TWMS J. App. and Eng. Math. V.14, N.3, 2024, pp. 921-933

ON DIAMETER AND GIRTH OF PRODUCT OF ZERO-DIVISOR GRAPHS

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ABSTRACT. Graph theory has become a hot topic in Mathematics due to the gradual research done in graph theory. Product of graphs enables the combination or decomposition of its elemental structures. In graph theory there are four standard products, each with its own set of applications and theoretical interpretations. In this article, we study these graph products of zero-divisor graphs of commutative rings and determine their structural properties such as connectivity, diameter and girth. We also determine when the graph product of zero-divisor graphs of ring $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are Eulerian.

Keywords: Zero-divisor graph, Graph product, Diameter, Girth, Eulerian.

AMS Subject Classification: 05C25, 05C76, 05C12.

1. Introduction

In recent days, Algebraic Graph Theory has become a very popular and rapidly growing area for its numerous theoretical development and countless applications to practical problems. As a research area, graph theory is still relatively young, but it is maturing rapidly with many deep results having been discovered over the last couple of decades. The connection between graph theory and ring theory was established by Beck [3] in 1988. Beck's main interest was the chromatic number $\chi(G(R))$ of the graph $G(R)$. After modifying the definition of Beck [3], Anderson and Livingston [2] defined the *zero-divisor graph* $\Gamma(R)$ as a graph whose vertex set is the set of zero-divisors of R, such that two distinct vertices a and b are adjacent if and only if $ab = 0$. In 2006, Redmond [14] gave an algorithm to find all commutative, reduced rings with unity which gives rise to a zero-divisor graph on n vertices for any $n \geq 1$ and a list of all commutative rings (up to isomorphism) which produce zero-divisor graph for $n = 6, 7, \ldots, 14$ vertices is also given. For further details regarding zero-divisor graphs and related works the reader is referred to [5, 15, 16, 17, 19, 20, 21].

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§ Manuscript received: June 22, 2022; accepted: September 23, 2022.

Nowadays, many researchers focused on the structural properties of the zero-divisor graphs of commutative rings that include diameter, girth, vertex degree, connectivity and many more. Anderson and Livingston [2] proved that $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $girth(\Gamma(R)) \leq 7$. They noticed that all of the examples they considered had girths of $3, 4$ or ∞. Based on this, they conjectured that if a zero-divisor graph has a cycle, then its girth is 3 or 4. They were able to prove this for Artinian ring (Theorem 2.4, [2]). The conjecture was proved independently by Mulay [11] and DeMeyer and Schneider [7]. In 2019, Akgunes and Nacaroglu [1] calculated the diameter and girth for $Z_p \times Z_q \times Z_r$. For more results on diameter and girth of zero-divisor graphs of commutative rings, readers may refer to [9, 12, 13].

By expanding small graphs, we construct many large graphs. So, it is essential to know which properties of small graphs can be transferred to the expanded ones. In (2011), Li et al. [10] proved that lexicograph of vertex transitive graphs is also vertex transitive as well as the lexicographic product of edge transitive graphs. In (2008), for cartesian product of two graphs, Spacapan [18] found the fewest number of vertices whose removal from the graph results in a disconnected or trivial graph.

This powerful idea motivated us to consider five kinds of graph products as the expander graphs. In this article, we discuss some structural properties: connectivity, diameter and girth of graph products of zero-divisor graphs of commutative rings.

Rest of the paper is arranged as follows: necessary notations and terminology are discussed in Section 2. In Section 3, we study the diameter and girth of cartesian product of zero-divisor graphs. In section 4 and Section 5, tensor product and strong product are investigated. Co-normal product and lexicographic product of zero-divisor graphs are discussed in Section 6 and Section 7. Finally, the paper is concluded in Section 8.

2. Terminology

In this section, we discuss some terms that are relatable to this article. The greatest distance between any pair of vertices is termed as *diameter* of graph G, denoted by $diam(G)$. *Girth* of graph G is defined as the length of the shortest cycle in G, provided G contains a cycle; otherwise girth of $G = \infty$. It is denoted by $girth(G)$. A graph is said to be *connected* if there is a path between any two arbitrary vertices, otherwise *disconnected.* The length of the shortest path between vertex x and y is known as *distance* between them denoted by $d(x, y)$. The *degree of vertex* x is the number of edges of G incident on x. The set of vertices adjacent to vertex x, is called the *neighborhood of vertex* x denoted by $N(x)$. A graph is *Eulerian* if it has a closed path containing every edge. In [4], it shows that a graph is Eulerian if and only if every vertex has even degree.

"Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. $\Gamma = (V, E)$, the product of them is a graph with vertex set $V = V_1 \times V_2$, and vertex (u_1, u_2) is adjacent to (v_1, v_2) in Γ if one of the relevant conditions happen depending on the product [8]:

(1) *Cartesian product.* u_1 is adjacent to v_1 in Γ_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2;
(2) *Tensor product.* u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2;
(3) *Strong product.* u_1 is adjacent to v_1 in Γ_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2 or u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2;
(4) *Lexicographic product.* u_1 is adjacent to v_1 in Γ_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2;

(5) *Co-normal product.* u_1 is adjacent to v_1 in Γ_1 or u_2 is adjacent to v_2 in Γ_2."

