
 &= \frac{(vt; q)_{\infty}}{(ut; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (ut; q)_n}{(q, vt; q)_n} q^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ctq^n \right) \sum_{k=n}^{\infty} \frac{(\alpha; q)_k x^k q^{-nk}}{(q; q)_{k-n}} \\
 &= \frac{(vt; q)_{\infty}}{(ut; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (ut, \alpha; q)_n}{(q, vt; q)_n} x^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ctq^n \right) \sum_{k=0}^{\infty} \frac{(q^n \alpha; q)_k}{(q; q)_k} (q^{-n} x)^k \\
 &= \frac{(vt; q)_{\infty}}{(ut; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (ut, \alpha; q)_n}{(q, vt; q)_n} x^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ctq^n \right) \frac{(\alpha x; q)_{\infty}}{(q^{-n} x; q)_{\infty}} \\
 &= \frac{(vt, \alpha x; q)_{\infty}}{(ut, x; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (ut, \alpha; q)_n}{(q, vt; q)_n (q^{-n} x; q)_n} x^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ctq^n \right) \\
 &= \frac{(vt, \alpha x; q)_{\infty}}{(ut, x; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(ut, \alpha; q)_n}{(q, vt, q/x; q)_n} q^n {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, ctq^n \right). \quad \text{(by using (9))}
 \end{aligned}$$

□

- We give special values for the parameters in equation (42) to reobtain Mehler’s formula for the q -Hahn polynomials $\phi_n^{(a)}(x|q)$ (equation (13)).

Proof. Setting $r = 1, s = 0, \alpha = a, \mathbf{a} = b, \mathbf{b} = 0, u = 1, v = 0, c = y,$ and $t = \tau$ in equation (42), we get

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \phi_n^{(a)}(x|q) \phi_n^{(b)}(y|q) \frac{t^n}{(q; q)_n} \\
 &= \frac{(ax; q)_{\infty}}{(t, x; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(t, a; q)_n q^n}{(q, q/x; q)_n} {}_1\phi_0 \left(\begin{matrix} b \\ 0 \end{matrix} ; q, ytq^n \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(ax; q)_\infty}{(t, x; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n q^{-\binom{n}{2}} (t, a; q)_n}{(q, q^{-n}x; q)_n} x^n {}_1\phi_0 \left(\begin{matrix} b \\ 0 \end{matrix}; q, ytq^n \right) \\
 &= \frac{1}{(t; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n q^{-\binom{n}{2}} (t, a; q)_n}{(q, q^{-n}x; q)_n} x^n {}_1\phi_0 \left(\begin{matrix} b \\ 0 \end{matrix}; q, ytq^n \right) \frac{(ax; q)_\infty}{(q^{-n}x; q)_\infty} \\
 &= \frac{1}{(t; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n q^{-\binom{n}{2}} (t; q)_n}{(q; q)_n} x^n {}_1\phi_0 \left(\begin{matrix} b \\ 0 \end{matrix}; q, ytq^n \right) \sum_{k=0}^\infty \frac{(a; q)_{n+k}}{(q; q)_k} (q^{-n}x)^k \\
 &= \frac{1}{(t; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n q^{-\binom{n}{2}} (t; q)_n}{(q; q)_n} x^n {}_1\phi_0 \left(\begin{matrix} b \\ 0 \end{matrix}; q, ytq^n \right) \sum_{k=n}^\infty \frac{(a; q)_k}{(q; q)_{k-n}} (q^{-n}x)^{k-n} \\
 &= \frac{1}{(t; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k}{(q; q)_k} x^k \sum_{n=0}^k \frac{(q^{-k}, t; q)_n q^n}{(q; q)_n} {}_1\phi_0 \left(\begin{matrix} b \\ 0 \end{matrix}; q, ytq^n \right) \\
 &= \frac{(byt; q)_\infty}{(t, yt; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k}{(q; q)_k} x^k \sum_{n=0}^k \frac{(q^{-k}, t, yt; q)_n}{(q, byt; q)_n} q^n \quad (\text{by using (1)}) \\
 &= \frac{(byt; q)_\infty}{(t, yt; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k}{(q; q)_k} (xt)^k \sum_{n=0}^k \begin{bmatrix} k \\ n \end{bmatrix} \frac{(t, b; q)_n}{(byt; q)_n} y^n \quad (\text{by using (36)}) \\
 &= \frac{(byt; q)_\infty}{(t, yt; q)_\infty} \sum_{n=0}^\infty \frac{(t, b; q)_n}{(q, byt; q)_n} y^n \sum_{k=n}^\infty \frac{(a; q)_k}{(q; q)_{k-n}} (xt)^k \\
 &= \frac{(byt; q)_\infty}{(t, yt; q)_\infty} \sum_{n=0}^\infty \frac{(t, a, b; q)_n}{(q, byt; q)_n} (xyt)^n \sum_{k=0}^\infty \frac{(aq^n; q)_k}{(q; q)_k} (xt)^k \\
 &= \frac{(byt; q)_\infty}{(t, yt; q)_\infty} \sum_{n=0}^\infty \frac{(t, a, b; q)_n}{(q, byt; q)_n} (xyt)^n \frac{(q^n axt; q)_\infty}{(xt; q)_\infty} \\
 &= \frac{(axt, byt; q)_\infty}{(t, xt, yt; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, t \\ axt, byt \end{matrix}; q, xyt \right).
 \end{aligned}$$

