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FUZZY TWO POINT BOUNDARY VALUE PROBLEM WITH EXTENSION PRINCIPLE

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Abstract. In this study, we have propose a new method for solving fuzzy two point boundary value problem (FTPBVP) with certain types of fuzzy boundary conditions commonly encountered. We have investigate linear differential equations with boundary values expressed by fuzzy numbers and we search for a fuzzy set of real functions as the solution. We have found the fuzzy solution which is obtained from solution of crisp problem by the application of the extension principle. Then we have validated the method by solving an example.

Keywords: fuzzy boundary value problem, fuzzy eigenfunction, fuzzy coefficients, extension principle.

AMS Subject Classification: 34B24; 34A07; 34B05; 47A75.

1. Introduction

In this paper we consider the FTPBVP

$$L = -\frac{d^2}{dx^2}$$

$$L\widehat{u} = \lambda \widehat{u}, \quad x \in [a,b] \tag{1}$$

which satisfies the conditions

$$\widehat{a}_1 \widehat{u}\,(a) = \widehat{a}_2 \widehat{u}'\,(a) \tag{2}$$

$$\widehat{b}_1 \widehat{u}\,(b) = \widehat{b}_2 \widehat{u}'\,(b) \tag{3}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ nonnegative triangular fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a}_1$ and $\widehat{a}_2$ and at least one of the numbers $\widehat{b}_1$ and $\widehat{b}_2$ are nonzero and $\widehat{u}$ is fuzzy function.

In recent years, the theory of fuzzy differential equations (FDEs) has gained a lot of interest because this theory is a natural way of real world problems under uncertainty. There are several approaches to studying fuzzy differential equations. One can provide a fuzzy solution to the problem by using the Hukuhara derivative, or a derivative in generalized sense [5, 15, 16, 18]. Other approach is based on Zadeh's extension principle.

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For a fuzzy initial value problem, the associated crisp problem is solved and in the solution the initial fuzzy value is substituted instead of the real constant. In the final solution, arithmetic operations are considered to be operations on fuzzy numbers [13, 14]. One can deal with FBVPs numerically as well. In this case, the fuzzy numerical solution can be given by means of α-levels [3]. For more information, we refer the authors in the references [4, 7, 9, 17].

The studies of FTPBVP with eigenvalue parameter have made with the Hukuhara derivative [10, 11, 12] and generalized Hukuhara derivative [6]. But in some cases the fuzzy solutions with Hukuhara derivative suffers certain disadvantages since the diameter of the solutions is unbounded as time increases [1, 2] and the fuzzy solutions with generalized Hukuhara derivative have some not an interval solutions which are associated with the existence of switch points [18]. For this reason, solution method on this problem has been carried out under limited conditions and limited domain.

In this study, we investigate the FTPBVP with fuzzy boundary values by using Zadeh's extension principle [8]. In this way the fuzzy problem is considered to be a set of crisp problems and arithmetic operations are considered to be operations on fuzzy numbers. Finally, we present an example in order to illustrate the main contributions of this study.

2. Preliminaries

2.1. **Solution for a crisp boundary value problem.** Let' s consider the fuzzy problem (1)-(3) as the crisp problem. Then we shall make use of solutions of (1) defined by initial conditions instead of boundary conditions in a manner similar to Titchmarsh [19, ch. 1]. This approach also enables us to obtain different solution method.

We define two solutions $\Phi_\lambda(x)$ and $\Psi_\lambda(x)$ of the equation (1) as follows. Let $\Phi_\lambda(x) = \Phi(x,\lambda)$ be the solution of equation (1) on $[a,b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \tag{4}$$

and $\Psi_\lambda(x) = \Psi(x,\lambda)$ be the solution of equation (1) on $[a,b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}. \tag{5}$$

Then we put this $\Phi_\lambda(x)$ and $\Psi_\lambda(x)$ functions as follows

$$W(\lambda) = \Phi_\lambda(x)\,\Psi'_\lambda(x) - \Phi'_\lambda(x)\,\Psi_\lambda(x) \tag{6}$$

which is independent of $x \in [a,b]$.

This two solutions $\Phi_\lambda(x)$ and $\Psi_\lambda(x)$ of the equation (1) are obtained by using the solution methodology of Gasilov et al. [8].