In this paper, we assume that R and S are two commutative rings and Γ_1 and Γ_2 are their zero-divisor graphs respectively. $N(u)$ and $N(u')$ represents the neighborhood of vertex $u \in V(\Gamma_1)$ and $u' \in V(\Gamma_2)$.

3. Cartesian product

In this section, we study the cartesian product of zero-divisor graphs. We calculated the upper bound for the diameter and girth of the graph. We also proved, under what conditions the graph is Eulerian. In the following theorem, we find the neighborhood of the vertex of cartesian product.

Theorem 3.1. *Let G be the Cartesian product of Γ_1 and Γ_2. Then $N(u,u') = (N(u) \times u') \cup (u \times N(u'))$, for any $(u,u') \in V(\Gamma_1 \times \Gamma_2)$.*

Proof. Let $(x,y) \in N(u,u')$. Therefore, (x,y) is adjacent to (u,u'). Thus, x is adjacent to u in Γ_1 and $y = u'$ or $x = u$ and y is adjacent to u' in Γ_2. Hence, $x \in N(u)$ and $y = u'$ or $x = u$ and $y \in N(u')$. Hence, $N(u,u') = (N(u) \times u') \cup (u \times N(u'))$. □

In the following results, we proved that the upper bound of diameter of zero-divisor graph is 6 and that of girth is 4.

Theorem 3.2. *Let G be the Cartesian product of Γ_1 and Γ_2. Then diam$(G)) \leq 6$.*

Proof. Let $(x,y),(u,v) \in V(\Gamma_1 \times \Gamma_2)$ be two distinct vertices. We have the following cases:

Case 1: When $x = u$, $y \neq v$, then following cases arise. If

Case 1.1: $yv = 0$, then $(x,y) - (a,b)$ is a path.

Case 1.2: $yv \neq 0$ and $y^2 = 0, v^2 = 0$, then $(x,y) - (x,yv) - (u,v)$ is a path.

Case 1.3: $yv \neq 0$ and $y^2 = 0, v^2 \neq 0$, then there exist $v^{'} \in V(\Gamma_2)$ such that $vv^{'} = 0$. Hence $(x,y) - (x,yv^{'}) - (u,v)$ is a path.

Case 1.4: $yv \neq 0$ and $y^2 \neq 0, v^2 = 0$, then there exist $y^{'} \in V(\Gamma_2)$ such that $yy^{'} = 0$. Hence $(x,y) - (x,y^{'}v) - (u,v)$ is a path.

Case 1.5: $yv \neq 0$ and $y^2 \neq 0, v^2 \neq 0$, then there exist $y^{'},v^{'} \in V(\Gamma_2)$ such that $yy^{'} = 0,\ vv^{'} = 0$. We have two sub cases:

(A). If $y^{'}v^{'} \neq 0$, then $(x,y) - (x,y^{'}v^{'}) - (x,v)$ is a path.

(B). If $y^{'}v^{'} = 0$, then $(x,y) - (x,y^{'}) - (x,v^{'}) - (x,v)$ is a path.

Case 2: When $x \neq u$ and $y = v$. We can proceed as in case 1.

Case 3: When $x \neq u$ and $y \neq v$, then we have following cases:

Case 3.1: Let $xu = 0$ and $yv \neq 0$. Then following sub cases arise:

(A). If $y^2 = 0$ and $v^2 = 0$, then $(x,y) - (u,y) - (u,yv) - (u,v)$ is a path.

(B). If $y^2 = 0$ and $v^2 \neq 0$, then there exist $v^{'} \in V(\Gamma_2)$ such that $vv^{'} = 0$. Hence $(x,y) - (u,y) - (u,yv^{'}) - (u,v)$ is a path.

(C). If $y^2 \neq 0$ and $v^2 = 0$, then there exist $y^{'} \in V(\Gamma_2)$ such that $yy^{'} = 0$. Hence $(x,y) - (u,y) - (u,y^{'}v) - (u,v)$ is a path.

(D). If $y^2 \neq 0$ and $v^2 \neq 0$ then there exist $y^{'},v^{'} \in V(\Gamma_2)$ such that $yy^{'} = 0,\ vv^{'} = 0$. Now, if $y^{'}v^{'} \neq 0$, then $(x,y) - (u,y) - (u,y^{'}v^{'}) - (u,v)$ is a path and if $y^{'}v^{'} \neq 0$, then $(x,y) - (u,y) - (u,y^{'}) - (u,v^{'}) - (u,v)$ is a path.

Case 3.2: Let $xu \neq 0$ and $yv = 0$. We can proceed as in case 3.2.

Case 3.3: Let $xu \neq 0$ and $yv \neq 0$. Then following sub cases arise:

(A). If $x^2 = y^2 = u^2 = v^2 = 0$, then $(x,y) - (xu,y) - (u,y) - (u,yv) - (u,v)$ is a path.

