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MODULAR PRODUCT OF SOFT DIRECTED GRAPHS

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Abstract. Soft set theory was proposed by D. Molodtsov as a mathematical framework for dealing with uncertain data. Many academics are now applying soft set theory in decision-making problems. In graph theory, a directed graph is a graph made up of vertices connected by directed edges, also known as arcs. Using directed graphs, it is possible to examine and find solutions to problems relating to social connections, shortest paths, electrical circuits etc. Soft directed graphs were introduced by applying the concept of soft set to directed graphs. They provide a parameterized point of view for directed graphs. In this work, we introduce the modular product and the restricted modular product of soft directed graphs. We prove that these products are also soft directed graphs and we develop the formulas for determining the vertex count, the arc count and the sum of degrees in them.

Keywords: Soft Graph, Soft Directed Graph, Modular Product.

AMS Subject Classification: 05C20, 05C76, 05C99

1. Introduction

D. Molodtsov [15] presented the innovative concept of soft set theory in 1999. This is a technique in mathematics for dealing with uncertainties. Many practical problems can be tackled using soft set theory. Authors like R. Biswas, P. K. Maji and A. R. Roy [13], [14] have delved deeper into the idea of soft sets and applied it to various decision-making situations. In 2014, R. K. Thumbakara and B. George [19] introduced the concept of soft graphs to provide a parameterized point of view for graphs. M. Akram and S. Nawas [1] updated R. K. Thumbakara and B. George's notion of the soft graph in 2015. They [2] also defined many varieties of soft graphs, such as regular soft graphs, soft trees, and soft bridges, as well as the notions of soft cut vertex, soft cycle and so on. M. Akram and S. Nawas [3] also introduced the notions of fuzzy soft graphs, strong fuzzy soft graphs,

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complete fuzzy soft graphs, and regular fuzzy soft graphs and investigated some of their properties. They [4] also described some applications of fuzzy soft graphs. M. Akram and F. Zafar introduced the notions of soft trees [5] and fuzzy soft trees [6]. More contributions to connected soft graphs came from J. D. Thenge, R. S. Jain and B. S. Reddy[16]. They [17] looked at the ideas of a soft graph's radius, diameter, and centre, as well as the concept of degree. They also addressed the notions of incidence and adjacency matrices of a soft graph in 2020 [18]. B. George, R. K. Thumbakara and J. Jose [7],[8], [20] discussed some soft graph operations and introduced notions such as soft semigraphs and soft hypergraphs.

Directed graphs arise in a natural way in many applications of graph theory. They can be used to analyze and resolve problems with electrical circuits, project timelines, shortest routes, social links, and many other issues. J. Jose, B. George and R.K. Thumbakara [12] introduced the notion of the soft directed graph by applying the concepts of soft set in a directed graph. They also introduced the concepts of indegree, outdegree, degree, adjacency matrix and incidence matrix in soft directed graphs and investigated their properties. The directed graph product [10] is a binary operation on directed graphs. It is a process that takes two directed graphs, $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ and creates a directed graph D having the characteristics listed below: The vertex set of D is the Cartesian product $V_1 \times V_2$. Two vertices (v_1, v_2) and (v_1', v_2') of D are joined by an arc, if and only if some conditions about v_1, v_1' in D_1 and/or v_2, v_2' in D_2 are satisfied. Analogous to the definitions of directed graph products, we can define product operations in soft directed graphs. In [12], some product operations of soft directed graphs like the cartesian product, restricted cartesian product, lexicographic product and restricted lexicographic product are studied. In this paper, we introduce the modular product and restricted modular product of soft directed graphs. We prove that these products are also soft directed graphs and we develop the formulas for determining the vertex count, the arc count and the sum of indegrees, outdegrees and degrees in them.

2. Preliminaries

2.1. **Directed Graphs.** [9],[11] A *directed graph* or *digraph* D^* consists of a non-empty finite set V of elements called *vertices* and a finite set A of ordered pairs of distinct vertices called *arcs*. We often write $D^* = (V, A)$ to represent a directed graph. The number of vertices and arcs in a directed graph D^* are called *order* and *size* respectively. The first vertex u of an arc (u, v) is called its *tail* and the second vertex v is called its *head*. If (u, v) is an arc then v is *adjacent from* u and u is *adjacent to* v. A vertex u is *incident* to an arc a if u is the head or tail of a. A directed graph $D^{**} = (U, F)$ is called a *subdigraph* of $D^* = (V, A)$ if $U \subseteq V$ and $F \subseteq A$. The *in-degree* of a vertex v denoted by *ideg* v is the number of vertices in D^* from which v is adjacent and *out-degree* of v denoted by *odeg* v is the number of vertices in D^* to which v is adjacent. The sum *ideg* v + *odeg* v is called the *degree* of the vertex v and is denoted by *deg* v. In a directed graph $D^* = (V, A)$, $\sum_{v \in V} ideg(v) = \sum_{v \in V} odeg(v)$ =Number of arcs in D^* and $\sum_{v \in V} deg(v) = 2$(Number of arcs in D^*).

Some directed graph products can be defined in a manner that is similar to how the corresponding graph products are defined [10]. Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs. Their *modular product* $D_1^* \bigcirc D_2^*$ is a directed graph with vertex set $V(D_1^* \bigcirc D_2^*) = V_1 \times V_2$ and arc set $A(D_1^* \bigcirc D_2^*)$ where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $D_1^* \bigcirc D_2^*$ if and only if

(1) (v_1, v_2) is an arc in D_1^* and (v_1', v_2') is an arc in D_2^* or
(2) (v_1, v_2) is not an arc in D_1^* and (v_1', v_2') is not an arc in D_2^*.

2.2. **Soft Set.** [20],[12] Let R be a set of parameters and U be an initial universe set. Then a pair (F, R) is called a *soft set* (over U) if and only F is a mapping of R into the power set of U. That is, $F : R \to \mathcal{P}(V)$.

2.3. **Soft Directed Graphs.** [12] We have defined a soft directed graph in a manner similar to how a soft graph is defined [1]. Let $D^* = (V, A)$ be a directed graph having vertex set V and arc set A and let P be a non-empty set. Let a subset R of $P \times V$ be an arbitrary relation from P to V. Define a mapping $J : P \to \mathcal{P}(V)$ by $J(x) = \{u \in V | xRu\}$ where $\mathcal{P}(V)$ denotes the powerset of V. Define another mapping $L : P \to \mathcal{P}(A)$ by $L(x) = \{(u, v) \in A | \{u, v\} \subseteq J(x)\}$ where $\mathcal{P}(A)$ denotes the powerset of E. Then $D = (D^*, J, L, P)$ is called a soft directed graph if it satisfies the following conditions:

(1) $D^* = (V, A)$ is a directed graph having vertex set V and arc set A,
(2) P is a nonempty set of parameters,
(3) (J, P) is a soft set over the vertex set V,
(4) (L, P) is a soft set over the arc set A,
(5) $(J(x), L(x))$ is a subdigraph of D^* for all $x \in P$.