Lemma 2.1. [19] *If $\lambda = \lambda_0$ is an eigenvalue, then $\Phi(x,\lambda_0)$ and $\Psi(x,\lambda_0)$ are linearly dependent and eigenfuntions corresponding to this eigenvalue .*

Theorem 2.1. [19] *The eigenvalues of the problem (1)-(3) are the zeros of the function $W(\lambda)$.*

In section 3, we will use a similar method of Titchmarsh to define a fuzzy solution for second order two point boundary values problems with fuzzy boundary values.

Before we introduce our approach to solve a FBVP, it is first necessary to review some concepts of fuzzy sets theory.

2.2. **Basic concepts of fuzzy sets.**

Definition 2.1. [16] *Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}} : E \to [0,1]$, where $\mu_{\widehat{A}}(x)$ represents the degree to which $x \in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the sembol $F_K(E)$.*

Definition 2.2. [17] *A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ level set is denoted by $[\widehat{u}]^{\alpha}$, i.e., $[\widehat{u}]^{\alpha} = \{x : \widehat{u}(x) \geq 0\}$, if it satisfies two axioms:*
i. *There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$,*
ii. *For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_{\alpha}^{-} \leq u_{\alpha}^{+} < +\infty$ such that $[\widehat{u}]^{\alpha}$ is equal to the closed interval $[u_{\alpha}^{-}, u_{\alpha}^{+}]$.*

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathbb{R}_F$.$F_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha-$ levels are nonempty compact convex subsets of $\mathbb{R}$. If $\widehat{u} \in \mathbb{R}_F$ and $\widehat{u}(t) = 0$ whenever $t < 0$, then $\widehat{u}$ is called a non-negative fuzzy real number and $\mathbb{R}_F^{+}$ denotes the set of all non-negative fuzzy real numbers. For all $\widehat{u} \in \mathbb{R}_F^{+}$ and each $\alpha \in (0,1]$, real number u_{α}^{-} is positive.

Definition 2.3. [1] *An arbitrary fuzzy number $\widehat{u}$ in the parametric form is represented by an ordered pair of functions $[u_{\alpha}^{-}, u_{\alpha}^{+}]$, $0 < \alpha \leq 1$ which satisfy the following requirements*
i. *u_{α}^{-} is bounded non-decreasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,*
ii. *u_{α}^{+} is bounded non- increasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,*
iii. *$u_{\alpha}^{-} \leq u_{\alpha}^{+}$, $0 < \alpha \leq 1$.*

Definition 2.4. [1] *For $\widehat{u}, \widehat{v} \in \mathbb{R}_F$, and $\lambda \in \mathbb{R}$, the sum $\widehat{u} \oplus \widehat{v}$ and the product $\lambda \odot \widehat{u}$ are defined for all $\alpha \in [0,1]$,*

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^{\alpha} &= [\widehat{u}]^{\alpha} + [\widehat{v}]^{\alpha} = \{x + y : x \in [\widehat{u}]^{\alpha}, y \in [\widehat{v}]^{\alpha}\}, \\ [\lambda \odot \widehat{u}]^{\alpha} &= \lambda \odot [\widehat{u}]^{\alpha} = \{\lambda x : x \in [\widehat{u}]^{\alpha}\}. \end{aligned}$$

Definition 2.5. [14] *A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$ level is of the form $\left[\widehat{A}\right]^{\alpha} = [(a_2 - a_1)\alpha + a_1, a_3 - (a_3 - a_2)\alpha]$, for all $\alpha \in [0,1]$, where $\left[\widehat{A}\right]^{0} = [a_1, a_3]$. A triangular fuzzy number is denoted by the triple (a_1, a_2, a_3).*

3. Solution Method of the FTPBVP

In this section, we are concerned with how to solve the FTPBVP. To do this, firstly we get two FIVPs by using fuzzy boundary conditions (2) and (3) from method of Titchmarsh. Then we solve these two FIVPs by using the method of Gasilov et al. [8]. Here λ is crisp number and $\lambda = p^2$, $p > 0$.