(B). If $y^2 = u^2 = v^2 = 0$ and $x^2 \neq 0$ then there exist $x^{'} \in V(\Gamma_1)$ such that $xx^{'} = 0$. Hence $(x, y) - (x^{'}u, y) - (u, y) - (u, yv) - (u, v)$ is a path.
(C). If $u^2 = v^2 = 0$ and $x^2 \neq 0,\ y^2 \neq 0$ then there exist $x^{'} \in V(\Gamma_1)$ and $y^{'} \in V(\Gamma_2)$ such that $xx^{'} = 0 = yy^{'}$. Hence $(x, y) - (x^{'}u, y) - (u, y) - (u, y^{'}v) - (u, v)$ is a path.
(D). If $y^2 = v^2 = 0$ and $x^2 \neq 0,\ u^2 \neq 0$ then there exist $x^{'}, u^{'} \in V(\Gamma_1)$ such that $xx^{'} = 0 = uu^{'}$. Now, if $x^{'}u^{'} \neq 0$, then $(x, y) - (x^{'}u^{'}, y) - (u, y) - (u, yv) - (u, v)$ is a path and if $x^{'}u^{'} = 0$, then $(x, y) - (x^{'}, y) - (u^{'}, y) - (u, y) - (u, yv) - (u, v)$ is a path.
(E). If $x^2 \neq 0, y^2 \neq 0, u^2 \neq 0, v^2 \neq 0$ then there exist $x^{'}, u^{'} \in V(\Gamma_1)$ and $y^{'}, v^{'} \in V(\Gamma_2)$ such that $xx^{'} = 0 = uu^{'}$ and $yy^{'} = 0 = vv^{'}$.
We have following sub cases:
(a). If $x^{'}u^{'} = 0$ and $y^{'}v^{'} \neq 0$, then $(x, y) - (x^{'}, y) - (u^{'}, y) - (u, y) - (u, y^{'}v^{'}) - (u, v)$ is a path.
(b). If $x^{'}u^{'} \neq 0$ and $y^{'}v^{'} = 0$, then $(x, y) - (x^{'}u^{'}, y) - (u, y) - (u, y^{'}) - (u, v^{'}) - (u, v)$ is a path.
(c). If $x^{'}u^{'} \neq 0$ and $y^{'}v^{'} \neq 0$, then $(x, y) - (x^{'}u^{'}, y) - (u, y) - (u, y^{'}v^{'}) - (u, v)$ is a path.
(d). If $x^{'}u^{'} = 0$ and $y^{'}v^{'} = 0$, then $(x, y) - (x^{'}, y) - (u^{'}, y) - (u, y) - (u, y^{'}) - (u, v^{'}) - (u, v)$ is a path.

Hence $diam(\Gamma) \leq 6$. □

Example 3.1. *Let Γ_1 and Γ_2 be the zero-divisor graphs of $\mathbb{Z}_{20}$ and $\mathbb{Z}_{63}$ respectively. Let G be the Cartesian product of Γ_1 and Γ_2 that is $G = \Gamma_1 \times \Gamma_2$. Then $d((2, 7), (5, 3)) = 6$*

$$(2, 7) - (10, 7) - (4, 7) - (5, 7) - (5, 9) - (5, 21) - (5, 3)$$

Theorem 3.3. *Let G be the Cartesian product of Γ_1 and Γ_2. If G has a cycle then $girth(G) \leq 4$.*

Proof. Suppose Γ_1 has a cycle of length 3. Let $a, b, c \in V(\Gamma_1)$ and $x \in V(\Gamma_2)$ such that $a - b - c - a$ is cycle of length 3 in Γ_1. Then $(a, x), (b, x), (c, x) \in V(G)$ and

$$(a, x) - (b, x) - (c, x) - (a, x)$$

is a cycle of length 3 in G.

Suppose Γ_1 has a cycle of length 4. Let $a, b, c, d \in V(\Gamma_1)$ and $x \in V(\Gamma_2)$ such that $a - b - c - d - a$ is cycle of length 4 in Γ_1. Then $(a, x), (b, x), (c, x), (d, x) \in V(G)$ and

$$(a, x) - (b, x) - (c, x) - (d, x) - (a, x)$$

is a cycle of length 4 in G.

Suppose, neither Γ_1 nor Γ_2 has a cycle. Let $a, b \in V(\Gamma_1)$ and $x, y \in V(\Gamma_2)$ with $a - b$ and $x - y$. Then $(a, x), (b, x), (a, y), (b, y) \in V(G)$ and

$$(a, x) - (a, y) - (b, y) - (b, x) - (a, x)$$

is a cycle of length 4 in G.

From above cases, we conclude that girth$(G) \leq 4$. □

Following theorem gives the condition, under which the cartesian product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ is Eulerian.

Theorem 3.4. *Let G be the cartesian product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then G is Eulerian if and only if $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_m)$ are Eulerian.*

Proof. Let G be the cartesian product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ and $(u,v) \in G$. Then degree of (u,v) is given by

$$deg(u,v) = deg(u) + deg(v)$$

To prove that G is Eulerian we need to show that sum of degrees of u and v are even for all $u \in \Gamma(\mathbb{Z}_n),\ v \in \Gamma(\mathbb{Z}_m)$. This is possible if degrees of both the vertices are either odd or even.

From theorem 3.1 [6], we observe that vertices of $\Gamma(\mathbb{Z}_n)$ have even degree if and only if either n is odd and square free or $n = 4$. Also, there does not exist any n for which degrees of all the vertices of $\Gamma(\mathbb{Z}_n)$ is odd. Hence we conclude that G is Eulerian if vertices of $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_m)$ have even degree, that is $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_m)$ are Eulerian. □

4. Tensor product

In this section, we study the Tensor product of zero-divisor graphs. We calculated the upper bound for the diameter and girth of the graph. We also proved, under what conditions the graph is Eulerian. In the following theorem, we find the neighborhood of the vertex of Tensor product.