If we represent $(J(x), L(x))$ by $M(x)$ then the soft directed graph D is also given by $\{M(x) : x \in P\}$. Then $M(x)$ corresponding to a parameter x in P is called a *directed part* or simply *dipart* of the soft directed graph D.

Let $D = (D^*, J, L, P)$ be a soft directed graph and let $M(x)$ be a dipart of D for some $x \in P$. Let v be a vertex of $M(x)$. Then dipart indegree of v in $M(x)$ denoted by *ideg* $v[M(x)]$ is defined as the number of vertices of $M(x)$ from which v is adjacent. That is, *ideg* $v[M(x)]$ is the number of arcs of $M(x)$ that have v as its head. Similarly, dipart outdegree of v in $M(x)$ denoted by *odeg* $v[M(x)]$ is defined as the number of vertices of $M(x)$ to which v is adjacent. That is, *odeg* $v[M(x)]$ is the number of arcs of $M(x)$ that have v as its tail. The dipart degree of v in $M(x)$ is defined as the sum, *ideg* $v[M(x)]$ + *odeg* $v[M(x)]$ and is denoted by *deg* $v[M(x)]$.

3. Modular Product of Soft Directed Graphs

Definition 3.1. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and $D_1 = (D_1^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D_2^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be two soft directed graphs of D_1^* and D_2^* respectively. Then the modular product of D_1 and D_2, which is represented by $D_1 \bigcirc D_2$ is defined as $D_1 \bigcirc D_2 = \{M_1(x_1) \bigcirc M_2(x_2) : (x_1, x_2) \in P_1 \times P_2\}$. Here $M_1(x_1) \bigcirc M_2(x_2)$ denotes the modular product of the diparts $M_1(x_1)$ of D_1 and $M_2(x_2)$ of D_2 which is defined as follows: $M_1(x_1) \bigcirc M_2(x_2)$ is a directed graph with vertex set $V(M_1(x_1) \bigcirc M_2(x_2)) = J_1(x_1) \times J_2(x_2)$ and arc set $A(M_1(x_1) \bigcirc M_2(x_2))$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $M_1(x_1) \bigcirc M_2(x_2)$ if and only if*

(1) *(v_1, v_2) is an arc in $M_1(x_1)$ and (v_1', v_2') is an arc in $M_2(x_2)$ or*
(2) *(v_1, v_2) is not an arc in $M_1(x_1)$ and (v_1', v_2') is not an arc in $M_2(x_2)$.*

Example 3.1. *Let $D_1^* = (V_1, A_1)$ be a directed graph which is shown in Fig. 1.*
Let $P_1 = \{v_2, v_6\} \subseteq V_1$ be a set of parameters. Define a mapping $J_1 : P_1 \to \mathcal{P}(V_1)$ by $J_1(x) = \{u \in V_1 \mid u = x$ or u is adjacent from $x\}, \forall x \in P_1$. That is, $J_1(v_2) = \{v_1, v_2, v_5\}$ and $J_1(v_6) = \{v_4, v_5, v_6\}$. Here (J_1, P_1) is a soft set over V_1. Define another mapping $L_1 : P_1 \to \mathcal{P}(A_1)$ by $L_1(x) = \{(u, v) \in A_1 \mid \{u, v\} \subseteq J_1(x)\}, \forall x \in P_1$. That is, $L_1(v_2) = \{(v_2, v_1), (v_2, v_5)\}$ and $L_1(v_6) = \{(v_6, v_4), (v_6, v_5)\}$. Here, (L_1, P_1) is a soft set over A_1. Then $M_1(v_2) = (J_1(v_2), L_1(v_2))$ and $M_1(v_6) = (J_1(v_6), L_1(v_6))$ are subdigraphs of D_1^* as shown in Fig. 2. Therefore $D_1 = \{M_1(v_2), M_1(v_6)\}$ is a soft directed graph of D_1^*.

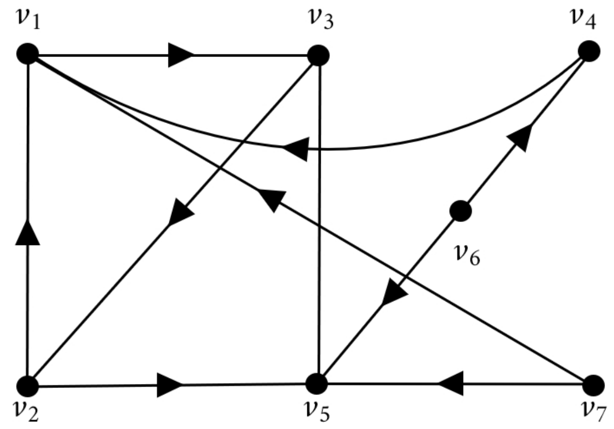


FIGURE 1. Directed Graph $D_1^* = (V_1, A_1)$

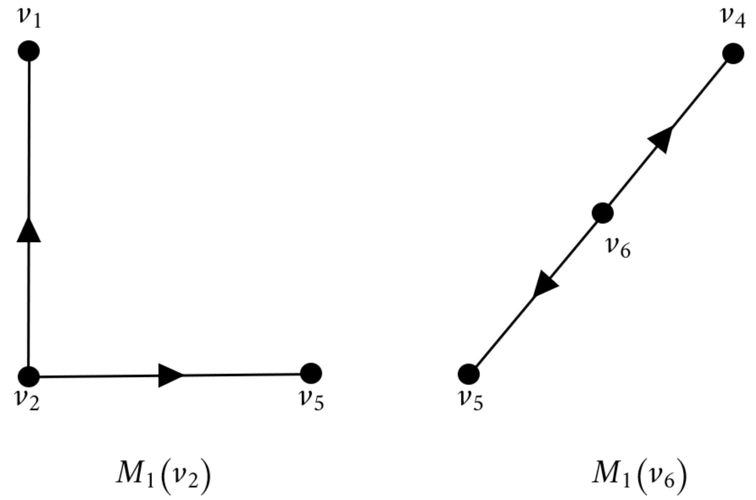


FIGURE 2. Soft Directed Graph $D_1 = \{M_1(v_2), M_1(v_6)\}$

Let $D_2^* = (V_2, A_2)$ be a directed graph which is shown in Fig. 3. Consider the parameter set $P_2 = \{u_2\} \subseteq V_2$. Define a mapping $J_2 : P_2 \to \mathcal{P}(V_2)$ by $J_2(x) = \{u \in V_2 \mid u = x$ or u is adjacent from x or u is adjacent to $x\}, \forall x \in P_2$. That is, $J_2(u_2) = \{u_1, u_2, u_5\}$. Here, (J_2, P_2) is a soft set over V_2. Define another mapping $L_2 : P_2 \to \mathcal{P}(A_2)$ by $L_2(x) = \{(u, v) \in A_2 \mid \{u, v\} \subseteq J_2(x)\}, \forall x \in P_2$. That is, $L_2(u_2) = \{(u_2, u_1), (u_2, u_5), (u_5, u_2)\}$. Here, (L_2, P_2) is a soft set over A_2. Then, $M_2(u_2) = (J_2(u_2), L_2(u_2))$ is a subdigraph of D_2^* as shown in Fig. 4. Therefore, $D_2 = \{M_2(u_2)\}$ is a soft directed graph of D_2^*. Then the modular product of these two soft directed graphs D_1 and D_2 is given by $D = D_1 \bigcirc D_2 = \{M_1(v_2) \bigcirc M_2(u_2), M_1(v_6) \bigcirc M_2(u_2)\}$ and is shown in Fig. 5.

