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A COMPARATIVE AND ILLUSTRATIVE STUDY FOR SOLVING SINGULARLY PERTURBED PROBLEMS WITH TWO PARAMETERS

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Abstract. This computational study concerns approximate solutions of singularly perturbed one-dimensional boundary-value problems having perturbed convection and diffusion terms. Such kinds of problems take different stands depending on the perturbation parameters. Typically, when the problem is convection-dominated, classical discretization methods suffer from numerical instability issues. Therefore, standard methods require special treatment in convection dominance. To this end, in this work, the standard Galerkin finite element method (GFEM) is stabilized with the streamline-upwind/Petrov–Galerkin (SUPG) formulation. Beyond that, an asymptotic approach, called the successive complementary expansion method (SCEM), is also proposed. Two test examples are provided to evaluate and compare the proposed methods' performances for various values of the convection and diffusion parameters.

Keywords: Asymptotic expansion, finite elements, singularly perturbed, stabilization, two parameters.

AMS Subject Classification: 34H15, 65L11.

1. Introduction

Almost everyone involved in mathematics, physics, or engineering sciences has probably encountered the term "singular." In this work, the term "singular," which might refer to different meanings depending on the context, addresses the "singularly perturbed ordinary differential equations" where the highest-order derivative is controlled by a positive small parameter. Such models have a broad range of applications, for example, in control theory, thermal processes, quantum mechanics, fluid/solid mechanics, population dynamics, chemically reactive processes, financial mathematics, etc. For further information on the areas where singularly perturbed differential equations find applications, one can refer to the survey papers [1], [2], and [3].

Numerical discretization methods or asymptotic approaches (perturbation techniques) can be employed to obtain approximate solutions to singularly perturbed differential equations for which analytical solutions cannot be obtained. Since each approach has several

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advantages and drawbacks compared to each other, sometimes several combinations of these methods, i.e., hybrid methods, can also be utilized. In the context of numerical methods, although the Galerkin finite element method (GFEM) is one of the most robust numerical methods with very established theory and broad literature, it fails to achieve accurate approximations in solving problems dominated by advection. Therefore, to solve such problems accurately, the standard GFEM formulation needs to be stabilized with special techniques. The streamline-upwind/Petrov–Galerkin (SUPG) [4] formulation is one of the most matured, famous, and efficient methods for stabilizing the classical GFEM. However, in some cases, one may need an analytical expression, even if it is implicit, to investigate the behavior of solutions asymptotically or to observe the characteristics of the solutions over different regions of the problem domain when the exact solution is not available. In such cases, asymptotic methods provide expressions through which the behavior of solutions can be studied. The successive complementary expansion method (SCEM) [5] is an example of asymptotic methods and was designed to obtain uniformly valid approximations to singularly perturbed differential equations.

There are numerous studies devoted to the theory and numerical treatment of singularly perturbed problems in the literature. One can refer to [1] by Kumar and Mittal and [2] by Kadalbajoo and Gupta, in which they reviewed various numerical methods and asymptotic approaches for solving singularly perturbed differential equations. For the theoretical aspects, asymptotic properties, and various techniques for solving singularly perturbed problems, one can also refer to the reference books by Cousteix and Mauss [6], Gie et al. [7], Miller et al. [8], Roos et al. [9], Kevorkian and Cole [10], O'Malley [11], and Bender and Orszag [12].

In this work, we study singularly perturbed boundary-value problems in the following form:

$$-\varepsilon u''(x) + \mu b(x) u'(x) + c(x) u(x) = f(x), \quad x \in \Omega = (0,1), \tag{1}$$

subject to Dirichlet boundary conditions

$$u(0) = \alpha, \quad u(1) = \beta, \tag{2}$$

where $0 < \varepsilon \ll 1$ and $0 < \mu \ll 1$ are the diffusion and convection parameters, respectively, and $u' = \frac{du}{dx}$ and $u'' = \frac{d^2u}{dx^2}$ throughout this paper. The functions $b(x)$, $c(x)$, and $f(x)$ are assumed to be sufficiently smooth functions on $\overline{\Omega}$ satisfying the following conditions:

$$0 < b_0 \le b(x), \quad 0 < c_0 \le c(x), \quad c(x) - \frac{\mu}{2} b'(x) \ge d_0 > 0, \quad \forall x \in \Omega, \tag{3}$$

where $b_0,\ c_0,\ d_0 \in \mathbb{R}$. The assumptions given by Eq. (3) guarantee the uniqueness of the solution to the model problem given by Eqs. (1)–(2). In the case of $\mu = 0$, the problem represents a class of reaction-diffusion equations with two boundary-layers of width $\mathcal{O}\left(\varepsilon^{1/2} |\ln \varepsilon^{1/2}|\right)$ at both endpoints of the interval $\overline{\Omega}$. If $\mu = 1$, then the problem represents a convection-diffusion equation having a boundary-layer at the right end of the interval $\overline{\Omega}$, i.e., at $x = 1$, of width $\mathcal{O}(\varepsilon |\ln \varepsilon|)$. If $0 < \varepsilon,\ \mu \ll 1$, then the magnitude of the dominance of these parameters with respect to each other determines the structure of boundary-layers occurring near the endpoints.

The following lines and paragraphs provide a brief overview of research on singularly perturbed differential equations with multi-parameters that has been done since the early 2000s. Roos and Uzelac [13] considered the model problem by employing a streamline-diffusion finite element method (SDFEM) on a Shishkin mesh. Linß and Ross [14] used an upwind finite difference scheme to approximate the solution of the problem. A finite difference method (FDM) consisting of central differences, standard, and mid-point upwinding schemes were employed by Gracia et al. [15]. They reported that their scheme is uniformly second-order. Kadalbajoo and Yadaw [16] studied the problem by employing a B-spline collocation method on a Shishkin mesh obtaining second-order uniform convergence. Patidar [17] handled the problem using a FDM by turning the original problem into a system of first-order differential equations. Later, Surla et al. [18] considered the problem by utilizing a quadratic spline collocation method. Linß [19] solved the problem using a SDFEM deriving a posteriori error analysis. Kadalbajoo and Jha [20] constructed a finite difference scheme using cubic splines obtaining a first-order convergence. They also presented some error estimates for their scheme.