Theorem 4.1. *Let G be the Tensor product of Γ_1 and Γ_2. Then $N(u,u') = (N(u) \times N(u'))$ for any $(u,u') \in V(\Gamma_1 \times \Gamma_2)$.*

Proof. Assume $(x,y) \in N(u,u')$. Then, (x,y) is adjacent to (u,u'). By definition of tensor product, x and u are adjacent and y and u' are adjacent too. Therefore, $x \in N(u)$ and $y \in N(u')$. It leads to $N(u,u') = N(u) \times N(u')$. □

In the following results, we discuss diameter and girth of tensor product of zero-divisor graphs.

Theorem 4.2. *Let G be the Tensor product of Γ_1 and Γ_2. Then G is connected if and only if for every pair of vertices in Γ_1 and Γ_2, there is a path of odd length.*

Proof. Let (x,y) and (u,v) be in G. If $x.u = 0$ and $y.v = 0$ for all $x,u \in V(\Gamma_2)$ and $y,v \in V(\Gamma_1)$, then G is connected. Let $x.u \neq 0$ and $x - x_1 - x_2 - \cdots - x_n - u$ is a connecting x and u. Then we have path

$$(x,y) - (x_1,v) - (x_2,y) - \cdots - (x_n,y) - (u,v), \text{ if } n \text{ is even}$$
$$(x,y) - (x_1,v) - (x_2,y) - \cdots - (x_n,v) - (u,y), \text{ if } n \text{ is odd}$$

Clearly, When n is odd there is no path connecting (x,y) and (u,v). That is length of path connecting x and u must be odd. Similarly we can prove the case for $x.u \neq 0$ and $y.v \neq 0$. □

Theorem 4.3. *Let G be the Tensor product of Γ_1 and Γ_2. If G is connected, then diam(G) is equal to the*

$$Max\{min_{(x,u)\in V(\Gamma_1)}(d(x,u)), min_{(y,v)\in V(\Gamma_2)}(d(y,v))\},$$

where $d(a,b)$ is the length of odd length path connecting vertices a and b.

Proof. From theorem 4.4, G is connected if there is a path of odd length connecting any two vertices of Γ_1 and Γ_2. Let (x,y) and $(u,v) \in G$. If G is connected then length of shortest path connecting (x,y) and (u,v) is $Max\{min(d(x,u)), min(d(y,v))\}$. Hence diam$(G)$ is

$$Max\{min_{(x,u)\in V(\Gamma_1)}(d(x,u)), min_{(y,v)\in V(\Gamma_2)}(d(y,v))\}.$$

□

FIGURE 1. Zero-divisor graph of $\mathbb{Z}_6$

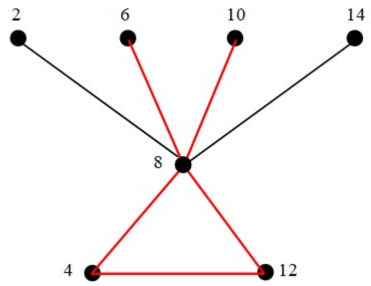

FIGURE 2. Zero-divisor graph of $\mathbb{Z}_{16}$

Example 4.1. *Let Γ_1 and Γ_2 be the zero-divisor graphs of $\mathbb{Z}_6$ and $\mathbb{Z}_{16}$ respectively and G the tensor product of Γ_1 and Γ_2. Then G is connected and $diam(G) = 5$ as $d((2,10),(3,6)) = 5$*

$$(2,10) - (3,8) - (2,4) - (3,12) - (2,8) - (3,6)$$

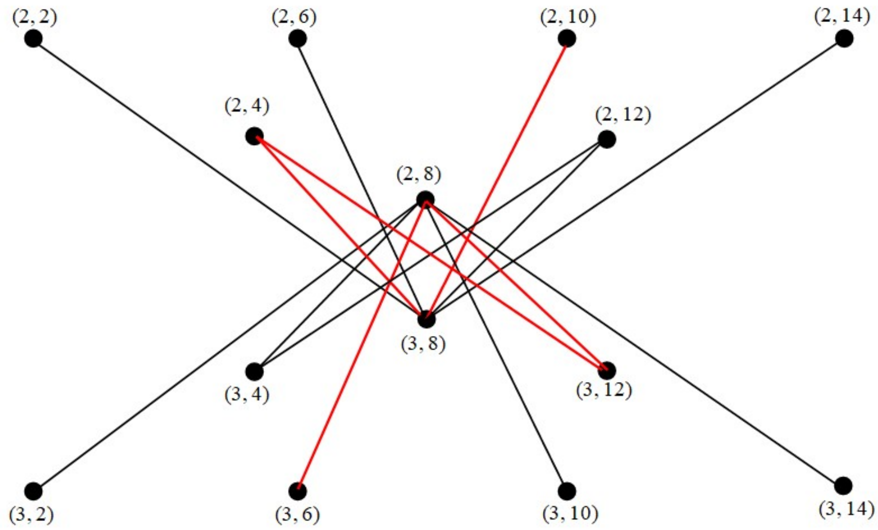

FIGURE 3. Tensor product of zero-divisor graph of $\mathbb{Z}_6$ and $\mathbb{Z}_{16}$