Theorem 3.1. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and D_1 and D_2 be two soft directed graphs of D_1^* and D_2^* respectively. Then the modular product of D_1 and D_2, which is represented by $D_1 \bigcirc D_2$ is a soft directed graph of $D_1^* \bigcirc D_2^*$.*

Proof. Let $D_1 = (D_1^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ be a soft directed graph of $D_1^* = (V_1, A_1)$ and $D_2 = (D_2^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be a soft directed graph of

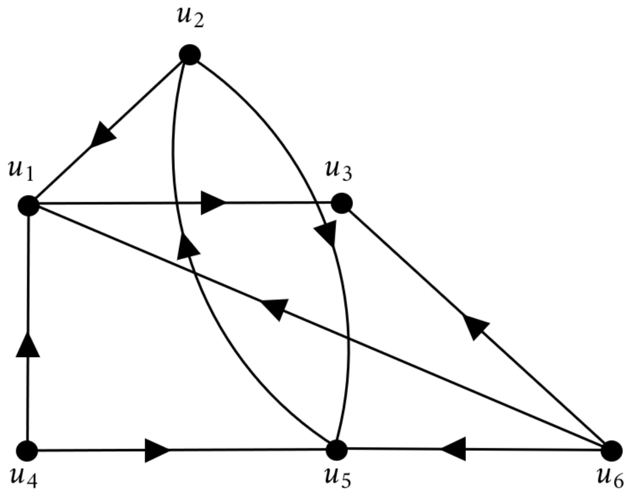


FIGURE 3. Directed Graph $D_2^* = (V_2, A_2)$

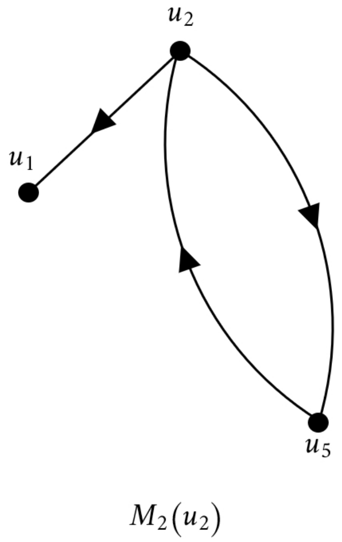


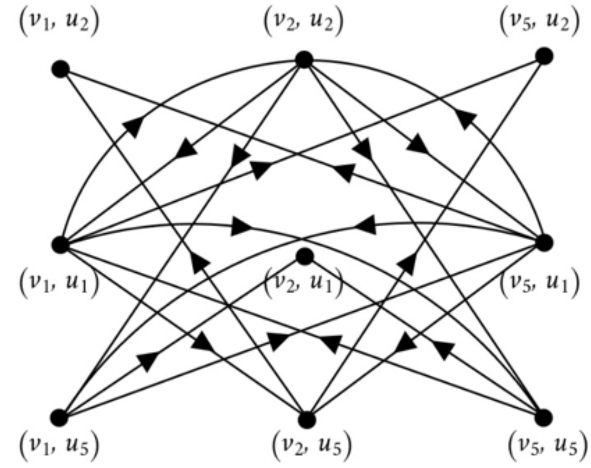
FIGURE 4. Soft Directed Graph $D_2 = \{M_2(u_2)\}$

$D_2^* = (V_2, A_2)$. Then the modular product $D_1 \bigcirc D_2$ is defined as $D_1 \bigcirc D_2 = \{M_1(x_1) \bigcirc M_2(x_2) : (x_1, x_2) \in P_1 \times P_2\}$. Here $M_1(x_1) \bigcirc M_2(x_2)$ denotes the modular product of the diparts $M_1(x_1)$ of D_1 and $M_2(x_2)$ of D_2 which is defined as follows: $M_1(x_1) \bigcirc M_2(x_2)$ is a directed graph with vertex set $V(M_1(x_1) \bigcirc M_2(x_2)) = J_1(x_1) \times J_2(x_2)$ and arc set $A(M_1(x_1) \bigcirc M_2(x_2))$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $M_1(x_1) \bigcirc M_2(x_2)$ if and only if

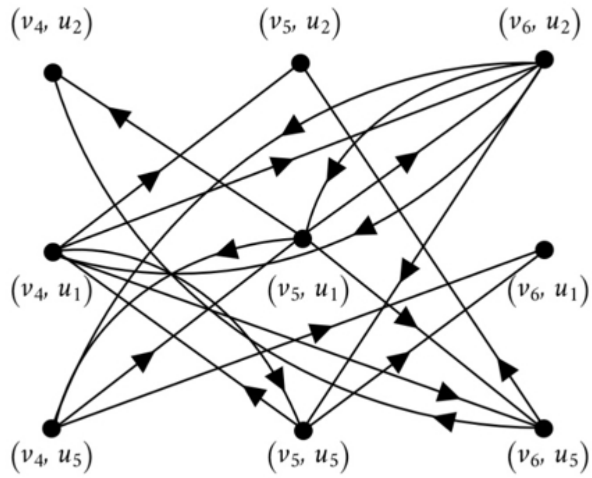
(1) (v_1, v_2) is an arc in $M_1(x_1)$ and (v_1', v_2') is an arc in $M_2(x_2)$ or
(2) (v_1, v_2) is not an arc in $M_1(x_1)$ and (v_1', v_2') is not an arc in $M_2(x_2)$.

The modular product $D_1^* \bigcirc D_2^*$ of the two directed graphs D_1^* and D_2^* is a directed graph with vertex set $V(D_1^* \bigcirc D_2^*) = V_1 \times V_2$ and arc set $A(D_1^* \bigcirc D_2^*)$ where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $D_1^* \bigcirc D_2^*$ if and only if

(1) (v_1, v_2) is an arc in D_1^* and (v_1', v_2') is an arc in D_2^* or



$M_1(v_2) \bigcirc M_2(u_2)$



$M_1(v_6) \bigcirc M_2(u_2)$

FIGURE 5. $D = D_1 \bigcirc D_2 = \{M_1(v_2) \bigcirc M_2(u_2), M_1(v_6) \bigcirc M_2(u_2)\}$

(2) (v_1, v_2) is not an arc in D_1^* and (v_1', v_2') is not an arc in D_2^*.