Wu et al. [21] treated the problem using upwind finite differences and provided a posteriori error estimate based on the maximum norm. Later, Das and Mehrmann [22] considered a time-dependent version of Eqs. (1)–(2) establishing error and convergence analyses. Chen et al. [23] studied a second-order non-monotone finite element scheme for the problem obtaining a priori error bounds. Brdar and Zarin [24] studied the model problem by employing a FEM on a Bakhvalov mesh providing error estimates in the energy norm. Later, again Brdar and Zarin [25] considered the problem on Duran/Duran–Shishkin type meshes establishing error estimates. Zarin [26] used an h-version of the standard FEM (also called the GFEM) to solve the problem comparing the results obtained on different types of meshes. Khandelwal and Khan [27] constructed a non-polynomial cubic spline method for solving the problem. Later on, Zahra and Daele [28] utilized cubic splines on Shishkin meshes.

In recent studies, Gupta et al. [29] handled a time-dependent version of Eqs. (1)–(2) by employing finite differences. They also provided stability and convergence analyses of their scheme, reporting that a first-order in time and a second-order convergence in space were achieved. Lu et al. [30] treated the problem using a rational spectral collocation method with a sinh transformation. O'Riordan et al. [31] used a parameter-uniform FDM on a Shishkin mesh providing comprehensive error analyses. Avijit and Natesan [32] employed a SUPG formulation on Shishkin/Duran–Shishkin meshes. Most recently, Kumar proposed a collocation method based on quintic B-spline basis functions on piecewise-uniform meshes [33].

In this work, the SCEM, which has previously been applied to a small number of 1D problems, is used to obtain asymptotic approximations for the model problem given by Eqs. (1)–(2). Later, the GFEM and SUPG formulations of the model problem are also provided. The methods are only formulated on the model problem to keep the study within a reasonable length. However, in sections where the relevant methods are employed, the necessary references that the interested reader might need are supplied. Here, we essentially concentrate on the behavior of solutions near steep gradients, stabilization of the traditional GFEM with the SUPG formulation to eliminate nonphysical oscillations, and demonstrating how the SCEM can be implemented to relatively "straightforward" problems in the form of Eqs. (1)–(2). Therefore, rather than obtaining theoretical findings such as convergence and stability analysis, we mainly aim visual comparison of the results obtained by employing the proposed methods and techniques. As a result, we intend to provide motivation for solving much more challenging problems by using these approaches and comparing the results obtained through them.

To the authors' best knowledge, singularly perturbed differential equations having multi-parameters are discussed for the first time here within the SCEM framework. It follows naturally that the proposed methods are compared for the first time in the multi-parameter context. The SCEM, GFEM, and SUPG formulations are also compared for the first time in the broad sense.

In Section 2, the SCEM, then the GFEM, and finally, the SUPG formulations are described on the model problem. In Section 3, two test problems, whose exact solutions are available, are considered. The analytical solutions and the SCEM approximations to the test problems are given/derived, and the results obtained with the proposed methods are compared. Finally, in the last section, the results are discussed in detail, and a comprehensive guide for possible future research is provided.

2. Treatment of The Problem

In this section, the asymptotic approach, SCEM, is explained first; later, the GFEM, and finally, the SUPG formulations of the model problem given by Eqs. (1)–(2) are introduced.

2.1. **Successive Complementary Expansion Method.**

The SCEM was introduced by Mauss and Cousteix in [5] for computing 1D Stokes–Oseen flow over a cylinder. Later on, the method was employed for computing several 2D problems arising in computational fluid dynamics (CFD), see, e.g., [34] and the references therein. For more on the method, one can refer to [6]. The interested reader may also refer to the material in [35–40] for several applications of the SCEM to various problems, including nonlinear, turning-point, and delay differential equations.

Consider the model problem given by Eqs. (1)–(2) again:

$$-\varepsilon u''(x) + \mu b(x) u'(x) + c(x) u(x) = f(x), \quad x \in \Omega = (0,1).$$

Dividing both sides of Eq. (1) by the convection parameter, μ, yields

$$\frac{-\varepsilon}{\mu} u''(x) + b(x) u'(x) + \frac{1}{\mu} c(x) u(x) = \frac{1}{\mu} f(x), \quad x \in \Omega = (0,1). \tag{4}$$

Letting $\frac{\varepsilon}{\mu} = \kappa$, one obtains

$$-\kappa u''(x) + b(x) u'(x) + \frac{1}{\mu} c(x) u(x) = \frac{1}{\mu} f(x), \quad x \in \Omega = (0,1), \tag{5}$$

with the same boundary conditions given by Eq. (2):

$$u(0) = \alpha, \quad u(1) = \beta. \tag{6}$$

As the parameter κ tends to zero, i.e., $\kappa \to 0^+$, although Eq. (5) has two boundary conditions introduced by Eq. (6), only one of them can be enforced since the order of Eq. (5) reduces by one. In order to overcome this challenge, we consider problem (5)–(6) on two (or more, depending on the problem) distinct regions called the *outer region* and *inner region (boundary-layer)*.