We consider two FIVPs involving a crisp differential equation but with fuzzy initial values:

$$\begin{cases} u'' + \lambda u = 0 \\ u(a) = \widehat{a}_2 \\ u'(a) = \widehat{a}_1 \end{cases} \tag{7}$$

and

$$\begin{cases} u'' + \lambda u = 0 \\ u(b) = \widehat{b}_2 \\ u'(b) = \widehat{b}_1. \end{cases} \tag{8}$$

We define two solutions $\widehat{\Phi}_\lambda(x) = \widehat{\Phi}(x,\lambda)$ and $\widehat{\Psi}_\lambda(x) = \widehat{\Psi}(x,\lambda)$ of the FIVPs (7) and (8) respectively.

These two solutions $\widehat{\Phi}_\lambda(x)$ and $\widehat{\Psi}_\lambda(x)$ of the equation (1) are well defined, that is, they are fuzzy numbers and these are obtained by using the solution methodology of Gasilov et al's as follows [8].

For problem (7), $\Phi_{1\lambda}(x) = \cos(px)$ and $\Phi_{2\lambda}(x) = \sin(px)$ are linear independent solutions. Then $s(x) = (\cos(px), \sin(px))$ and $m = (\widehat{a}_2, \widehat{a}_1)$. Consider

$$w(x) = s(x)\,N^{-1} \tag{9}$$

where $N = \begin{pmatrix} \Phi_{1\lambda}(a) & \Phi_{2\lambda}(a) \\ \Phi'_{1\lambda}(a) & \Phi'_{2\lambda}(a) \end{pmatrix} = \begin{pmatrix} \cos(pa) & \sin(pa) \\ -p\sin(pa) & p\cos(pa) \end{pmatrix}$ and

$N^{-1} = \frac{1}{p}\begin{pmatrix} p\cos(pa) & -\sin(pa) \\ p\sin(pa) & \cos(pa) \end{pmatrix}$, $s(x) = (\cos(px), \sin(px))$ and

$w(x) = (w_1(x), w_2(x))$. From equation (9) we find

$$(w_1(x), w_2(x)) = \left(\cos(px), \frac{1}{p}\sin(px)\right)$$

Then the solutions of (7) can be represented as

$$\widehat{\Phi}_\lambda(x) = w_1(x)\,\widehat{a}_2 + w_2(x)\,\widehat{a}_1 = \cos(px)\,\widehat{a}_2 + \frac{1}{p}\sin(px)\,\widehat{a}_1. \tag{10}$$

Similarly we find $\widehat{\Psi}_\lambda(x)$ such that

$$\widehat{\Psi}_\lambda(x) = \cos(px - pb)\,\widehat{b}_2 + \frac{1}{p}\sin(px - pb)\,\widehat{b}_1. \tag{11}$$

Then (7) and (8) have a unique solution $\widehat{\Phi}_\lambda(x)$ and $\widehat{\Psi}_\lambda(x)$ respectively from [8].

So, putting (10) and (11) solutions the above equation in (6), we get crisp Wronskian function as

$$W(\lambda) = \left(a_2 b_2 p + \frac{a_1 b_1}{p}\right)\sin(pb) + (a_1 b_2 - a_2 b_1)\cos(pb) \tag{12}$$

Here, we get crisp Wonskian function because λ is crisp number.

Definition 3.1. [10] *The values of the parameter λ are called the eigenvalues of (1)-(3) if the equation (1) has the nontrivial solutions satisfying (2)-(3). These corresponding solutions are called fuzzy eigenfunctions of (1)-(3) fuzzy problem.*

Theorem 3.1. [19] *The roots of equation (12) coincide with the eigenvalues of the fuzzy boundary value problem (1)-(3).*

The method separation variables for solving the heat equation often leads transformation of partial differential equation to ordinary differential equation. The ODE thus obtained will be in the form of an eigenvalue problem such as (1)-(3). So here certain types of boundary conditions commonly encountered in the fuzzy problem are discussed below.