Let the parameter set be $P_{D_1 \bigcirc D_2} = P_1 \times P_2$. Define a mapping $J_{D_1 \bigcirc D_2}$ from $P_{D_1 \bigcirc D_2}$ to $\mathcal{P}[V(D_1^* \bigcirc D_2^*)]$ by $J_{D_1 \bigcirc D_2}(x_1, x_2) = J_1(x_1) \times J_2(x_2), \forall (x_1, x_2) \in P_1 \times P_2$ where $\mathcal{P}[V(D_1^* \bigcirc D_2^*)]$ denotes the power set of $V(D_1^* \bigcirc D_2^*)$. Then $(J_{D_1 \bigcirc D_2}, P_{D_1 \bigcirc D_2})$ is a soft set over $V(D_1^* \bigcirc D_2^*)$. Define another mapping $L_{D_1 \bigcirc D_2}$ from $P_{D_1 \bigcirc D_2}$ to $\mathcal{P}[A(D_1^* \bigcirc D_2^*)]$ by $L_{D_1 \bigcirc D_2}(x_1, x_2) = \{((u, v), (y, z)) \in A(D_1^* \bigcirc D_2^*) \mid \{(u, v), (y, z)\} \in J_{D_1 \bigcirc D_2}(x_1, x_2)\}, \forall (x_1, x_2) \in P_1 \times P_2$, where $\mathcal{P}[A(D_1^* \bigcirc D_2^*)]$ denotes the power set of $A(D_1^* \bigcirc D_2^*)$. Then $(L_{D_1 \bigcirc D_2}, P_{D_1 \bigcirc D_2})$ is a soft set over $A(D_1^* \bigcirc D_2^*)$. Also if we denote $(J_{D_1 \bigcirc D_2}(x_1, x_2), L_{D_1 \bigcirc D_2}(x_1, x_2))$ by $M_{D_1 \bigcirc D_2}(x_1, x_2)$, then $M_{D_1 \bigcirc D_2}(x_1, x_2)$ is a subdigraph of $D_1^* \bigcirc D_2^*, \forall (x_1, x_2) \in P_1 \times P_2$, since $J_1(x_1) \times J_2(x_2) \subseteq V_1 \times V_2$ and any arc in $L_{D_1 \bigcirc D_2}(x_1, x_2)$ is also an arc in $A(D_1^* \bigcirc D_2^*)$. Then $D_1 \bigcirc D_2$ can be represented by the 4-tuple $(D_1^* \bigcirc$

$D_2^*, J_{D_1 \bigcirc D_2}, L_{D_1 \bigcirc D_2}, P_{D_1 \bigcirc D_2})$ and also by $\{M_{D_1 \bigcirc D_2}(x_1, x_2) : (x_1, x_2) \in P_1 \times P_2\}$ and $D_1 \bigcirc D_2$ is a soft directed graph of $D_1^* \bigcirc D_2^*$ since the following conditions are satisfied:

(1) $D_1^* \bigcirc D_2^* = (V(D_1^* \bigcirc D_2^*), A(D_1^* \bigcirc D_2^*))$ is a directed graph having vertex set $V(D_1^* \bigcirc D_2^*)$ and arc set $A(D_1^* \bigcirc D_2^*)$,
(2) $P_{D_1 \bigcirc D_2} = P_1 \times P_2$ is the set of parameters which is nonempty,
(3) $(J_{D_1 \bigcirc D_2}, P_{D_1 \bigcirc D_2})$ is a soft set over $V(D_1^* \bigcirc D_2^*)$,
(4) $(L_{D_1 \bigcirc D_2}, P_{D_1 \bigcirc D_2})$ is a soft set over $A(D_1^* \bigcirc D_2^*)$,
(5) $M_{D_1 \bigcirc D_2}(x_1, x_2) = (J_{D_1 \bigcirc D_2}(x_1, x_2), L_{D_1 \bigcirc D_2}(x_1, x_2))$ is a subdigraph of $D_1^* \bigcirc D_2^*, \forall (x_1, x_2) \in P_{D_1 \bigcirc D_2} = P_1 \times P_2$.

□

Theorem 3.2. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and $D_1 = (D_1^*, J_1, L_1, P_1)$ and $D_2 = (D_2^*, J_2, L_2, P_2)$ be two soft directed graphs of D_1^* and D_2^* respectively. Then the modular product of D_1 and D_2, which is represented by $D_1 \bigcirc D_2$ contains $\sum_{(x_i,x_j) \in P_1 \times P_2} |J_1(x_i)||J_2(x_j)|$ vertices and $\sum_{(x_i,x_j) \in P_1 \times P_2} (|J_1(x_i)|(|J_1(x_i)| - 1) - |L_1(x_i)|)$ $(|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)|) + \sum_{(x_i,x_j) \in P_1 \times P_2} |L_1(x_i)||L_2(x_j)|$ arcs, if we count the vertices and arcs as many times they appear in different diparts of $D_1 \bigcirc D_2$.*

Proof. By definition, $D_1 \bigcirc D_2 = \{M_1(x_1) \bigcirc M_2(x_2) : (x_1, x_2) \in P_1 \times P_2\}$. The parameter set of $D_1 \bigcirc D_2$ is $P_1 \times P_2$. Consider the dipart $M_1(x_i) \bigcirc M_2(x_j)$ of $D_1 \bigcirc D_2$ corresponding to the parameter $(x_i, x_j) \in P_1 \times P_2$. The vertex set of $M_1(x_i) \bigcirc M_2(x_j)$ is $J_1(x_i) \times J_2(x_j)$ which contains $|J_1(x_i)||J_2(x_j)|$ elements. This is true for all diparts of $D_1 \bigcirc D_2$. Therefore total number of vertices in $D_1 \bigcirc D_2$ is $\sum_{(x_i,x_j) \in P_1 \times P_2} |J_1(x_i)||J_2(x_j)|$, if we count the vertices as many times they appear in different diparts of $D_1 \bigcirc D_2$. Also we know, $((v_q, v_r), (v_s, v_t))$ is an arc in $M_1(x_i) \bigcirc M_2(x_j)$ if and only if

(1) (v_q, v_s) is an arc in $M_1(x_i)$ and (v_r, v_t) is an arc in $M_2(x_j)$ or
(2) (v_q, v_s) is not an arc in $M_1(x_i)$ and (v_r, v_t) is not an arc in $M_2(x_j)$.