Theorem 4.4. *Let G be the Tensor product of Γ_1 and Γ_2. If G has a cycle then $girth(G) \leq 4$.*

Proof. We have the following cases:

(1) When $|\Gamma_1| = |\Gamma_2| = 2$. Then G does not have a cycle.
(2) When $|\Gamma_2| \geq 2$ and $|\Gamma_1| \geq 3$. Let $a, b, c \in V(\Gamma_1)$ and $x, y \in V(\Gamma_2)$ with $a - b - c$ and $x - y$. Then $(a,x), (b,x), (c,x), (a,y), (b,y), (c,y) \in V(G)$ and

$$(a,x) - (b,y) - (c,x) - (b,y) - (a,x)$$

is a cycle of length 4.

□

Following theorem gives the condition, under which the Tensor product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ is Eulerian.

Theorem 4.5. *Let G be the Tensor product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then G is Eulerian if either $\Gamma(\mathbb{Z}_n)$ or $\Gamma(\mathbb{Z}_m)$ is Eulerian.*

Proof. Let G be the Tensor product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ and $(u, v) \in G$. Then degree of (u, v) is given by

$$deg(u, v) = deg(u) * deg(v)$$

Clearly, G is Eulerian if degree of u or v is even for all $u \in V(\Gamma(\mathbb{Z}_n)),\ v \in V(\Gamma(\mathbb{Z}_m))$. That is, G is Eulerian if either of $\Gamma(\mathbb{Z}_n)$ or $\Gamma(\mathbb{Z}_m)$ is Eulerian. □

5. Strong product

In this section, we study the Strong product of zero-divisor graphs. We calculated the upper bound for the diameter and girth of the graph. We also proved, under what conditions the graph is Eulerian. In the following theorem, we find the neighborhood of the vertex of Strong product. From theorem 3.1 and 4.1, we conclude the following theorems:

Theorem 5.1. *Let G be the Strong product of Γ_1 and Γ_2. Then, if runs for any $(u, u^{'}) \in V(G), N(u, u^{'}) = (N(u) \times u^{'}) \cup (u \times N(u^{'})) \cup (N(u) \times N(u^{'}))$.*

Proof. For any $(x, y) \in N(u, u^{'})$, where $(u, u^{'}) \in V(G)$, x is adjacent to u in Γ_1 and $y = u^{'}$ or $x = u$ and y is adjacent to $u^{'}$ in Γ_2 or x is adjacent to u in Γ_1 and y is adjacent to $u^{'}$ in Γ_2. Therefore, $N(u, u^{'}) = (N(u) \times u^{'}) \cup (u \times N(u^{'})) \cup (N(u) \times N(u^{'}))$. □

In the following results, we discuss diameter and girth of Strong product of zero-divisor graphs.

Theorem 5.2. *Let G be the Strong product of Γ_1 and Γ_2. Then $diam(G) \leq 3$.*

Proof. Let $(x, y), (u, v) \in V(\Gamma_1 \times \Gamma_2)$ be two distinct vertices. Suppose they are not connected. Then we have the following cases:

Case 1: When $x = u$ and $y \neq v$. Same as Case 1 of theorem 3.2.
Case 2: When $x \neq u$ and $y = v$. Same as Case 2 of theorem 3.2.
Case 3: When $x \neq u$ and $y \neq v$ then following cases arise:
Case 3.1: Let $xu = yv = 0$. Then $(x, y) - (u, v)$ is a path.
Case 3.2: Let $xu = 0$ and $yv \neq 0$. Then following sub cases arise:
(A). If $y^2 = v^2 = 0$, then $(x, y) - (x, yv) - (u, v)$ is a path.
(B). If $y^2 = 0$ and $v^2 \neq 0$ then there exist $v^{'} \in V(\Gamma_2)$ such that $vv^{'} = 0$. Hence $(x, y) - (x, yv^{'}) - (u, v)$ is a path.
(C). If $y^2 \neq 0$ and $v^2 = 0$ then there exist $y^{'} \in V(\Gamma_2)$ such that $yy^{'} = 0$. Hence $(x, y) - (x, y^{'}v) - (u, v)$ is a path.
(D). If $y^2 \neq 0$ and $v^2 \neq 0$ then there exist $y^{'}, v^{'} \in V(\Gamma_2)$ such that $yy^{'} = 0$ and $vv^{'} = 0$ respectively.
If $y^{'}v^{'} \neq 0$, then $(x, y) - (x, y^{'}v^{'}) - (u, v)$ is a path.
If $y^{'}v^{'} = 0$, then $(x, y) - (x, y^{'}) - (u, v^{'}) - (u, v)$ is a path.
Case 3.3: When $xu \neq 0$ and $yv = 0$. Same as case 3.2
Case 3.4: When $xu \neq 0$ and $yv \neq 0$. Then we have following sub cases:
(A). If $x^2 = y^2 = u^2 = v^2 = 0$, then $(x, y) - (xu, yv) - (u, v)$ is a path.
(B). If $y^2 = u^2 = v^2 = 0$ and $x^2 \neq 0$ then there exist $x^{'} \in V(\Gamma_1)$ such that $xx^{'} = 0$. Hence $(x, y) - (x^{'}u, yv) - (u, v)$ is a path.