i) Dirichlet Boundary Condition

For this boundary conditions, we take $\widehat{a}_1 = 1$, $\widehat{a}_2 = 0$, $a = 0$ and $\widehat{b}_1 = 1$, $\widehat{b}_2 = 0$, $b \neq 0$. So we get Dirichlet type boundary conditions

$$\widehat{u}(0) = 0, \qquad \widehat{u}(b) = 0.$$

The problem here is to find the nonnegative values of λ and the nontrivial fuzzy solutions $\widehat{u}(x)$ of (1). Then from equations (10) and (11), we get $\widehat{\Phi}_\lambda(x)$ and $\widehat{\Psi}_\lambda(x)$ as follows

$$\widehat{\Phi}_\lambda(x) = \frac{1}{p}\sin(px)$$

and

$$\widehat{\Psi}_\lambda(x) = \frac{1}{p}\sin(px - pb).$$

For this eigenfunctions, we are only interested in the function itself and not the constants in front of them; therefore, we generally drop that coefficients, so these fuzzy solutions are crisp solutions because there are no fuzzy coefficients such as $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$. Thus, from the equation (12) and Theorem (2.1), we get Wronskian function as

$$W(\lambda) = \frac{1}{p}\sin(pb) = 0.$$

Hence for nontrivial solution p must be positive roots of the equaton

$$\sin(pb) = 0$$

from which we have

$$p = \frac{n\pi}{b} \quad (n = 1, 2, ...).$$

Therefore, except for the constant factor

$$\Phi_\lambda(x) = \sin\left(\frac{n\pi}{b}x\right) \quad (n = 1, 2, ...)$$

and

$$\Psi_\lambda(x) = \frac{1}{p}\sin\left(\frac{n\pi}{b}x - \frac{n\pi}{b}b\right) = \sin\left(\frac{n\pi}{b}x\right) \quad (n = 1, 2, ...)$$

and the corresponding values of λ are

$$\lambda = p^2 = \left(\frac{n\pi}{b}\right)^2 \quad (n = 1, 2, ...).$$

Thus, the discrete values

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2 \quad (n = 1, 2, ...)$$

of λ for which problem (1) has nontrivial solutions are called eigenvalues of that problem and the solutions

$$\Phi_{\lambda_n}(x) = \sin\left(\frac{n\pi}{b}x\right) \quad (n = 1, 2, ...)$$

and

$$\Psi_{\lambda_n}(x) = \frac{1}{p}\sin\left(\frac{n\pi}{b}x - \frac{n\pi}{b}b\right) = \sin\left(\frac{n\pi}{b}x\right) \quad (n = 1, 2, ...)$$

are the corresponding eigenfunctions and these are not fuzzy.

ii) Neumann Boundary Condition

For this boundary conditions, we take $\widehat{a}_1 = 0$, $\widehat{a}_2 = 1$, $a = 0$ and $\widehat{b}_1 = 0$, $\widehat{b}_2 = 1$, $b \neq 0$. So we get Neumann type boundary conditions

$$\widehat{u}'(0) = 0, \qquad \widehat{u}'(b) = 0.$$

Thus, we similarly get

$$\widehat{\Phi}_\lambda(x) = \cos(px)$$

and

$$\widehat{\Psi}_\lambda(x) = \cos(px - pb).$$

Then the Wronskian function is

$$W(\lambda) = p\sin(pb) = 0$$

where we have

$$p = \frac{n\pi}{b} \text{ and } \lambda = p^2 = \left(\frac{n\pi}{b}\right)^2 \quad (n = 1, 2, ...).$$

Thus the solutions

$$\Phi_{\lambda_n}(x) = \cos\left(\frac{n\pi}{b}x\right) \quad (n = 1, 2, ...)$$

and

$$\Psi_{\lambda_n}(x) = \frac{1}{n\pi}\cos\left(\frac{n\pi}{b}x - \frac{n\pi}{b}b\right) = \frac{1}{n\pi}\cos\left(\frac{n\pi}{b}x\right) \quad (n = 1, 2, ...)$$

are the corresponding eigenfunctions and these are not fuzzy.

iii) Mixed Dirichlet-Neumann Boundary Condition

The first version:

For this boundary conditions, we take $\widehat{a}_1 = 1$, $\widehat{a}_2 = 0$, $a = 0$ and $\widehat{b}_1 = 0$, $\widehat{b}_2 = 1$, $b \neq 0$. Thus, we get the first version of Mixed type boundary conditions

$$\widehat{u}(0) = 0, \qquad \widehat{u}'(b) = 0.$$

Therefore, we similarly get

$$\widehat{\Phi}_\lambda(x) = \frac{1}{p}\sin(px)$$

and

$$\widehat{\Psi}_\lambda(x) = \cos(px - pb).$$

Then the Wronskian function is

$$W(\lambda) = \cos(pb) = 0$$

where we have

$$p = \frac{\left(n - \frac{1}{2}\right)\pi}{b} \text{ and } \lambda = p^2 = \left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}\right)^2 \quad (n = 1, 2, ...)$$

Thus, the solutions

$$\Phi_{\lambda_n}(x) = \sin\left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}\right) \quad (n = 1, 2, ...)$$

and

$$\Psi_{\lambda_n}(x) = \cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}x - \frac{\left(n - \frac{1}{2}\right)\pi}{b}b\right) = \sin\left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}x\right) \quad (n = 1, 2, ...)$$

are the corresponding eigenfunctions and these are not fuzzy.