□

Theorem 6.2. *We have*

$$\begin{aligned}
 &\sum_{n=0}^\infty \phi_n^{(a,b)}(x, y, c|q) P_n(u, v) \frac{t^n}{(q; q)_n} \\
 &= \frac{(v/u, yut; q)_\infty}{(xut; q)_\infty} \sum_{j=0}^\infty \frac{(xut; q)_j (v/u)^j}{(q, yut; q)_j} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cutq^j \right), \quad (43)
 \end{aligned}$$

provided $|xut| < 1$.

Proof.

$$\begin{aligned}
 &\sum_{n=0}^\infty \phi_n^{(a,b)}(x, y, c|q) P_n(u, v) \frac{t^n}{(q; q)_n} \\
 &= \sum_{n=0}^\infty {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \{P_n(x, y)\} P_n(u, v) \frac{t^n}{(q; q)_n}
 \end{aligned}$$

$$\begin{aligned}
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} P_n(u, v) \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} v^k u^{n-k} \right\} \\
 &\hspace{15em} \text{(by using (17))} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} v^k}{(q; q)_k} \sum_{n=k}^{\infty} \frac{P_n(x, y) u^{n-k} t^n}{(q; q)_{n-k}} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} v^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{P_{n+k}(x, y) u^n t^{n+k}}{(q; q)_n} \right\} \\
 &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (vt)^k P_k(x, y)}{(q; q)_k} \sum_{n=0}^{\infty} \frac{P_n(x, q^k y)}{(q; q)_n} (ut)^n \right\} \\
 &= {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (vt)^k P_k(x, y)}{(q; q)_k} \frac{(q^k y ut; q)_{\infty}}{(xut; q)_{\infty}} \right\} \\
 &\hspace{15em} \text{(by using (16))} \\
 &= {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (vt)^k P_k(x, y)}{(q; q)_k} \frac{(yut; q)_{\infty}}{(yut; q)_k} \frac{(q^k y ut; q)_{\infty}}{(xut; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (vt)^k}{(q; q)_k} {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right) \left\{ \frac{P_k(x, y)}{(yut; q)_k} \frac{(yut; q)_{\infty}}{(xut; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (vt)^k}{(q; q)_k} (ut)^{-k} \frac{(yut; q)_{\infty}}{(xut; q)_{\infty}} \sum_{j=0}^k \frac{(q^{-k}, xut; q)_j q^j}{(q, yut; q)_j} \\
 &\quad \times {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cutq^j \right) \quad \text{(by using (32))} \\
 &= \frac{(yut; q)_{\infty}}{(xut; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(xut; q)_j (-1)^j q^{\binom{j}{2}} q^j}{(q, yut; q)_j} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cutq^j \right) \\
 &\quad \times \sum_{k=j}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (v/u)^k q^{-kj}}{(q; q)_{k-j}} \\
 &= \frac{(v/u, yut; q)_{\infty}}{(xut; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(xut; q)_j (v/u)^j}{(q, yut; q)_j} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cutq^j \right).
 \end{aligned}$$

□

- Setting $v/u = \lambda$ and $ut \rightarrow t$ in equation (43), we get the following corollary:

Corollary 6.1. (Srivastava-Agarwal type generating function for $\phi_n^{(a,b)}(x, y, c|q)$).
 We have

$$\sum_{n=0}^{\infty} \phi_n^{(a,b)}(x, y, c|q) (\lambda; q)_n \frac{t^n}{(q; q)_n}$$

$$= \frac{(\lambda, yt; q)_\infty}{(xt; q)_\infty} \sum_{j=0}^{\infty} \frac{(xt; q)_j \lambda^j}{(q, yt; q)_j} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^j \right), \quad |xt| < 1. \quad (44)$$

- We give special values for the parameters in equation (44) to get Srivastava-Agarwal type generating function for $\phi_n^{(a)}(x)$ (equation (15)).
- Letting $r = s + 1$, $x = y$, $y = 0$, $c = x$ in equation (44) and using Cauchy identity (1), we recover Srivastava-Agarwal type generating function for the polynomials $\phi_n^{(a,b)}(x, y|q)$ (equation (27)).