Let the expression

$$y(x,\kappa) = y_0(x,\kappa) + \kappa y_1(x,\kappa) + \mathcal{O}\left(\kappa^2\right) \tag{7}$$

be an asymptotic approximation for the outer region assuming that the problem has a boundary-layer about the point $x = 0$. Substituting the asymptotic expansion given by Eq. (7) into Eq. (5), one finds

$$-\kappa\left(y_0'' + \kappa y_1''\right) + b(x)\left(y_0' + \kappa y_1'\right) + \frac{1}{\mu} c(x)\left(y_0 + \kappa y_1\right) = \frac{1}{\mu} f(x). \tag{8}$$

Balancing the terms with respect to the orders of κ, the following initial-value problems are obtained:

$$b(x)y_0' + \frac{1}{\mu}c(x)y_0 = \frac{1}{\mu}f(x), \quad y_0(1) = \beta, \qquad \mathcal{O}(1), \tag{9}$$

$$-y_0'' + b(x)y_1' + \frac{1}{\mu}c(x)y_1 = 0, \quad y_1(1) = 0, \qquad \mathcal{O}(\kappa). \tag{10}$$

Now, we desire to obtain a complementary solution that is valid for the inner region. Considering a stretching-variable (boundary-layer) transformation given as

$$\overline{x} = \frac{x}{\kappa}, \tag{11}$$

and substituting it into Eq. (5), by employing the chain rule, i.e.,

$$\frac{d}{dx} = \frac{d\overline{x}}{dx}\frac{d}{d\overline{x}} = \frac{1}{\kappa}\frac{d}{d\overline{x}} \tag{12}$$

and

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \frac{1}{\kappa^2}\frac{d^2}{d\overline{x}^2}, \tag{13}$$

one obtains

$$-\kappa\frac{\Psi''(\overline{x},\kappa)}{\kappa^2} + \overline{b}(\overline{x})\frac{\Psi'(\overline{x},\kappa)}{\kappa} + \overline{c}(\overline{x})\Psi(\overline{x},\kappa) = \frac{1}{\mu}\overline{f}(\overline{x}). \tag{14}$$

Here, the functions $\overline{b}(\overline{x})$, $\overline{c}(\overline{x})$, and $\overline{f}(\overline{x})$ are the mappings of the functions $b(x)$, $c(x)$, and $f(x)$ under the transformation $\overline{x} = \frac{x}{\kappa}$, respectively. Rearranging Eq. (14) yields the following regularly perturbed boundary-value problem:

$$-\Psi''(\overline{x},\kappa) + \overline{b}(\overline{x})\Psi'(\overline{x},\kappa) + \kappa\overline{c}(\overline{x})\Psi(\overline{x},\kappa) = \frac{\kappa}{\mu}\overline{f}(\overline{x}), \tag{15}$$

with Dirichlet boundary conditions

$$\Psi(0,\kappa) = \alpha - y_0(0), \quad \Psi\left(\frac{1}{\kappa},\kappa\right) = 0. \tag{16}$$

Proposing an asymptotic approximation for Eq. (15) in the following form

$$\Psi(\overline{x},\kappa) = \Psi_0(\overline{x},\kappa) + \kappa\Psi_1(\overline{x},\kappa) + \mathcal{O}\left(\kappa^2\right), \tag{17}$$

and balancing the terms of the resulting equation with respect to the orders of κ yields the following boundary-value problems:

$$-\Psi_0''(\overline{x},\kappa) + \overline{b}(\overline{x})\Psi_0'(\overline{x},\kappa) = 0, \tag{18}$$

with boundary conditions

$$\Psi_0(0,\kappa) = \alpha - y_0(0), \quad \Psi_0\left(\frac{1}{\kappa},\kappa\right) = 0, \tag{19}$$

and

$$-\Psi_1''(\overline{x},\kappa) + \overline{b}(\overline{x})\Psi_1'(\overline{x},\kappa) + \kappa\overline{c}(\overline{x})\Psi_0(\overline{x},\kappa) = \frac{1}{\mu}\overline{f}(\overline{x}), \tag{20}$$

with boundary conditions

$$\Psi_1(0,\kappa) = -y_1(0), \quad \Psi_1\left(\frac{1}{\kappa},\kappa\right) = 0, \tag{21}$$

where the functions Ψ_0 and Ψ_1 are the zeroth- and first-order complementary SCEM approximations to be obtained as solutions to problems given by Eqs. (18)–(19) and Eqs. (20)–(21), respectively.

Then, the zeroth- and first-order SCEM approximations are given as follows:

$$\Phi_0^{\text{SCEM}}(x,\overline{x},\kappa) = y_0(x,\kappa) + \Psi_0(\overline{x},\kappa), \tag{22}$$

$$\Phi_1^{\text{SCEM}}(x,\overline{x},\kappa) = \Phi_0^{\text{SCEM}}(\overline{x},\kappa) + \kappa\left[y_1(x,\kappa) + \Psi_1(\overline{x},\kappa)\right], \tag{23}$$

where y_0 and y_1 are the solutions to the initial-value problems given by Eqs. (9)–(10).

For completeness, the nth-order SCEM approximation, in generalized form, can be given as follows [5,6]:

$$\Phi_n^{\text{SCEM}}(x,\overline{x},\kappa) = \sum_{i=0}^{n}\delta_i(\kappa)\left[y_i(x,\kappa) + \Psi_i(\overline{x},\kappa)\right], \tag{24}$$

where $\delta_i(\kappa)$ are asymptotic sequences. Adding further SCEM terms iteratively by using Eq. (24), one can obtain more accurate approximations.

2.2. **Galerkin Finite Element Method.** Consider the model problem given by Eqs. (1)–(2) to describe the GFEM formulation on it briefly. A variational formulation of problem can be formed multiplying both sides of Eq. (1) by a test function, $w(x) \in \mathcal{H}_0^1(\Omega)$, and integrating it over the computational domain, Ω, as follows:

$$-\varepsilon\int_\Omega \frac{d^2u(x)}{dx^2}w(x)\,dx + \mu\int_\Omega b(x)\frac{du(x)}{dx}w(x)\,dx + \int_\Omega c(x)u(x)w(x)\,dx = \int_\Omega f(x)w(x)\,dx, \tag{25}$$

where the Sobolev spaces $\mathcal{H}_0^1$ and $\mathcal{H}^1$ are defined as

$$\mathcal{H}_0^1(\Omega) = \{u \in \mathcal{H}^1(\Omega): \ u|_{\partial\Omega} = 0\} \tag{26}$$

and

$$\mathcal{H}^1(\Omega) = \{u : \Omega \to \mathbb{R} \ | \ u, u' \in L^2(\Omega)\}. \tag{27}$$