The Second Version:

For these boundary conditions, we take $\widehat{a}_1 = 0$, $\widehat{a}_2 = 1$, $a = 0$ and $\widehat{b}_1 = 1$, $\widehat{b}_2 = 0$, $b \neq 0$. So we get the second version of Mixed type boundary conditions

$$\widehat{u}'(0) = 0, \qquad \widehat{u}(b) = 0.$$

Thus, we similarly get

$$\widehat{\Phi}_\lambda(x) = \cos(px)$$

and

$$\widehat{\Psi}_\lambda(x) = \frac{1}{p}\sin(px - pb).$$

Then the Wronskian function is

$$W(\lambda) = -\cos(pb) = 0$$

where we have

$$p = \frac{\left(n - \frac{1}{2}\right)\pi}{b} \text{ and } \lambda = p^2 = \left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}\right)^2 \quad (n = 1, 2, ...)$$

Thus, the solutions

$$\Phi_{\lambda_n}(x) = \cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}\right) \quad (n = 1, 2, ...)$$

and

$$\Psi_{\lambda_n}(x) = \frac{1}{p}\sin\left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}x - \frac{\left(n - \frac{1}{2}\right)\pi}{b}b\right) = \frac{1}{p}\cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{b}x\right) \quad (n = 1, 2, ...)$$

are the corresponding eigenfunctions and these are not fuzzy.

The third version:

For this boundary conditions, we take $\widehat{a}_1 = 1$, $\widehat{a}_2 = \widehat{h}$, $a = 0$ and $\widehat{b}_1 = 1$, $\widehat{b}_2 = \widehat{H}$, $b \neq 0$ where $\widehat{h}$ and $\widehat{H}$ are triangular fuzzy numbers. Then we get the third version of Mixed type boundary conditions

$$\begin{aligned} \widehat{u}(0) &= \widehat{h}\widehat{u}'(0) \\ \widehat{u}(b) &= \widehat{H}\widehat{u}'(b). \end{aligned}$$

So similarly to the operations in the first version, we get

$$\widehat{\Phi}_\lambda(x) = \cos(px)\widehat{h} + \frac{1}{p}\sin(px) \tag{13}$$

and

$$\widehat{\Psi}_\lambda(x) = \cos(px - pb)\widehat{H} + \frac{1}{p}\sin(px - pb). \tag{14}$$

We note that there are fuzzy coefficients $\widehat{h}$ and $\widehat{H}$ in equations (13) and (14). Then the Wronskian function is

$$W(\lambda) = \left(hHp + \frac{1}{p}\right)\sin(p) + (H - h)\cos(p) = 0. \tag{15}$$

The values p satisfying the equation (15) are eigenvalues of the problem. Then putting this values in equation (13) and (14) respectively, we get fuzzy eigenfunctions.

To sum up, there are other different boundary conditions. So fuzzy solutions of equation (1) are found only for some types boundary conditions of Mixed Dirichlet-Neumann boundary conditions. For other type boundary conditions we get only crisp solutions. Now we examine the following problem for Mixed Dirichlet-Neumann boundary conditions.

Example 3.1. *Consider the two point fuzzy boundary value problem*

$$-\widehat{u}'' = \lambda\widehat{u} \tag{16}$$

$$\widehat{2}\widehat{u}(0) = \widehat{1}\widehat{u}'(0) \tag{17}$$

$$\widehat{4}\widehat{u}(1) = \widehat{3}\widehat{u}'(1) \tag{18}$$

where $\widehat{1} = (0, 1, 2)$, $\widehat{2} = (1, 2, 3)$, $\widehat{3} = (2, 3, 4)$, $\widehat{4} = (3, 4, 5)$, *and* $\lambda = p^2$, $p > 0$.