(C). If $y^2 = v^2 = 0$ and $x^2 \neq 0, u^2 \neq 0$ then there exist $x', u' \in V(\Gamma_1)$ such that $xx' = 0$ and $uu' = 0$.
If $x'u' \neq 0$ then $(x, y) - (x'u', yv) - (u, v)$ is a path.
If $x'u' = 0$, then $(x, y) - (x', yv) - (u', v) - (u, v)$ is a path.
(D). If $u^2 = v^2 = 0$ and $x^2 \neq 0, y^2 \neq 0$ then there exist $x' \in V(\Gamma_1)$, $y' \in V(\Gamma_2)$ such that $xx' = 0$ and $yy' = 0$. Hence $(x, y) - (x'u, y'v) - (u, v)$ is a path.
(E). If $x^2 \neq 0, y^2 \neq 0, u^2 \neq 0, v^2 \neq 0$ then there exist $x', u' \in V(\Gamma_1)$ and $y', v' \in V(\Gamma_2)$ such that $xx' = 0 = uu'$ and $yy' = 0 = vv'$.
We have following sub cases:
(a). If $x'u' = 0$ and $y'v' \neq 0$, then $(x, y) - (x', y) - (u', y'v') - (u, v)$ is a path.
(b). If $x'u' \neq 0$ and $y'v' = 0$, then $(x, y) - (x'u', y') - (u, v') - (u, v)$ is a path.
(c). If $x'u' \neq 0$ and $y'v' \neq 0$, then $(x, y) - (x'u', y'v') - (u, v)$ is a path.
(d). If $x'u' = 0$ and $y'v' = 0$, then $(x, y) - (x', y') - (u', v') - (u, v)$ is a path.

Hence $diam(G) \leq 3$. □

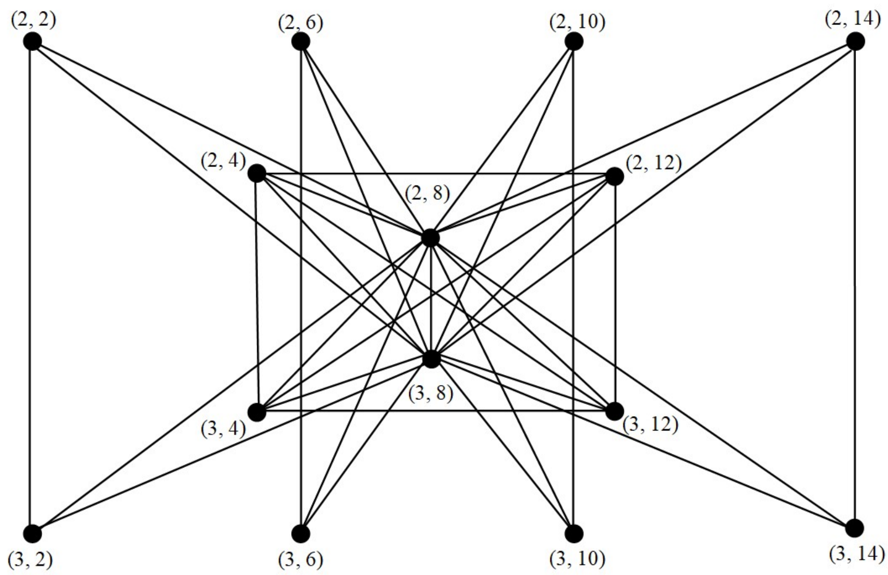

FIGURE 4. Strong product of zero-divisor graph of $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$

Theorem 5.3. *Let G be the Strong product of Γ_1 and Γ_2. If G has a cycle then $girth(G) \leq 4$.*

Proof. Proof follows from theorem 3.3. □

Theorem 5.4. *Let G be the Strong product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then G is Eulerian if both $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_m)$ are Eulerian.*

Proof. Let G be the Strong product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ and $(u, v) \in G$. Then degree of (u, v) is given by

$$deg(u, v) = deg(u) + deg(v) + deg(u) * deg(v)$$

Clearly, only possibility for degree of (u, v) to be even is when degrees of both vertices u and v are even. That is, G is Eulerian if and only if both $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_m)$ are Eulerian. □

6. Co-normal product

In this section, we study the Co-normal product of zero-divisor graphs. We calculated the diameter and girth of the graph. We also proved, under what conditions the graph is Eulerian. In the following theorem, we find the neighborhood of the vertex of Co-normal product.

Theorem 6.1. *Let G be the Co-normal product of Γ_1 and Γ_2. Then for any $(u,u^{'}) \in V(G), N(u,u^{'}) = (N(u) \times V(\Gamma_2)) \cup (V(\Gamma_1) \times N(u^{'}))$.*

Proof. If (x,y) is adjacent to $(u,u^{'})$, x, u are adjacent in Γ_1 or y, $u^{'}$ are adjacent in Γ_2. Thus, $N(u,u^{'}) = (N(u) \times V(\Gamma_2)) \cup (V(\Gamma_2) \times N(u^{'}))$. □

In the following theorems, we discuss some results on diameter and girth of Co-normal product of zero-divisor graphs.