The space of the square-integrable functions, $L^2(\Omega)$, is defined as follows:

$$L^2(\Omega) = \left\{u : \Omega \to \mathbb{R} \ | \ \int_\Omega u^2 d\Omega < \infty\right\}. \tag{28}$$

If *integration by parts* applies to the first integral in Eq. (25), then, one finds the following weak formulation:

$$\varepsilon\int_0^1\left(\frac{du(x)}{dx}\frac{dw(x)}{dx}\right)dx - u'(1)w(1) + u'(0)w(0) + \mu\int_0^1 b(x)\frac{du(x)}{dx}w(x)\,dx + \int_0^1 c(x)u(x)w(x)\,dx = \int_0^1 f(x)w(x)\,dx. \tag{29}$$

For compactness, the weak formulation can be given as: find $u \in \mathcal{H}_0^1$ such that

$$a(u,w) = (f,w), \quad \forall w \in \mathcal{H}_0^1, \tag{30}$$

where $a(u,w)$ is a bilinear form defined by the inner product in $L^2(\Omega)$ as

$$a(u,v) = \varepsilon\left(u',w'\right) + \mu\left(bu',w\right) + (cu,w). \tag{31}$$

The weak formulation is discretized by dividing the computational domain, Ω, into a finite number of open intervals (elements) as follows:

$$\Omega_j = (x_{j-1},x_j), \quad j = 1,2,\ldots,N, \tag{32}$$

where N denotes the number of elements, $x_0 = 0$ and $x_N = 1$. And finally, by replacing the infinite-dimensional function space $\mathcal{H}_0^1$ with a finite-dimensional subspace, i.e., $S_0^h = X^h \bigcap \mathcal{H}_0^1$, the discrete weak formulation reads: find $u^h \in S_0^h$ such that

$$a\left(u^h, w^h\right) = \left(f^h, w^h\right), \quad \forall w^h \in \mathcal{S}_0^h, \tag{33}$$

where the finite element space of first-order polynomials defined on Ω is given as follows:

$$X^h = \left\{w^h \in \mathcal{C}\left(\Omega\right), w^h|_{\Omega_j} \in \mathcal{P}_1\left(\Omega_j\right) \;\; \forall j = 1, 2, \ldots, N\right\}. \tag{34}$$

In these formulations, the superscript "h" indicates the function belongs to a finite-dimensional space.

2.3. **Streamline-upwind/Petrov–Galerkin Method.** The GFEM is, unfortunately, not sufficient alone to handle convection-dominated problems accurately because the approximations generated by using the method involve spurious oscillations. In 1982, Brooks and Hughes [4] introduced the SUPG formulation as a modification to the standard GFEM for solving incompressible flow problems. Later, Tezduyar and Hughes [41–43] extended the method for computing compressible flow problems. The method has been developed since then, and its development is still ongoing. Interested readers can refer to the material in Linß and Stynes [44], Stynes and Tobiska [45], Tezduyar et al. [46, 47], Teafonov et al. [48], and Yin et al. [49] for more on the SUPG formulations.

To introduce the SUPG formulation, the residual function, $\mathcal{R}$, is required:

$$\mathcal{R}\left(u\right) = -\varepsilon u''\left(x\right) + \mu b\left(x\right) u'\left(x\right) + c\left(x\right) u\left(x\right) - f\left(x\right). \tag{35}$$

Then, the SUPG formulation reads: find $u^h \in S_0^h$ such that

$$a\left(u^h, w^h\right) + \sum_{e=1}^{N} \tau \left(\mathcal{R}\left(u^h\right), b^h \left(w'\right)^h\right) = \left(f^h, w^h\right), \quad \forall w^h \in \mathcal{S}_0^h, \tag{36}$$

where e is the element counter and τ is the stabilization parameter that plays a critical role in the accuracy of SUPG computations. In this study, for simplicity, we set the stabilization parameter as suggested in [50]:

$$\tau = \frac{h^e}{2|b^h(x)|}, \tag{37}$$

where the term h^e represents the element length associated with the element e. One can find more on the stabilization parameters in [50, 54], where the authors review and discuss a variety of stabilization parameters.

3. Test Computations

In this section, two test examples are studied to provide several comparisons of the approximations obtained by employing the proposed methods, i.e., the SCEM, GFEM, and SUPG. All computations are performed using Python, and the finite element computations are carried out in the FEniCS 2019.1.0 environment. For more on the FEniCS Project, see [51, 52].

Example 1. Consider the following singularly perturbed reaction-convection-diffusion problem [13, 16]:

$$-\varepsilon u''\left(x\right) + \mu u'\left(x\right) + u\left(x\right) = \cos\left(\pi x\right), \quad x \in \left(0, 1\right), \tag{38}$$

with boundary conditions

$$u\left(0\right) = 0, \quad u\left(1\right) = 0. \tag{39}$$

Problem (38)–(39) has an exact solution given as follows:

$$u(x) = \rho_1 \cos(\pi x) + \rho_2 \sin(\pi x) + Ae^{\lambda_1 x} + Be^{\lambda_2 (x-1)}, \tag{40}$$

where

$$\rho_1 = \frac{\varepsilon\pi^2 + 1}{\mu^2\pi^2 + (\varepsilon\pi^2+1)^2}, \quad \rho_2 = \frac{\mu\pi}{\mu^2\pi^2 + (\varepsilon\pi^2+1)^2}, \tag{41}$$

$$A = -\rho_1 \frac{1+e^{-\lambda_2}}{1-e^{\lambda_1-\lambda_2}}, \quad B = \rho_1 \frac{1+e^{\lambda_1}}{1-e^{\lambda_1-\lambda_2}}, \tag{42}$$

and

$$\lambda_1 = \frac{\mu - \sqrt{\mu^2+4\varepsilon}}{2\varepsilon}, \quad \lambda_2 = \frac{\mu + \sqrt{\mu^2+4\varepsilon}}{2\varepsilon}. \tag{43}$$

The zeroth-order SCEM approximation to the solution of Eqs. (38)–(39) is obtained as follows:

$$\Phi_0^{\text{SCEM}}(x, \overline{x}, \kappa) = \frac{-e^{-x/\mu} + \pi\mu \sin \pi x + \cos \pi x}{(\pi^2\mu^2+1)} + \frac{e^{-1/\mu} - \pi\mu \sin \pi - \cos \pi}{(\pi^2\mu^2+1)\left(1-e^{-1/\kappa}\right)}\left(e^{\frac{x-1}{\kappa}} - e^{-1/\kappa}\right), \tag{44}$$

where $\kappa = \frac{\varepsilon}{\mu}$.