Now, each arc in $M_1(x_i) \bigcirc M_2(x_j)$ was made by just one of these two requirements and both of them can not be true at the same time. So to get the total number of arcs in $M_1(x_i) \bigcirc M_2(x_j)$, we add the number of arcs generated by each condition. Consider the first condition for adjacency, i.e.,(v_q, v_s) is an arc in $M_1(x_i)$ and (v_r, v_t) is an arc in $M_2(x_j)$. There are $|L_1(x_i)|$ arcs in $M_1(x_i)$ and $|L_2(x_j)|$ arcs in $M_2(x_j)$. So we can choose a pair of arcs a_k and a_l such that one is from $M_1(x_i)$ and the other is from $M_2(x_j)$ in $|L_1(x_i)||L_2(x_j)|$ different ways. Suppose that a_k is the arc (v_q, v_s) in $M_1(x_i)$ and a_l is the arc (v_r, v_t) in $M_2(x_j)$. Then this pair of arcs gives an arc $((v_q, v_r), (v_s, v_t))$ in $M_1(x_i) \bigcirc M_2(x_j)$. That is, we get $|L_1(x_i)||L_2(x_j)|$ arcs such that the first condition of adjacency is satisfied. Now consider the second condition for adjacency, i.e., (v_q, v_s) is not an arc in $M_1(x_i)$ and (v_r, v_t) is not an arc in $M_2(x_j)$. We can choose two different vertices v_q and v_s in $M_1(x_i)$ such that (v_q, v_s) is not an arc in $M_1(x_i)$ in $(|J_1(x_i)|(|J_1(x_i)| - 1) - |L_1(x_i)|)$ different ways. Similarly we can choose two different vertices v_r and v_t in $M_2(x_j)$ such that (v_r, v_t) is not an arc in $M_2(x_j)$ in $(|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)|)$ different ways. Let v_q and v_s be two vertices in $M_1(x_i)$ such that (v_q, v_s) is not an arc in $M_1(x_i)$ and let v_r and v_t be two vertices in $M_2(x_j)$ such that (v_r, v_t) is not an arc in $M_2(x_j)$. From this we get an arc $((v_q, v_r), (v_s, v_t))$ in $M_1(x_i) \bigcirc M_2(x_j)$. Hence totally the second condition for adjacency gives $(|J_1(x_i)|(|J_1(x_i)| - 1) - |L_1(x_i)|)\,(|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)|)$ arcs in $M_1(x_i) \bigcirc M_2(x_j)$. Hence the total number of arcs in $M_1(x_i) \bigcirc M_2(x_j)$ is $(|J_1(x_i)|(|J_1(x_i)| - 1) - |L_1(x_i)|)(|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)|) + |L_1(x_i)||L_2(x_j)|$. This is true for all diparts of $D_1 \bigcirc D_2$. Therefore total number of arcs in $D_1 \bigcirc D_2$ is $\sum_{(x_i,x_j) \in P_1 \times P_2}(|J_1(x_i)|(|J_1(x_i)| -$

$1)-|L_1(x_i)|)(|J_2(x_j)|(|J_2(x_j)|-1)-|L_2(x_j)|)+\sum_{(x_i,x_j)\in P_1\times P_2}|L_1(x_i)||L_2(x_j)|$, if we count the arcs as many times they appear in different diparts of $D_1 \bigcirc D_2$. □

Corollary 3.1. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and $D_1 = (D_1^*, J_1, L_1, P_1)$ and $D_2 = (D_2^*, J_2, L_2, P_2)$ be two soft directed graphs of D_1^* and D_2^* respectively. Then*

$$(i) \sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(|J_1(x_i)|(|J_1(x_i)|-1)-|L_1(x_i)|\right)\left(|J_2(x_j)|(|J_2(x_j)|-1)-|L_2(x_j)|\right)$$

$$+\sum_{(x_i,x_j)\in P_1\times P_2} |L_1(x_i)||L_2(x_j)|$$

$$(ii) \sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} deg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} 2\left(|J_1(x_i)|(|J_1(x_i)|-1)-|L_1(x_i)|\right)\left(|J_2(x_j)|(|J_2(x_j)|-1)-|L_2(x_j)|\right)$$

$$+\sum_{(x_i,x_j)\in P_1\times P_2} 2|L_1(x_i)||L_2(x_j)|,$$

where $ideg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)], odeg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)]$ and $deg(u,v)[M_{D_1\bigcirc D_2}(x_i, x_j)]$ denote the dipart in-degree, dipart out-degree and dipart degree respectively, of the vertex (u,v), in the dipart $M_{D_1\bigcirc D_2}(x_i,x_j)$ of $D_1 \bigcirc D_2$.

Proof. (i) Consider any dipart $M_{D_1\bigcirc D_2}(x_i,x_j) = (J_{D_1\bigcirc D_2}(x_i,x_j), L_{D_1\bigcirc D_2}(x_i,x_j))$ of $D_1 \bigcirc D_2$ which is given by $M_1(x_i)\times M_2(x_j)$. By *theorem* 3.2, we have number of arcs in $M_1(x_i)\times M_2(x_j)$ is $(|J_1(x_i)|(|J_1(x_i)|-1)-|L_1(x_i)|)(|J_2(x_j)|(|J_2(x_j)|-1)-|L_2(x_j)|)+|L_1(x_i)||L_2(x_j)|$. Hence, we have

$$\sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(|J_1(x_i)|(|J_1(x_i)|-1)-|L_1(x_i)|\right)\left(|J_2(x_j)|(|J_2(x_j)|-1)-|L_2(x_j)|\right)+$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} |L_1(x_i)||L_2(x_j)|,$$

since each arc in $M_{D_1\bigcirc D_2}(x_i,x_j)$ contributes 1 each to the sums $\sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)]$ and $\sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)]$.
This is true for all the diparts $M_{D_1\bigcirc D_2}(x_i,x_j)$ of $D_1 \bigcirc D_2$. Hence,

$$\sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2}\ \sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(|J_1(x_i)|(|J_1(x_i)|-1)-|L_1(x_i)|\right)\left(|J_2(x_j)|(|J_2(x_j)|-1)-|L_2(x_j)|\right)+$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} |L_1(x_i)||L_2(x_j)|.$$

(ii) Since $deg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] = ideg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] +$ $odeg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)]$ and by part (i) of this theorem we have,

$$\sum_{(x_i,x_j)\in P_1\times P_2}\ \sum_{(u,v)\in J_{D_1\bigcirc D_2}(x_i,x_j)} deg(u,v)[M_{D_1\bigcirc D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} 2\left(|J_1(x_i)|(|J_1(x_i)|-1)-|L_1(x_i)|\right)\left(|J_2(x_j)|(|J_2(x_j)|-1)-|L_2(x_j)|\right)$$

$$+\sum_{(x_i,x_j)\in P_1\times P_2} 2|L_1(x_i)||L_2(x_j)|.$$

□

4. Restricted Modular Product of Soft Directed Graphs

Definition 4.1. *Let $D^* = (V,A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be two soft directed graphs of D^* such that $P_1 \cap P_2 \neq \phi$. Then the restricted modular product of D_1 and D_2, which is represented by $D_1 \bigodot D_2$, is defined as $D_1 \bigodot D_2 = \{M_1(x) \bigcirc M_2(x) : x \in P_1 \cap P_2\}$. Here $M_1(x) \bigcirc M_2(x)$ denotes the modular product of the diparts $M_1(x)$ of D_1 and $M_2(x)$ of D_2 which is defined as follows: $M_1(x) \bigcirc M_2(x)$ is a directed graph with vertex set $V(M_1(x)\bigcirc M_2(x)) = J_1(x)\times J_2(x)$ and arc set $A(M_1(x)\bigcirc M_2(x))$, where $((v_1,v_1'),(v_2,v_2'))$ is an arc in $M_1(x) \bigcirc M_2(x)$ if and only if*

(1) *(v_1,v_2) is an arc in $M_1(x)$ and (v_1',v_2') is an arc in $M_2(x)$ or*
(2) *(v_1,v_2) is not an arc in $M_1(x)$ and (v_1',v_2') is not an arc in $M_2(x)$.*

Example 4.1. *Let $D^* = (V,A)$ be a directed graph which is shown in Fig. 6. Let*