From (16)-(18) problem, we get two FIVPs involving a crisp differential equation (16) but with fuzzy initial values as follows:

$$\Phi'' + p^2\Phi = 0, \quad \Phi(0) = \widehat{1}, \quad \Phi'(0) = \widehat{2} \tag{19}$$

and

$$\Psi'' + p^2\Psi = 0, \quad \Psi(1) = \widehat{3}, \quad \Psi'(1) = \widehat{4} \tag{20}$$

For (19), $\Phi_{1\lambda}(x) = \cos(px)$ *and* $\Phi_{2\lambda}(x) = \sin(px)$ *are linear independent solutions. Then* $s(x) = (\cos(px), \sin(px))$ *and* $N = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ *and* $w(x) = s(x)N^{-1} = \cos(px) + \frac{1}{p}\sin(px)$.

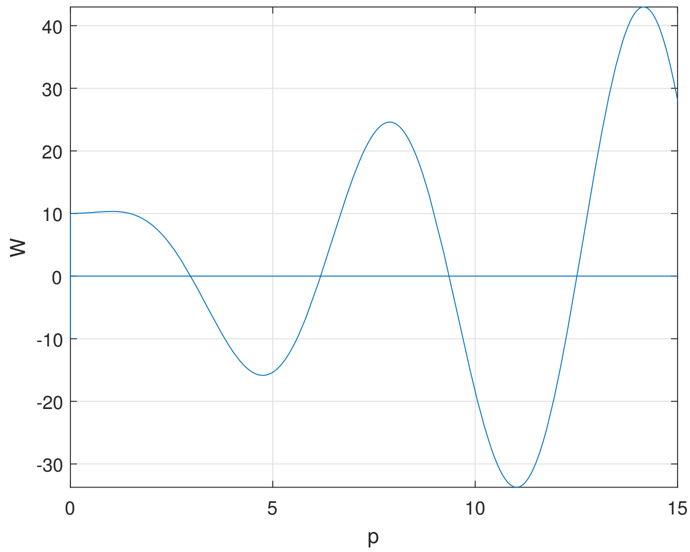

FIGURE 1. The function $W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (6-4)\cos(p)$

So from equation (9), we get the following solution for (19):

$$\widehat{\Phi}_\lambda(x) = w_1(x)\widehat{a}_2 + w_2(x)\widehat{a}_1 = \cos(px)(0,1,2) + \frac{1}{p}\sin(px)(1,2,3) \tag{21}$$

and similarly we get $\widehat{\Psi}_\lambda(x)$ *as follows*

$$\widehat{\Psi}_\lambda(x) = \cos(px - pb)(2,3,4) + \frac{1}{p}\sin(px - pb)(4,5,6). \tag{22}$$

Then putting the solutions (21) and (22) the above equation (6), we get Wronskian function as

$$W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (6-4)\cos(p) \tag{23}$$

From Theorem (2.1), equation (23) has a nontrivial solution if and only if $W(\lambda) = 0$.

	p_n	λ_n
$n=1$	2.9709	8.8262
$n=2$	6.1827	38.2257
$n=3$	9.3557	87.5291
$n=4$	12.514	156.6
$n\approx$	$n\pi$	$(n\pi)^2$

Table 1. Eigenvalues of the fuzzy problem

For the purposes of this example we found the first five numerically and then we will use the approximation of the remaining eigenvalues. We can see from the Fig. 1 that the graphs intersect at infinitely many points $p_n \approx n\pi \quad (n = 5, 6, 7...)$*, where the error in this approximation approaches zero as* $n \to \infty$*. Given this estimate, we can use Matlab program to compute* p_n *more accurately.*

From the equations (21) and (22)

$$\widehat{\Phi}_{\lambda_n}(x) = \cos(p_n x)(0,1,2) + \frac{1}{p_n}\sin(p_n x)(1,2,3) \tag{24}$$

and

$$\widehat{\Psi}_{\lambda_n}(x) = \cos(p_n x - p_n)(2,3,4) + \frac{1}{p}\sin(p_n x - p_n)(4,5,6) \tag{25}$$