As can be seen in Figure 1, the GFEM solution exhibits spurious oscillations near the right boundary, i.e., around the point $x = 1$. The SUPG formulation generates approximations so close to the exact solution for $\varepsilon = 10^{-4}$ and $\mu = 1$. It can be seen in Figure 2 that

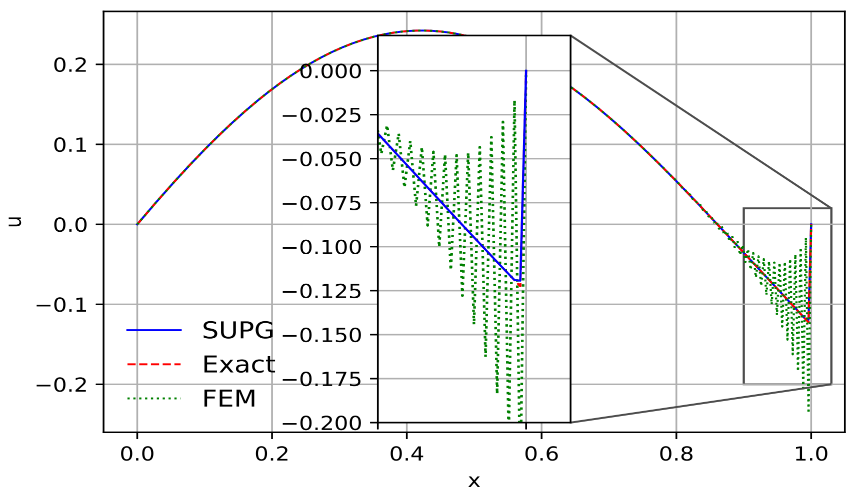

FIGURE 1. Comparison of GFEM and SUPG approximations for solving problem (38)–(39): $\varepsilon = 10^{-4}$, $\mu = 1$, and $N = 256$.

as the convection parameter becomes smaller, the accuracy of the SUPG approximations decreases while the GFEM approximations gives still highly accurate results. One can point out from Figure 3 that when the diffusion parameter ε is considerably smaller than the diffusion parameter μ, the SUPG produces more accurate approximations than the GFEM. However, since the dominance of μ means stronger gradients near the endpoints,

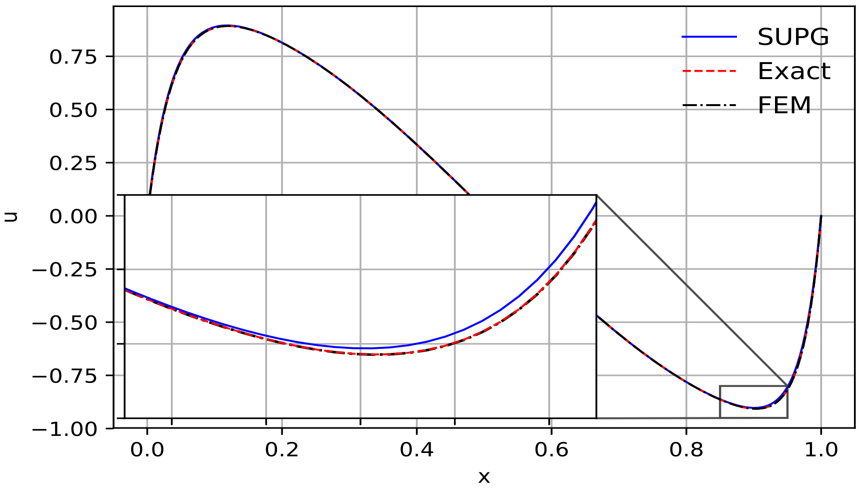

FIGURE 2. Comparison of GFEM and SUPG approximations for solving problem (38)–(39): $\varepsilon = 10^{-3}$, $\mu = 10^{-2}$, and $N = 256$.

especially near the point $x = 0$, the SUPG method is insufficient in resolving steep gradients accurately. Increasing the number of elements would result in better boundary-layer representations for the SUPG. Adaptive mesh algorithms and graded meshes can also be used, see, for example [25, 49, 53] and the references therein. Alternatively, the stabilized formulation can be further supplemented with appropriate shock-capturing (also called discontinuity-capturing) terms (see, e.g., [50]).

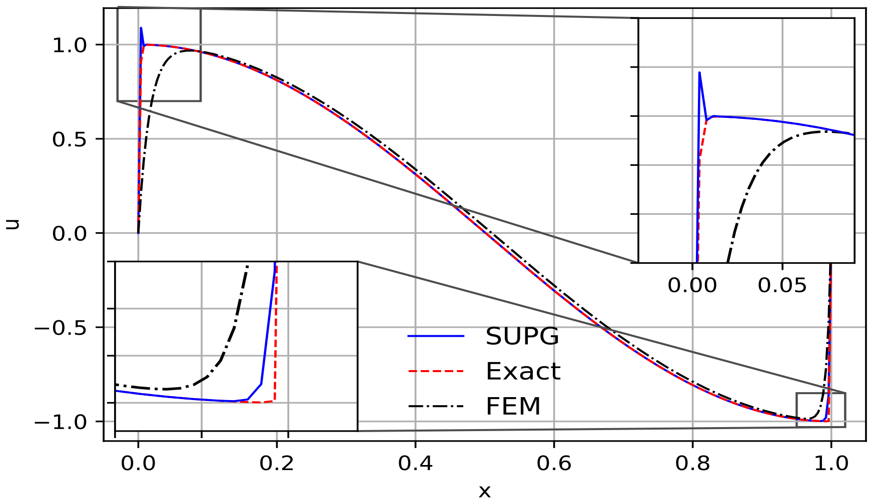

FIGURE 3. Comparison of GFEM and SUPG approximations for solving problem (38)–(39): $\varepsilon = 10^{-6}$, $\mu = 10^{-3}$, and $N = 256$.