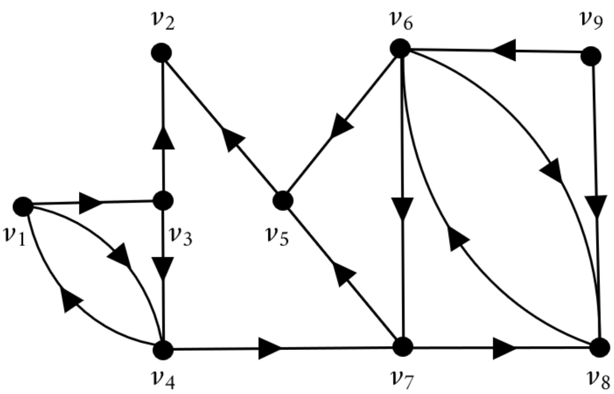


Figure 6. Directed Graph $D^* = (V,A)$

$P_1 = \{v_3, v_6\} \subseteq V$ be a set of parameters. Define a mapping $J_1 : P_1 \to \mathcal{P}(V)$ by

$J_1(x) = \{u \in V \mid u = x$ or u is adjacent from x or u is adjacent to $x\}, \forall x \in P_1$. That is, $J_1(v_3) = \{v_1, v_2, v_3, v_4\}$ and $J_1(v_6) = \{v_5, v_6, v_7, v_8, v_9\}$. Here (J_1, P_1) is a soft set over V. Define another mapping $L_1 : P_1 \to \mathcal{P}(A)$ by $L_1(x) = \{(u, v) \in A \mid \{u, v\} \subseteq J_1(x)\}, \forall x \in P_1$. That is, $L_1(v_3) = \{(v_3, v_2), (v_1, v_3), (v_1, v_4), (v_4, v_1), (v_3, v_4)\}$ and $L_1(v_6) = \{(v_6, v_5), (v_7, v_5), (v_6, v_7), (v_9, v_6), (v_7, v_8), (v_9, v_8), (v_6, v_8), (v_8, v_6)\}$. Here, (L_1, P_1) is a soft set over A. Then $M_1(v_3) = (J_1(v_3), L_1(v_3))$ and $M_1(v_6) = (J_1(v_6), L_1(v_6))$ are subdigraphs of D^* as shown in Fig. 7. Therefore $D_1 = \{M_1(v_3), M_1(v_6)\}$ is a soft directed graph of D^*.

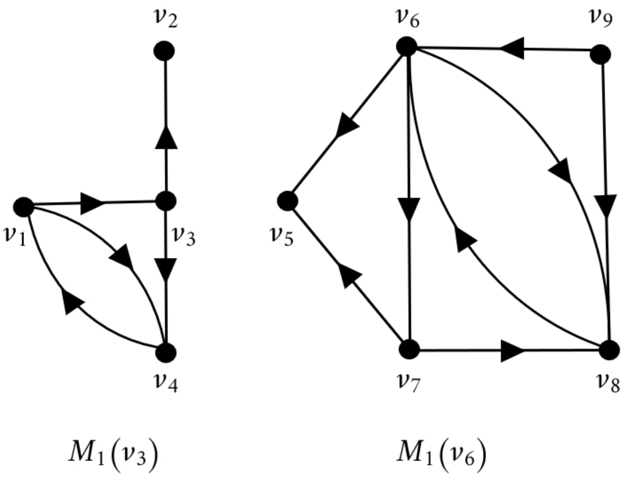


FIGURE 7. Soft Directed Graph $D_1 = \{M_1(v_3), M_1(v_6)\}$

Consider another parameter set $P_2 = \{v_3, v_9\} \subseteq V$. Define a mapping $J_2 : P_2 \to \mathcal{P}(V)$ by $J_2(x) = \{u \in V \mid u = x$ or u is adjacent from $x\}, \forall x \in P_2$. That is, $J_2(v_3) = \{v_2, v_3, v_4\}$ and $J_2(v_9) = \{v_6, v_8, v_9\}$. Here, (J_2, P_2) is a soft set over V. Define another mapping $L_2 : P_2 \to \mathcal{P}(A)$ by $L_2(x) = \{(u, v) \in A \mid \{u, v\} \subseteq J_2(x)\}, \forall x \in P_2$. That is, $L_2(v_3) = \{(v_3, v_2), (v_3, v_4)\}$ and $L_2(v_9) = \{(v_9, v_6), (v_9, v_8), (v_6, v_8), (v_8, v_6)\}$. Here, (L_2, P_2) is a soft set over A. Then, $M_2(v_3) = (J_2(v_3), L_2(v_3))$ and $M_2(v_9) = (J_2(v_9), L_2(v_9))$ are subdigraphs of D^* as shown in Fig. 8. Therefore, $D_2 = \{M_2(v_3), M_2(v_9)\}$ is a soft directed graph of D^*. Then the restricted modular product of these two soft directed graphs D_1 and D_2 is given by $D = D_1 \bigodot D_2 = \{M_1(v_3) \bigcirc M_2(v_3)\}$ and is shown in Fig. 9.

Theorem 4.1. *Let $D^* = (V, A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be two soft directed graphs of D^* such that $P_1 \cap P_2 \neq \phi$. Then the restricted modular product of D_1 and D_2, which is represented by $D_1 \bigodot D_2$ is a soft directed graph of $D^* \bigcirc D^*$.*

Proof. Let $D^* = (V, A)$ be a directed graph having vertex set V and arc set A. Also, let $D_1 = (D^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be soft directed graphs of $D^* = (V, A)$ such that $P_1 \cap P_2 \neq \phi$. Then the restricted modular product $D_1 \bigodot D_2$ is defined as $D_1 \bigodot D_2 = \{M_1(x) \bigcirc M_2(x) : x \in P_1 \cap P_2\}$. Here $M_1(x) \bigcirc M_2(x)$ denotes the modular product of the diparts $M_1(x)$ of D_1 and $M_2(x)$ of D_2 which is defined as follows: $M_1(x) \bigcirc M_2(x)$ is a directed graph with vertex set $V(M_1(x) \bigcirc M_2(x)) = J_1(x) \times J_2(x)$ and arc set $A(M_1(x) \bigcirc M_2(x))$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $M_1(x) \bigcirc M_2(x)$ if and only if