are fuzzy eigenfunctions associated with λ_n*.*

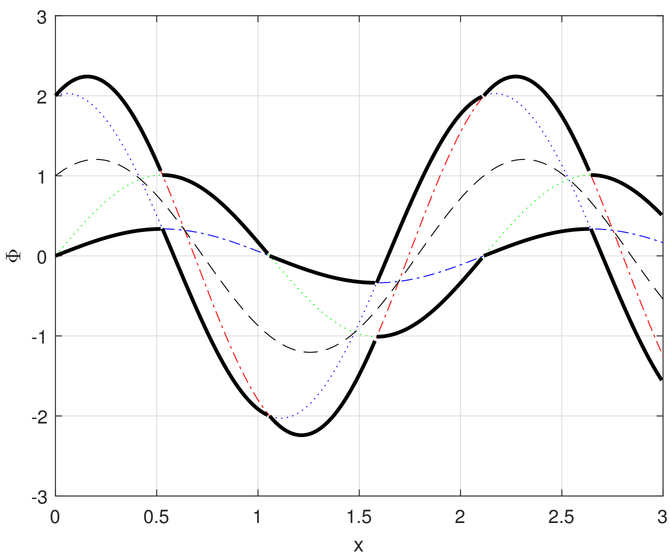

FIGURE 2. The fuzzy solution $\widehat{\Phi}_{\lambda_1}(x)$ for Example 1. The dashed black line represents the crisp solution. The black lines represents the upper and lower boundaries of the band

In particular, we select $p_1 = 2.9709$ *in Table 1. If we substitute this value respectively in (24) and (25), we have the following equations.*

$$\widehat{\Phi}_{\lambda_1}(x) = \cos(2.9709x)(0,1,2) + \frac{1}{2.9709}\sin(2.9709x)(1,2,3) \tag{26}$$

and

$$\widehat{\Psi}_{\lambda_1}(x) = \cos(2.9709x - 2.9709)(2,3,4) + \frac{1}{2.9709}\sin(2.9709x - 2.9709)(4,5,6) \qquad (27)$$

The fuzzy solutions (26) and (27) form a band in the xy-plane Fig. 2 and Fig. 3. The

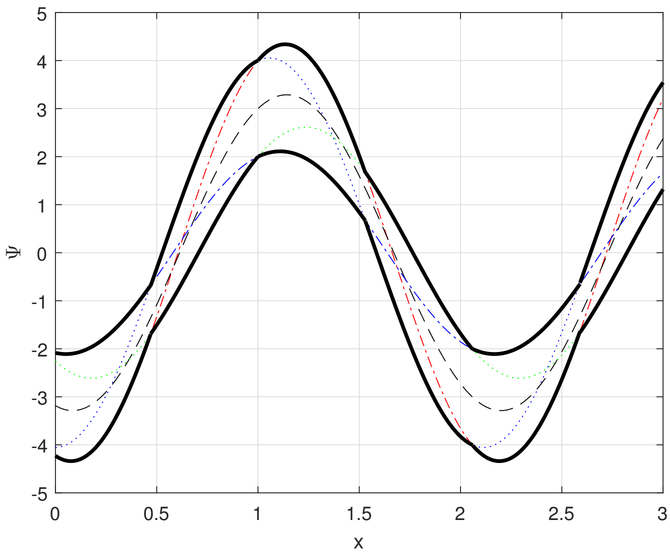

FIGURE 3. The fuzzy solution $\widehat{\Psi}_{\lambda_1}(x)$ for Example 1. The dashed black line represents the crisp solution. The black lines represents the upper and lower boundaries of the band

functions cos *and* sin *take both positive and negative values in the interval* $0 < x < 1$. *So,* $\overline{1}$ *and* $\underline{1}$, $\overline{2}$ *and* $\underline{2}$, $\overline{3}$ *and* $\underline{3}$, $\overline{4}$ *and* $\underline{4}$ *by turns dominate in generating the upper and lower boundary values of the band. So similarly for all values, the eigenfunctions (24) and (25) form a band in the xy-plane. That is, the eigenfunctions (24) and (25) are fuzzy eigenfunctions of the problem (16) - (18).*

Remark 3.1. [8] *At present there is no concept of a fuzzy derivative that corresponds to the derivative of a real function that is fully understood. For this reason, applying a single method to solve all the fuzzy differential equations would not be successful. When solving a practical problem, we have to use concepts of derivative and solution that correspond to the problem in question.*