In Figure 4, it can be observed that the asymptotic approach SCEM generates highly accurate approximations for sufficiently small values of the diffusion parameter ε. While the SUPG approximations are relatively close to the exact solution away from the points where the steep gradients observed, the GFEM exhibits oscillatory behavior again. It

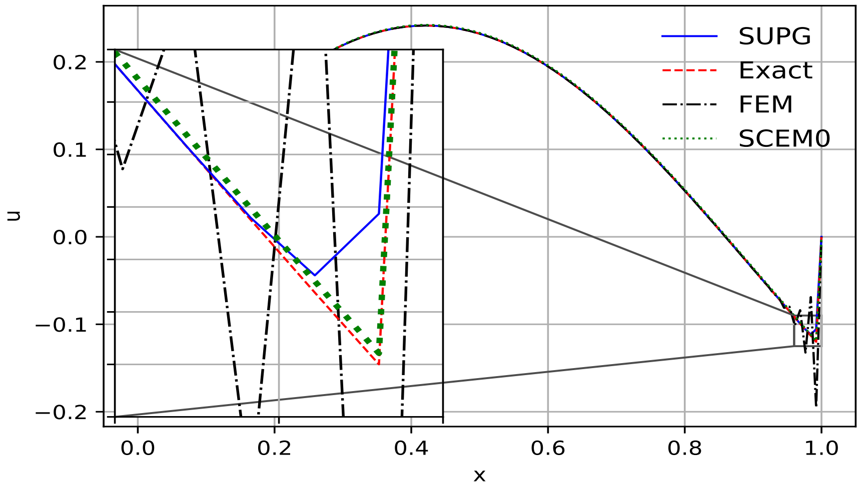

FIGURE 4. Comparison of the proposed methods for solving problem (38)–(39): $\varepsilon = 10^{-3}$, $\mu = 1$, and $N = 128$.

can be seen in Figure 5 that the decrease of convection parameter μ results in inaccuracy in the SCEM approximations, not surprisingly, while the exact, GFEM, and SUPG approximations are in quite good agreement with each other.

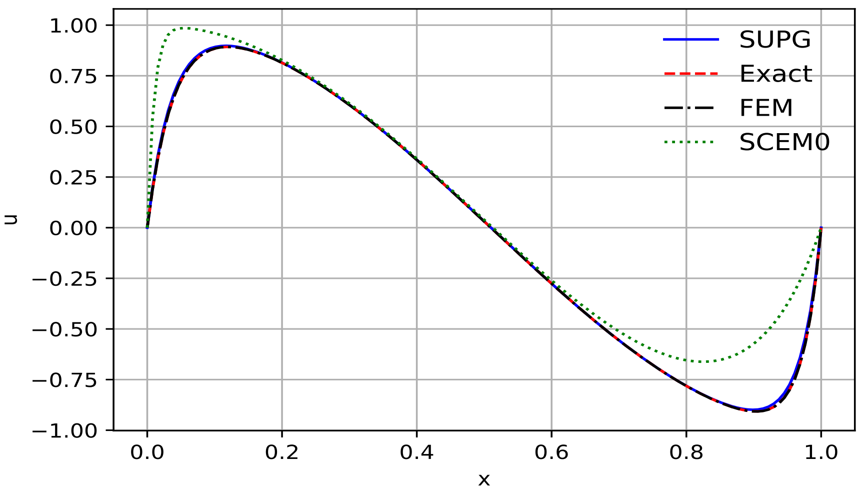

FIGURE 5. Comparison of the proposed methods for solving problem (38)–(39): $\varepsilon = 10^{-3}$, $\mu = 10^{-2}$, and $N = 128$.

Table 1 presents maximum pointwise errors (E) in approximations for solving Eqs. (38)–(39) with respect to various values of convection and diffusion parameters. The superiority of the SCEM is clearly observed for dominating values of the convection parameter.

Table 1. Maximum pointwise errors (E) in solving Eqs. (38)–(39) for various values of convection and diffusion parameters; $N = 256$.

$\varepsilon,\ \mu$	E^{SCEM0}	E^{GFEM}	E^{SUPG}
$\varepsilon = 10^{-1},\ \mu = 1$	0.10936201	0.00001170	0.00173478
$\varepsilon = 10^{-2},\ \mu = 1$	0.01265051	0.00064438	0.00846143
$\varepsilon = 10^{-3},\ \mu = 1$	0.00127747	0.04348281	0.02328801
$\varepsilon = 10^{-4},\ \mu = 10^{-1}$	0.00390387	0.31006775	0.16536862
$\varepsilon = 10^{-5},\ \mu = 10^{-2}$	0.03204730	0.32643732	0.16933287
$\varepsilon = 10^{-6},\ \mu = 10^{-3}$	0.06933179	0.28630863	0.17860562

Example 2. Consider the following singularly perturbed reaction-convection-diffusion problem [20]:

$$-\varepsilon u''(x) - \mu u'(x) + u(x) = x, \quad x \in (0,1), \tag{45}$$

with boundary conditions

$$u(0) = 1, \quad u(1) = 0. \tag{46}$$

Problem (45)–(46) has an exact solution given as follows:

$$u(x) = x + \mu + \frac{(1+\mu) + (1-\mu)e^{\lambda_2}}{e^{\lambda_2 - \lambda_1}} e^{\lambda_1 x} - \frac{(1+\mu) + (1-\mu)e^{\lambda_1}}{e^{\lambda_2 - \lambda_1}} e^{\lambda_2 x}, \tag{47}$$

where

$$\lambda_1 = \frac{-\mu - \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}, \quad \lambda_2 = \frac{-\mu + \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon} \tag{48}$$