Theorem 6.2. *Let G be the Co-normal product of Γ_1 and Γ_2. Then $diam(G) =$*

$$\begin{cases} 2, & \textit{if } diam(\Gamma_1) = 1 \textit{ or } diam(\Gamma_2) = 1 \\ Min\{diam(\Gamma_1), diam(\Gamma_2)\}, & \textit{otherwise.} \end{cases}$$

Proof. Let $(x,y),(u,v) \in V(G)$ be two distinct vertices. Suppose they are not connected. Then we have the following cases:

Case 1: When $x = u$, $y \neq v$, then there exist $x^{'} \in V(\Gamma_1)$ such that $xx^{'} = 0$. Hence $(x,y) - (x^{'},y) - (u,v)$ is a path.

Case 2: When $x \neq u$, $y = v$, then there exist $y^{'} \in V(\Gamma_2)$ such that $yy^{'} = 0$. Hence $(x,y) - (x,y^{'}) - (u,v)$ is a path.

Case 3: When $x \neq u$, $y \neq v$, then we have following cases:

Case 3.1: If $xu = 0$ or $yv = 0$, then $(x,y) - (u,v)$ is a path..

Case 3.2: If $xu \neq 0$ and $yv \neq 0$. Let $x - x_1 - \cdots - x_n - u$ is a path connecting x and u and $y - y_1 - \cdots - y_m - v$ is a path connecting y and v. Without loos of generality, let $n < m$. Then $(x,y) - (x_1,y) - (x_2,y) \cdots - (x_n,y) - (u,v)$ is the shortest path connecting (x,y) and (u,v).

Hence, $diam(G) =$

$$\begin{cases} 2, & \text{if } diam(\Gamma_1) = 1 \text{ or } diam(\Gamma_2) = 1 \\ Min\{diam(\Gamma_1), diam(\Gamma_2)\}, & \text{otherwise.} \end{cases}$$

□

Example 6.1. *Let Γ_1 and Γ_2 be the zero-divisor graphs of $\mathbb{Z}_6$ and $\mathbb{Z}_8$ respectively. Let G be the conormal product of Γ_1 and Γ_2 that is $G = \Gamma_1 \times \Gamma_2$. Then $diam(\Gamma_1) = 1$, $diam(\Gamma_2) = 2$ and $diam(G) = 2$ as there is no path of length 1 connecting $(3,2)$ and $(3,6)$. Shortest path connecting $(3,2)$ and $(3,6)$ is $(3,2) - (3,4) - (3,6)$.*

Theorem 6.3. *Let G be the Co-normal product of Γ_1 and Γ_2. Then $girth(G) = 3$.*

Proof. Let $a,b \in \Gamma_1$ and $x,y \in \Gamma_2$ with $a - b$ and $x - y$. Then $(a,x),(a,y),(b,x),(b,y) \in V(G)$ and

$$(a,x) - (b,y) - (a,y) - (a,x)$$

is a cycle of length 3. □

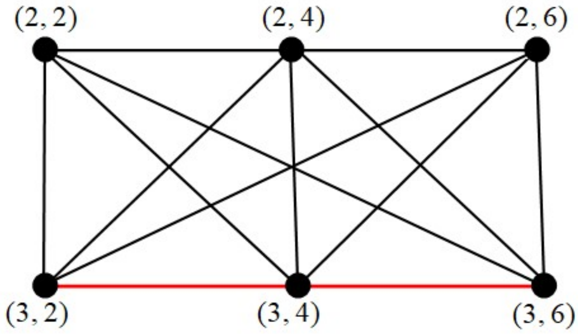

FIGURE 5. Co-normal product of $\mathbb{Z}_6$ and $\mathbb{Z}_8$

The next result discusses the Eulerian property of Co-normal product of zero-divisor graphs.

Theorem 6.4. *Let G be the Co-normal product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then G is Eulerian if:*

(1) *both m and n are odd;*
(2) *n is odd and square free and m is even;*
(3) *m is odd and square free and n is even;*
(4) $n = m = 4$

Proof. Let G be the Co-normal product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ and $(u, v) \in G$. Then degree of (u, v) is given by

$$deg(u, v) = deg(u)|V(\Gamma(\mathbb{Z}_m))| + deg(v)|V(\Gamma(\mathbb{Z}_n))|$$

We have following cases:
a.) When $|V(\Gamma(\mathbb{Z}_n))|$ and $|V(\Gamma(\mathbb{Z}_m))|$ are even that is, when both n and m are odd. Clearly in this case, $deg(u, v)$ is even.
b.) When $|V(\Gamma(\mathbb{Z}_n))|$ is even and $|V(\Gamma(\mathbb{Z}_m))|$ is odd. Then $deg(u, v)$ is even when n is square free. So G is Eulerian when n is odd and square free and m is even.
c.) When $|V(\Gamma(\mathbb{Z}_m))|$ is even and $|V(\Gamma(\mathbb{Z}_n))|$ is odd. Then $deg(u, v)$ is even when m is square free. So G is Eulerian when m is odd and square free and n is even.
c.) When $|V(\Gamma(\mathbb{Z}_n))|$ and $|V(\Gamma(\mathbb{Z}_m))|$ are odd that is, n and m are even. So G is Eulerian when degree of u and v are both odd or both even. Since n and m are even, we have $n = m = 4$. □

7. Lexicographic product

In this section, we study the Lexicographic product of zero-divisor graphs. We calculated the diameter and girth of the graph. We also proved, under what conditions the graph is Eulerian. In the following theorem, we find the neighborhood of the vertex of Lexicographic product.