7. A TRANSFORMATION FORMULA INVOLVING GENERATING FUNCTIONS FOR $\phi_n^{(a,b)}(x, y, c|q)$

In this section, we derive the transformation formula for generalized q -hypergeometric polynomials $\phi_n^{(a,b)}(x, y, c|q)$ by using the homogeneous q -shift operator ${}_r\Phi_s$.

Theorem 7.1. *Let the coefficients $A(n)$ and $B(n)$ satisfy the following relation:*

$$\sum_{n=0}^{\infty} A(n) P_n(u, v) = \sum_{n=0}^{\infty} B(n) \frac{(q^n vt; q)_\infty}{(q^n ut; q)_\infty}. \quad (45)$$

Then

$$\sum_{n=0}^{\infty} A(n) \phi_n^{(a,b)}(u, v, c|q) = \sum_{n=0}^{\infty} B(n) \frac{(q^n vt; q)_\infty}{(q^n ut; q)_\infty} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^n \right), \quad (46)$$

where each of the series in (45) and (46) are absolutely convergent.

Proof. Let $f(u, v, c)$ be the right-hand side of (46). By using (45) and (30) we find that

$$\begin{aligned} f(u, v, c) &= \sum_{n=0}^{\infty} B(n) \frac{(q^n vt; q)_\infty}{(q^n ut; q)_\infty} {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, ctq^n \right) \\ &= \sum_{n=0}^{\infty} B(n) {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cDuv \right) \left\{ \frac{(q^n vt; q)_\infty}{(q^n ut; q)_\infty} \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cDuv \right) \left\{ \sum_{n=0}^{\infty} B(n) \frac{(q^n vt; q)_\infty}{(q^n ut; q)_\infty} \right\} \\ &= {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cDuv \right) \left\{ \sum_{n=0}^{\infty} A(n) P_n(u, v) \right\} \\ &= \sum_{n=0}^{\infty} A(n) {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cDuv \right) \{P_n(u, v)\} \\ &= \sum_{n=0}^{\infty} A(n) \phi_n^{(a,b)}(u, v, c|q), \end{aligned} \quad (48)$$

which is precisely the left-hand side of (46). The proof of Theorem 7.1 is completed. \square

- When $r = 1$, $s = 0$, $a_1 = a$, $u = 1$, $v = 0$, $c = u$ in equations (45) and (46), we get the transformation formula for the polynomials $\phi_n^{(a)}(x|q)$.

Corollary 7.1. *Let the coefficients $A(n)$ and $B(n)$ satisfy the following relationship:*

$$\sum_{n=0}^{\infty} A(n) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n t; q)_{\infty}}. \quad (49)$$

Then

$$\sum_{n=0}^{\infty} A(n) \phi_n^{(a)}(u|q) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n t; q)_{\infty}} \frac{(autq^n; q)_{\infty}}{(utq^n; q)_{\infty}}, \quad (50)$$

where each of the series in (49) and (50) are absolutely convergent.

- Setting $r = s + 1$, $v = 0$, $u = v$, $c = u$ in equations (45) and (46), we get the transformation formula for the polynomials $\phi_n^{(a,b)}(x, y|q)$.

Corollary 7.2. *Let the coefficients $A(n)$ and $B(n)$ satisfy the following relationship:*

$$\sum_{n=0}^{\infty} A(n)v^n = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n vt; q)_{\infty}} \quad (51)$$

Then

$$\sum_{n=0}^{\infty} A(n) \phi_n^{(a,b)}(u, v|q) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n vt; q)_{\infty}} {}_{s+1}\phi_s \left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, utq^n \right), \quad (52)$$

where each of the series in (51) and (52) are absolutely convergent.

8. CONCLUSIONS

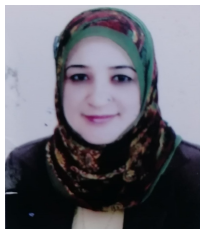
- (1) Several previously specified operators can be obtained by providing special values to the generalized homogeneous q -shift operator ${}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_{xy} \right)$, for more information, see [3, 5, 7].
- (2) The q -Hahn polynomials $\phi_n^{(a)}(x|q)$ are a special case of the generalized q -hypergeometric polynomials $\phi_n^{(a,b)}(x, y, c|q)$.
- (3) The generalized q -hypergeometric polynomials $\phi_n^{(a,b)}(x, y, c|q)$ are a generalization for the generalized Al-Salam-Carlitz q -polynomials $\phi_n^{(a,b)}(x, y|q)$.
- (4) The polynomials identities for $\phi_n^{(a,b)}(x, y, c|q)$ are a generalization of the q -Hahn polynomials $\phi_n^{(a)}(x|q)$ and the generalized Al-Salam-Carlitz q -polynomials $\phi_n^{(a,b)}(x, y|q)$.

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