The zeroth- and first-order SCEM approximations to the solution of problem (45)–(46) are obtained as follows:

$$\Phi_0^{\text{SCEM}}(x, \overline{x}, \kappa) = -e^{\frac{x-1}{\mu}}(\mu + 1) + (\mu + x) + \frac{(1-\mu) + (\mu+1)e^{-1/\mu}}{1 - e^{-1/\kappa}} \left(e^{-x/\kappa} - e^{-1/\kappa}\right), \tag{49}$$

$$\Phi_1^{\text{SCEM}}(x, \overline{x}, \kappa) = \Phi_0^{\text{SCEM}}(x, \overline{x}, \kappa) + \kappa \frac{(\mu+1)(x-1)e^{\frac{x-1}{\mu}}}{\mu^2} - \kappa \frac{(\mu+1)e^{-1/\mu} e^{x\lambda_2^s/\kappa}\left(e^{x\lambda_1^s/\kappa} - e^{x\lambda_2^s/\kappa}\right)}{\mu^2\left(e^{\lambda_1^s/\kappa} - e^{\lambda_2^s/\kappa}\right)} + \kappa \frac{(\mu+1)e^{-1/\mu} e^{x\lambda_2^s/\kappa}}{\mu^2}, \tag{50}$$

where

$$\lambda_1^s = \frac{-\mu + \sqrt{\mu^2 + 4\mu}}{2\mu}, \quad \lambda_2^s = \frac{-\mu - \sqrt{\mu^2 + 4\mu}}{2\mu}, \tag{51}$$

and $\kappa = \frac{\varepsilon}{\mu}$.

In Figure 6, it is easily observed that the SUPG formulation is superior to the GFEM in the vicinity of the boundary-layer while the GFEM generates highly accurate approximations far away the boundary-layer. For sufficiently small values of the parameters ε and μ, when convection does not dominate diffusion, as it can be seen in Figure 7, the GFEM and SUPG both produce accurate approximations to the exact solution even near the endpoints.

One can point out from Figure 8 that the first-order SCEM approximation denoted by "SCEM1" produces approximations that are not distinguishable from the exact solution for solving problem (45)–(46) for sufficiently small values of ε parameters. In the case of

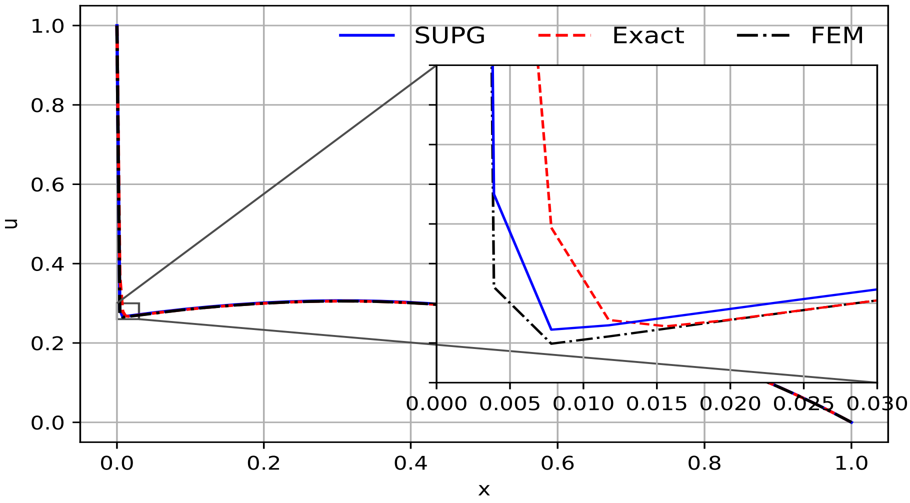

FIGURE 6. Comparison of GFEM and SUPG for solving problem (45)–(46): $\varepsilon = 2 \times 10^{-3}$, $\mu = 1$, and $N = 256$.

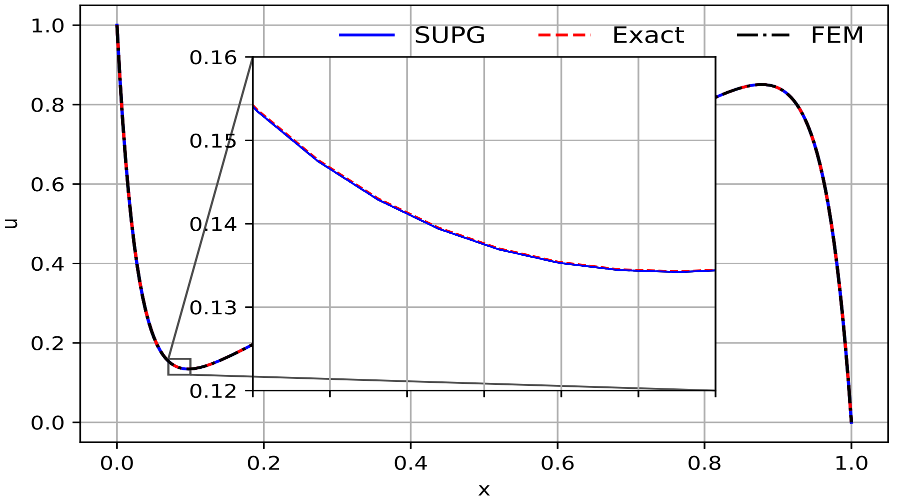

FIGURE 7. Comparison of GFEM and SUPG for solving problem (45)–(46): $\varepsilon = 10^{-3}$, $\mu = 10^{-2}$, and $N = 256$.

smaller values of the convection parameter μ, while the GFEM and SUPG approximations are quite accurate, the SCEM approximations significantly differ from the exact solution due to the asymptotic nature of the SCEM.

Finally, Table 2 shows maximum pointwise errors (E) in approximations for solving Eqs. (45)–(46) with respect to various values of convection and diffusion parameters. It is obvious from the table that as the diffusion parameter ε dominates the diffusion parameter μ, the SCEM yields more accurate approximations.