If one consider a BVP with crisp dynamics but with fuzzy boundary values, such as Example 3 in [8] *, non-derivative approach is more effective. In cases with fuzzy dynamics, use of the generalized fuzzy derivative is more appropriate.*

4. CONCLUSION

In this study we researched the FTPBVP by using two different method as Titchmarch and Gomes et al.. From this methods, we first found the crisp solution, fuzzified it and then checked to see if it satisfied the FTPBVP. So we obtained the eigenvalues and the fuzzy eigenfunctions for certain types of boundary conditions. But we found clasic eigenfunctions for Neumann and Dirichlet boundary conditions and fuzzy eigenfunctions for some types boundary conditions of Mixed Dirichlet-Neumann boundary conditions. Next, we gived an example that provides the properties of fuzzy eigenfunctions.

In the future work we plan to make the parameter in the boundary conditions for eigenvalue problems.Then we will apply this method for Sturm-Liouville problem.

References

[1] Diamond, P. and Kloeden, P., (1994), Metric Spaces of Fuzzy Sets, World Scientific, Singapore.
[2] Kaleva, O., (1987), Fuzzy differential equations, Fuzzy sets and systems, 24(3), pp. 301-317.
[3] Allahviranloo, T. and Khalilpour, K., (2011), A numerical method for two-point fuzzy boundary value problems, World Applied Sciences Journal, 13(10), pp. 2137–2147.
[4] Anastassiou, G., (2000), Handbook of analytic - computational methods in applied mathematics, Boca Raton: Chapman and Hall/CRC.
[5] Bede, B. and Stefanini, L., (2013), Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems, 230, pp. 119–141.
[6] Ceylan, T. and Altınışık, N., (2018), Eigenvalue problem with fuzzy coefficients of boundary conditions, Scholars Journal of Physics, Mathematics and Statistics, 5(2), pp. 187–193.
[7] Jameel, A. F., Anakira, N. R., Alomari, A. K., Alsharo, D. M. and Saaban, A., (2019), New semianalytical method for solving two-point nth order fuzzy boundary value problem, International Journal of Mathematical Modelling and Numerical Optimisation, 9(1), pp. 12-31.
[8] Gasilov, N., Amrahov, S. E. and Fatullayev, A. G., (2011), Linear differential equations with fuzzy boundary values, CoRR, vol. abs/1107.4157, pp. 696–700.
[9] Gomes, L. T., Barros, L. C. and Bede, B., (2015), Fuzzy Differential Equations in Various Approaches, London: Springer Cham.
[10] Gültekin Çitil, H. and Altınışık, N., (2017), On the eigenvalues and the eigenfunctions of the Sturm-liouville fuzzy boundary value problem, J. Math. Comput. Sci., 7(4), pp. 786–805.
[11] Gültekin Çitil, H. and Altınışık, N., (2018), The examination of eigenvalues and eigenfunctions of the sturm-liouville fuzzy problem according to boundary conditions, International Journal of Mathematical Combinatorics, 1, pp. 51–60.
[12] Gültekin Çitil, H., (2019), Comparisons of the exact and the approximate solutions of second-order fuzzy linear boundary value problems, Miskolc Mathematical Notes, 20(2), pp. 823–837.
[13] Buckley, J. J. and Feuring, T., (2000), Fuzzy differential equations, Fuzzy Sets Syst., 110, pp. 43–54.
[14] Buckley, J. J. and Feuring, T., (2001), Fuzzy initial value problem for Nth-order linear differential equations, Fuzzy Sets Syst., 121, pp. 247–255.
[15] Khastan, A. and Nieto, J. J., (2010), A boundary value problem for second order fuzzy differential equations, Fuzzy sets and systems, 72(9-10), pp. 3583–3593.
[16] Puri, M. L. and Ralescu, D. A., (1983), Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications, 91(2), pp. 552–558.
[17] Sadeqi, I., Moradlou, M. and Salehi, M., (2013), On approximate cauchy equation in felbin's type fuzzy normed linear spaces, Iranian Journal of Fuzzy Systems, 10(3), pp. 51–63.
[18] Stefanini, L. and Bede, B., (2009), Generalized hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Analysis: Theory, Methods and Applications, 71(3-4), pp. 1311–1328.
[19] Titchmarsh, E. C., (1962), Eigenfunction expansions associated with second-order differential equations I, 2nd edn., Oxford University Press, London.

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