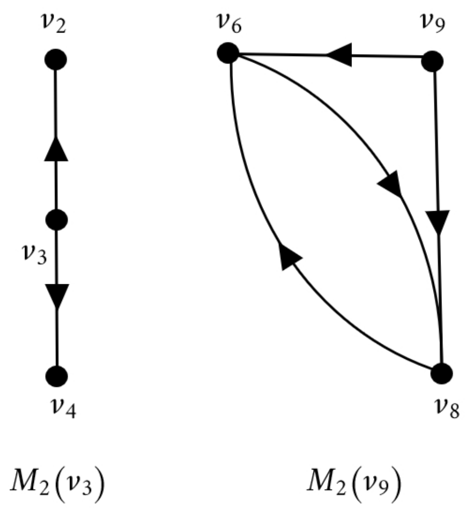


FIGURE 8. Soft Directed Graph $D_2 = \{M_2(v_3), M_2(v_9)\}$

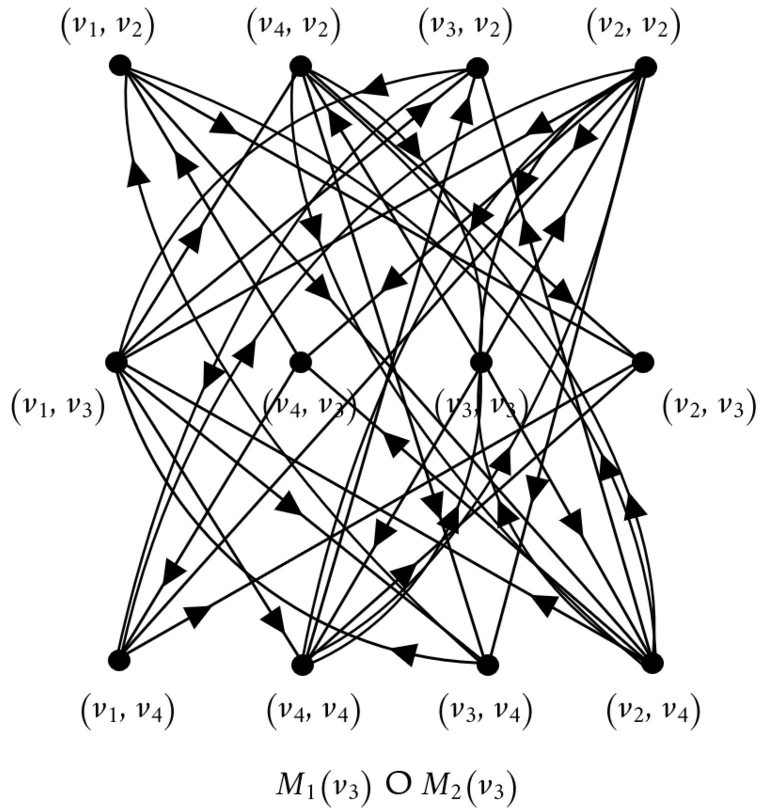


FIGURE 9. $D = D_1 \odot D_2 = \{M_1(v_3) \bigcirc M_2(v_3)\}$

(1) (v_1, v_2) is an arc in $M_1(x)$ and (v_1', v_2') is an arc in $M_2(x)$ or
(2) (v_1, v_2) is not an arc in $M_1(x)$ and (v_1', v_2') is not an arc in $M_2(x)$.

The modular product $D^* \bigcirc D^*$ is a directed graph with vertex set $V(D^* \bigcirc D^*) = V \times V$ and arc set $A(D^* \bigcirc D^*)$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $D^* \bigcirc D^*$ if and only if

(1) (v_1, v_2) as well as (v'_1, v'_2) are arcs in D^* or
(2) (v_1, v_2) as well as (v'_1, v'_2) are not arcs in D^*.

Let the parameter set be $P_{D_1 \bigodot D_2} = P_1 \cap P_2$. Define a mapping $J_{D_1 \bigodot D_2}$ from $P_{D_1 \bigodot D_2}$ to $\mathcal{P}[V(D^* \bigcirc D^*)]$ by $J_{D_1 \bigodot D_2}(x) = J_1(x) \times J_2(x), \forall x \in P_1 \cap P_2$ where $\mathcal{P}[V(D^* \bigcirc D^*)]$ denotes the power set of $V(D^* \bigcirc D^*)$. Then $(J_{D_1 \bigodot D_2}, P_{D_1 \bigodot D_2})$ is a soft set over $V(D^* \bigcirc D^*)$. Define another mapping $L_{D_1 \bigodot D_2}$ from $P_{D_1 \bigodot D_2}$ to $\mathcal{P}[A(D^* \bigcirc D^*)]$ by $L_{D_1 \bigodot D_2}(x) = \{((u, v), (y, z)) \in A(D^* \bigcirc D^*) \mid \{(u, v), (y, z)\} \in J_{D_1 \bigodot D_2}\}, \forall x \in P_1 \cap P_2$, where $\mathcal{P}[A(D^* \bigcirc D^*)]$ denotes the power set of $A(D^* \bigcirc D^*)$. Then $(L_{D_1 \bigodot D_2}, P_{D_1 \bigodot D_2})$ is a soft set over $A(D^* \bigcirc D^*)$. Also if we denote $(J_{D_1 \bigodot D_2}(x), L_{D_1 \bigodot D_2}(x))$ by $M_{D_1 \bigodot D_2}(x)$, then $M_{D_1 \bigodot D_2}(x)$ is a subdigraph of $D^* \bigcirc D^*, \forall x \in P_1 \cap P_2$, since $J_1(x) \times J_2(x) \subseteq V \times V$ and any arc in $L_{D_1 \bigodot D_2}(x)$ is also an arc in $A(D^* \bigcirc D^*)$. Then $D_1 \bigodot D_2$ can be represented by the 4-tuple $(D^* \bigcirc D^*, J_{D_1 \bigodot D_2}, L_{D_1 \bigodot D_2}, P_{D_1 \bigodot D_2})$ and also by $\{M_{D_1 \bigodot D_2}(x) : x \in P_1 \cap P_2\}$ and $D_1 \bigodot D_2$ is a soft directed graph of $D^* \bigcirc D^*$ since the following conditions are satisfied:

(1) $D^* \bigcirc D^* = (V(D^* \bigcirc D^*), A(D^* \bigcirc D^*))$ is a directed graph having vertex set $V(D^* \bigcirc D^*)$ and arc set $A(D^* \bigcirc D^*)$,
(2) $P_{D_1 \bigodot D_2} = P_1 \cap P_2$ is the set of parameters which is nonempty,
(3) $(J_{D_1 \bigodot D_2}, P_{D_1 \bigodot D_2})$ is a soft set over $V(D^* \bigcirc D^*)$,
(4) $(L_{D_1 \bigodot D_2}, P_{D_1 \bigodot D_2})$ is a soft set over $A(D^* \bigcirc D^*)$,
(5) $M_{D_1 \bigodot D_2}(x) = (J_{D_1 \bigodot D_2}(x), L_{D_1 \bigodot D_2}(x))$ is a subdigraph of $D^* \bigcirc D^*, \forall x \in P_{D_1 \bigodot D_2} = P_1 \cap P_2$.