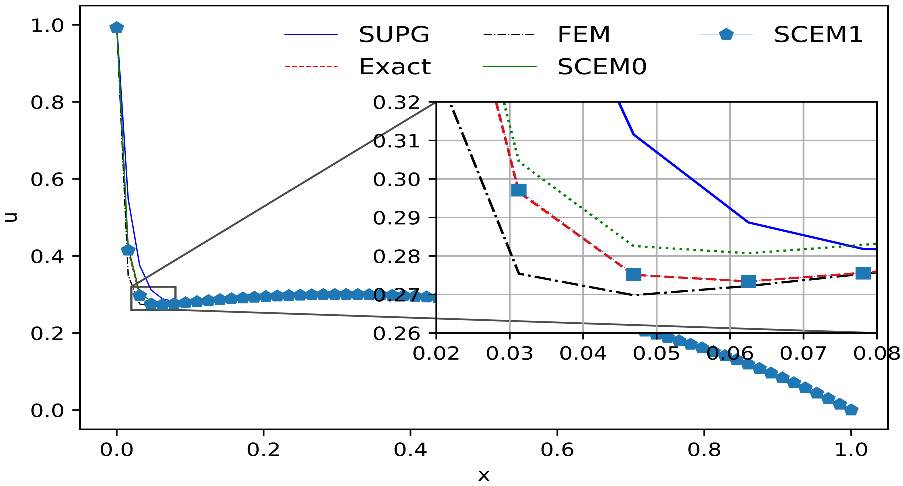

FIGURE 8. Comparison of the proposed methods for solving problem (45)–(46): $\varepsilon = 10^{-2}$, $\mu = 1$, and $N = 64$.

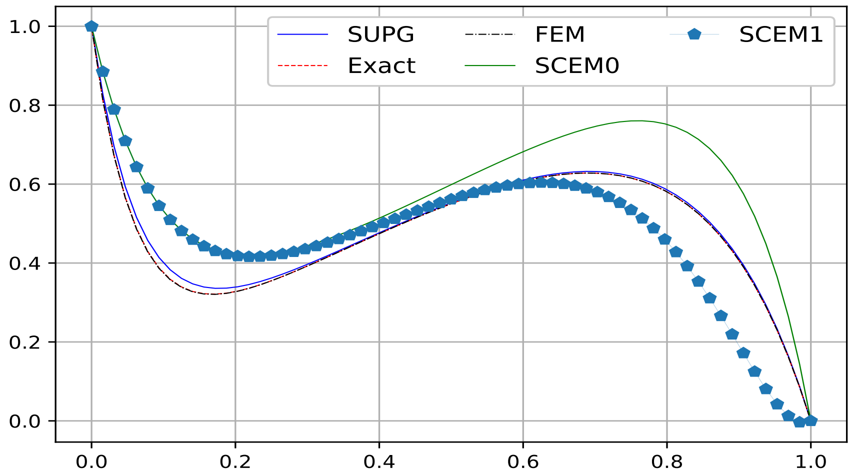

FIGURE 9. Comparison of the proposed methods for solving problem (45)–(46): $\varepsilon = 10^{-2}$, $\mu = 10^{-1}$, and $N = 64$.

4. CONCLUSIONS

In this study, singularly perturbed second-order boundary-value problems having two perturbation parameters have been considered computationally by employing the well-known GFEM and a stabilized version of the GFEM, i.e., the so-called SUPG formulation. Additionally, an asymptotic approach, called the SCEM, has also been used in order to compare the results. Two numerical experiments, whose exact solutions are available, have been adopted as test problems. The numerical experiments have shown that the SCEM approximations are highly accurate for decreasing values of the diffusion parameter, ε. However, it has been noticed that the SCEM approximations dramatically differ from the exact solutions when ε is not kept sufficiently small and when μ is relatively larger than

TABLE 2. Maximum pointwise errors (E) in solving Eqs. (45)–(46) for various values of convection and diffusion parameters; $N = 256$.

$\varepsilon,\ \mu$	E^{SCEM0}	E^{SCEM1}	E^{GFEM}	E^{SUPG}
$\varepsilon = 10^{-1},\ \mu = 1$	0.07221196	0.00732341	0.00004110	0.00575656
$\varepsilon = 10^{-2},\ \mu = 1$	0.00823766	0.00140589	0.00351315	0.00823766
$\varepsilon = 10^{-3},\ \mu = 1$	0.00077757	0.00004314	0.25237231	0.13515676
$\varepsilon = 10^{-4},\ \mu = 10^{-1}$	0.00398704	0.00068698	0.30684000	0.16366155
$\varepsilon = 10^{-5},\ \mu = 10^{-2}$	0.03233408	0.00599067	0.32350747	0.16783162
$\varepsilon = 10^{-6},\ \mu = 10^{-3}$	0.06939281	0.01829833	0.28601186	0.17877407

ε. In this regard, the SCEM should be used with smaller convection parameters since it is an asymptotic method.

On the other hand, spurious oscillations have been observed in approximations obtained with the standard GFEM for decreasing values of ε. This issue may be sorted out by increasing the number of elements. However, it is a well-known fact that, in this case, the memory usage and the CPU time may increase significantly for complex problems, particularly in solving CFD problems. It has been also observed that the SUPG has performed much better than the GFEM for decreasing values ε and, not surprisingly, for increasing values of μ.

It has been found that the SUPG formulation requires additional treatment in order to yield better solution profiles near sharp gradients, see Figure 3. This issue can be resolved by adding an appropriate discontinuity-capturing term to the SUPG-stabilized formulation. On the other hand, unsurprisingly, the SCEM approximations do not exhibit any oscillatory behavior, and the method does not require any special treatment for even decreasing values of the diffusion parameter, ε.

The comparisons that have been made in this study to evaluate the performance of various methods for solving 1D convection-dominated problems can be enhanced by including theoretical aspects such as error estimations and convergence analyses. The test computations can be extended to include much more challenging problems, e.g., the Navier–Stokes equations at high Reynolds numbers.

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