Theorem 7.1. *Let G be the Lexicographic product of two zero-divisor graphs Γ_1 and Γ_2 of the rings R and S, respectively. Then, $N(u, u') = (N(u) \times V(\Gamma_2)) \cup (u \times N(u'))$, for any $(u, u') \in V(G)$.*

Proof. Assume $(x, y) \in N(u, u')$. Therefore, x and u are adjacent in Γ_1 or $x = u$ in Γ_1 and y and u' are adjacent in Γ_1. Therefore, $N(u, u') = (N(u) \times V(\Gamma_2)) \cup (u \times N(u'))$, for any $(u, u') \in V(G)$. □

In the following results, we discuss some results on diameter and girth of Lexicographic product of zero-divisor graphs.

Theorem 7.2. *Let G be the Lexicographic product of Γ_1 and Γ_2. Then $diam(G) \leq 3$.*

Proof. Let $(x, y), (u, v) \in V(\Gamma_1 \times \Gamma_2)$ be two distinct vertices. Suppose they are not connected then $xu \neq 0$. Now we have the following cases:

Case 1: When $x = u$ and $yv \neq 0$. Then following cases arise:
Case 1.1: If $y^2 = 0 = v^2$, then $(x, y) - (x, yv) - (u, v)$ is a path.
Case 1.2: If $y^2 = 0, v^2 \neq 0$ then there exist $v' \in V(\Gamma_2)$ such that $vv' = 0$. Hence $(x, y) - (x, yv') - (u, v)$ is a path.
Case 1.3: If $y^2 \neq 0, v^2 = 0$ then there exist $y' \in V(\Gamma_2)$ such that $yy' = 0$. Hence $(x, y) - (x, y'v) - (u, v)$ is a path.
Case 1.4: If $y^2 \neq 0, v^2 \neq 0$ then there exist $y', v' \in V(\Gamma_2)$ such that $yy' = 0$ and $vv' = 0$.
If $y'v' \neq 0$, then $(x, y) - (x, y'v') - (u, v)$ is a path.
If $y'v' = 0$. Then $(x, y) - (x, y') - (u, v') - (u, v)$ is a path.

Case 2: When $x \neq u$ and $yv = 0$.
Case 2.1: If $x^2 = 0 = u^2$, then $(x, y) - (xu, y) - (u, v)$ is a path.
Case 2.2: If $x^2 = 0, u^2 \neq 0$ then there exist $u' \in V(\Gamma_1)$ such that $uu' = 0$. Hence $(x, y) - (xu', y) - (u, v)$ is a path.
Case 2.3: If $x^2 \neq 0, u^2 = 0$ then there exist $x' \in V(\Gamma_1)$ such that $xx' = 0$. Hence $(x, y) - (x'u, y) - (u, v)$ is a path.
Case 2.4: If $x^2 \neq 0, u^2 \neq 0$ then there exist $x', u' \in V(\Gamma_1)$ such that $xx' = 0$ and $uu' = 0$.
If $x'u' \neq 0$, then $(x, y) - (x'u', y) - (u, v)$ is a path.
If $x'u' = 0$. Then $(x, y) - (x', y) - (u', v) - (u, v)$ is a path.

Case 3: When $x \neq u$ and $yv \neq 0$, then from Case 2, there is a path connecting (x, y) and (u, v) of length at most 3.

□

Theorem 7.3. *Let G be the Lexicographic product of Γ_1 and Γ_2. Then $girth(G) = 3$.*

Proof. Proof follows from theorem 6.3. □

In the following result, we find the value of m and n for which the zero-divisor graph of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ is Eulerian.

Theorem 7.4. *Let G be the Lexicographic product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then G is Eulerian if:*

(1) *m is odd and square free;*
(2) *$n = m = 4$*

Proof. Let G be the Lexicographic product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ and $(u, v) \in G$. Then degree of (u, v) is given by

$$deg(u, v) = deg(u)|V(\Gamma(\mathbb{Z}_m))| + deg(v)$$

We have two cases:
a.) When $|V(\Gamma(\mathbb{Z}_m))|$ is even. We know that $|V(\Gamma(\mathbb{Z}_m))|$ is even for odd m and in this case degree of v is even when m is square free. So G is Eulerian when m is odd and square free and n is arbitrary.
b.) When $|V(\Gamma(\mathbb{Z}_m))|$ is odd. We know that $|V(\Gamma(\mathbb{Z}_m))|$ is odd for even m. So G is

Eulerain when degree of u and v are both odd or both even. Since m is even, we have $m = 4$ and hence $n = 4$. So G is Eulerian, when $n = m = 4$ and in this case G have only one vertex namely $(2, 2)$. $\square$

8. Conclusion

We have studied the graph products namely, cartesian product, tensor product, strong product, co-normal product and lexicographic product, of zero-divisor graphs of commutative rings. Some structural properties: connectivity, diameter and girth of these graph products are discussed. We also discussed when the graph product of zero-divisor graphs of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are Eulerian.

9. Acknowledgement

The authors would like to express their sincere thanks to the referees for their remarks and suggestions that resulted in the present shape of the manuscript.

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