□

Theorem 4.2. *Let $D_1^* = (V, A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1)$ and $D_2 = (D^*, J_2, L_2, P_2)$ be two soft directed graphs of D^*. Then $D_1 \bigodot D_2$ contains $\sum_{x \in P_1 \cap P_2} |J_1(x)|\,|J_2(x)|$ vertices and $\sum_{x \in P_1 \cap P_2} |L_1(x)||L_2(x)| + \sum_{x \in P_1 \cap P_2} (|J_1(x)|(|J_1(x)| - 1) - |L_1(x)|)\,(|J_2(x)|(|J_2(x)| - 1) - |L_2(x)|)$ arcs, if we count the vertices and arcs as many times they appear in different diparts of $D_1 \bigodot D_2$.*

Proof. By the definition of the restricted modular product, $D_1 \bigodot D_2 = \{M_1(x) \bigcirc M_2(x) : x \in P_1 \cap P_2\}$. The parameter set of $D_1 \bigodot D_2$ is $P_1 \cap P_2$. Consider the dipart $M_1(x) \bigcirc M_2(x)$ of $D_1 \bigodot D_2$ corresponding to the parameter $x \in P_1 \cap P_2$. The vertex set of $M_1(x) \bigcirc M_2(x)$ is $J_1(x) \times J_2(x)$ which contains $|J_1(x)||J_2(x)|$ elements. This is true for all diparts of $D_1 \bigodot D_2$. Therefore, total number of vertices in $D_1 \bigodot D_2$ is $\sum_{x \in P_1 \cap P_2} |J_1(x)||J_2(x)|$, if we count the vertices as many times they appear in different diparts of $D_1 \bigodot D_2$. Also we know, $((v_q, v_r), (v_s, v_t))$ is an arc in $M_1(x) \bigcirc M_2(x)$ if and only if

(1) (v_q, v_s) is an arc in $M_1(x)$ and (v_r, v_t) is an arc in $M_2(x)$ or
(2) (v_q, v_s) is not an arc in $M_1(x)$ and (v_r, v_t) is not an arc in $M_2(x)$.

Now, each arc in $M_1(x) \bigcirc M_2(x)$ was made by just one of these two requirements and both of them can not be true at the same time. So to get the total number of arcs in $M_1(x) \bigcirc M_2(x)$, we add the number of arcs generated by each condition. Consider the first condition for adjacency, i.e.,(v_q, v_s) is an arc in $M_1(x)$ and (v_r, v_t) is an arc in $M_2(x)$. There are $|L_1(x)|$ arcs in $M_1(x)$ and $|L_2(x)|$ arcs in $M_2(x)$. So we can choose a pair of arcs a_k and a_l such that one is from $M_1(x)$ and the other is from $M_2(x)$ in $|L_1(x)||L_2(x)|$ different ways. Suppose that a_k is the arc (v_q, v_s) in $M_1(x)$ and a_l is the arc (v_r, v_t) in $M_2(x)$. Then this pair of arcs gives an arc $((v_q, v_r), (v_s, v_t))$ in $M_1(x) \bigcirc M_2(x)$. That is, we get $|L_1(x)||L_2(x)|$ arcs such that the first condition of adjacency is satisfied. Now consider the second condition for adjacency, i.e., (v_q, v_s) is not an arc in $M_1(x)$ and (v_r, v_t) is not an arc in $M_2(x)$. We can choose two different vertices v_q and v_s in

$M_1(x)$ such that (v_q, v_s) is not an arc in $M_1(x)$ in $(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)$ different ways. Similarly we can choose two different vertices v_r and v_t in $M_2(x)$ such that (v_r, v_t) is not an arc in $M_2(x)$ in $(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)$ different ways. Let v_q and v_s be two vertices in $M_1(x)$ such that (v_q, v_s) is not an arc in $M_1(x)$ and let v_r and v_t be two vertices in $M_2(x)$ such that (v_r, v_t) is not an arc in $M_2(x)$. From this we get an arc $((v_q, v_r), (v_s, v_t))$ in $M_1(x) \bigcirc M_2(x)$. Hence totally the second condition for adjacency gives $(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)\,(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)$ arcs in $M_1(x) \bigcirc M_2(x)$. Hence the total number of arcs in $M_1(x) \bigcirc M_2(x)$ is $(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)$ $(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)+|L_1(x)||L_2(x)|$. This is true for all diparts of $D_1 \bigodot D_2$. Therefore total number of arcs in $D_1 \bigodot D_2$ is $\sum_{x\in P_1\cap P_2}(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)$ $(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)+\sum_{x\in P_1\cap P_2}|L_1(x)||L_2(x)|$, if we count the arcs as many times they appear in different diparts of $D_1 \bigodot D_2$. □

Corollary 4.1. *Let $D^* = (V, A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1)$ and $D_2 = (D^*, J_2, L_2, P_2)$ be two soft directed graphs of D^*. Then*

$$(i) \sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} ideg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} odeg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} (|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)\,(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)+$$

$$\sum_{x\in P_1\cap P_2} |L_1(x)||L_2(x)|$$

$$(ii) \sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} deg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} 2\,(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)\,(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)+$$

$$\sum_{x\in P_1\cap P_2} 2|L_1(x)||L_2(x)|,$$

where $ideg(u,v)[M_{D_1 \bigodot D_2}(x)], odeg(u,v)[M_{D_1 \bigodot D_2}(x)]$ and $deg(u,v)[M_{D_1 \bigodot D_2}(x)]$ denote the dipart in-degree, dipart out-degree and dipart degree respectively, of the vertex (u,v), in the dipart $M_{D_1 \bigodot D_2}(x)$ of $D_1 \bigodot D_2$.

Proof. (i) Consider any dipart $M_{D_1 \bigodot D_2}(x) = (J_{D_1 \bigodot D_2}(x), L_{D_1 \bigodot D_2}(x))$ of $D_1 \bigodot D_2$ which is given by $M_1(x) \times M_2(x)$. By *theorem* 4.2, we have number of arcs in $M_1(x) \times M_2(x)$ is $(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)\,(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)+|L_1(x)||L_2(x)|$. Hence, we have

$$\sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} ideg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$\sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} odeg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)\,(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|)+|L_1(x)||L_2(x)|,$$

since each arc in $M_{D_1 \bigodot D_2}(x)$ contributes 1 each to the sums $\sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} ideg(u,v)[M_{D_1 \bigodot D_2}(x)]$ and

$\sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} odeg(u,v)[M_{D_1 \bigodot D_2}(x)]$.
This is true for all the diparts $M_{D_1 \bigodot D_2}(x)$ of $D_1 \bigodot D_2$. Hence,

$$\sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} ideg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} odeg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} (|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)\,(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|) +$$

$$\sum_{x\in P_1\cap P_2} |L_1(x)||L_2(x)|.$$

(ii) Since $deg(u,v)[M_{D_1 \bigodot D_2}(x)] = ideg(u,v)[M_{D_1 \bigodot D_2}(x)] + odeg(u,v)[M_{D_1 \bigodot D_2}(x)]$ and by part (i) of this theorem we have,

$$\sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \bigodot D_2}(x)} deg(u,v)[M_{D_1 \bigodot D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} 2\,(|J_1(x)|(|J_1(x)|-1)-|L_1(x)|)\,(|J_2(x)|(|J_2(x)|-1)-|L_2(x)|) +$$

$$\sum_{x\in P_1\cap P_2} 2|L_1(x)||L_2(x)|.$$

□

5. Conclusion

Soft directed graph generates a series of representations of a relationship given by a directed graph, through parameterization. In this paper, we introduced the modular product and the restricted modular product of soft directed graphs. We proved that these products are also soft directed graphs and we developed the formulas for determining the vertex count, the arc count and the sum of indegrees, outdegrees and degrees in them.

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