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FURTHER RESULTS ON K-PRODUCT CORDIAL LABELING

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Abstract. Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0, 1, ..., k-1)$. In this paper, we investigate the k-product cordial behaviour of $G + \overline{K_t}$. In addition, we find an upper bound of the size of connected k-product cordial graphs.

Keywords: Cordial labeling, product cordial labeling, k-product cordial labeling, 4-product cordial graph.

AMS Subject Classification: 05C78.

1. Introduction

The concept of labelings of graphs has gained a lot of popularity during the last six decades due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [15] published a pioneering paper on graph labeling problems. Thereafter, many types of graph labeling techniques have been studied by several authors. Gallian [2] in his survey beautifully classified them into graceful labeling and harmonious labelings, variations of graceful labelings, variations of harmonious labelings, magic type labelings, anti-magic type labelings and miscellaneous labelings. The concept of cordial labeling was due to Cahit [1]. Sundaram et al. [16] extended the concept of cordial labeling and introduced product

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cordial labeling. Also, they proved that if G is a product cordial graph of even order then $G + K_1$ is also product cordial. Many researchers have shown interest on this topic and proved that several classes of graphs admit product cordial labeling [2].

Ponraj et al. [14] further extended the concept of product cordial labeling and introduced 'k-product cordial labeling'. They proved that k-product cordial labeling of stars and bistars further they studied the 4-product cordial labeling behavior of paths, complete graphs and combs. Javed and Jamil [4] proved that rhombic grid graphs are 3-total edge product cordial graphs. Jeyanthi and Maheswari [12] proved that if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph then $G_1 \cup G_2$ is also 3-product cordial graph and also gave the maximum number of edges in a 3-product cordial graph of order p is $\frac{p^2-3p+6}{3}$ if $p \equiv 0(mod\ 3)$, $\frac{p^2-2p+7}{3}$ if $p \equiv 1(mod\ 3)$ and $\frac{p^2-p+4}{3}$ if $p \equiv 2(mod\ 3)$. For further results on 3-product and 4-product cordial labeling an interested reader can refer to [2].

Inspired by the concept of k-product cordial labeling and the results in [14], we further studied on k-product cordial labeling and showed that the following graphs admit k-product cordial labeling: union of graphs [5]; cone and double cone graphs [6]; fan and double fan graphs [7]; powers of paths [8]; Napier bridge graphs [9]; paths [10] and product of graphs [11]. In this paper, we further investigate the k-product cordial labeling of some graphs. We organize this paper as follows: In the next section, we give the definitions and notations which are useful for the present study. In Section 3, we establish the k-product cordial labeling of $G+\overline{K_t}$ and also we find an upper bound of the size of connected k-product cordial graphs. It is followed by conclusion.

2. Definitions and Terminology

In this section, we give the definitions and terminology used in this paper. All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [3].

Definition 2.1.[1] Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Then f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1.

Definition 2.2.[16] Let f be a function from $V(G)$ to $\{0, 1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i = 0, 1)$.

Definition 2.3.[14] Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. Then f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0, 1, ..., k-1)$.

Definition 2.4.[3] The union $G_1 \cup G_2$ of two graphs G_1 and G_2 has the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

Definition 2.5.[3] The join G_1+G_2 of two graphs G_1 and G_2 has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) + E(G_2) = E(G_1) \cup E(G_2) \cup$ $\{uv : u \in V(G_1)\ and\ v \in V(G_2)\}$.

Definition 2.6.[13] The Euler's phi function $\phi(n)$ denotes the number of positive integers less than or equal to n and relatively prime to n.

Definition 2.7.[3] The complement of a graph G is a graph $\overline{G}$, whose vertex set is same as the vertex set G and two distinct vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G.

3. Main Results

Theorem 3.1. *If G is a (p,q) k-product cordial graph with $p \equiv 0(mod\ k)$ then $G+\overline{K_t}$ is a k-product cordial graph for every $1 \leq t \leq \phi(k)$, where $\phi(k)$ is Euler's ϕ function. Here, the order and size of the graph are denoted by p and q respectively. We take $p \geq 0$ and $q \geq 0$.*

Proof. Let f be a k-product cordial labeling of G. Since G has $p \equiv 0(mod\ k)$ vertices then $p = nk$. Therefore, $v_f(i) = n$ for $0 \leq i \leq k-1$ which implies that $|v_f(i) - v_f(j)| = 0$ for $0 \leq i,j \leq k-1$. Since f is a k-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1$ for $0 \leq i,j \leq k-1$.
Define a injection function $g : V(\overline{K_t}) \to \{i : i < k \text{ and } g.c.d.\ (i,k) = 1\}$.
Since $v_g(i) = 0$ *or* 1 then $|v_g(i) - v_g(j)| \leq 1$ for $0 \leq i,j \leq k-1$.
Now define a function $h : V(G+\overline{K_t}) \to \{0,1,2,...,k-1\}$ by

$$h(v) = \begin{cases} f(v) & \text{if } v \in V(G) \\ g(v) & \text{if } v \in V(\overline{K_t}). \end{cases}$$

Therefore, $v_h(i) = v_f(i) + v_g(i)$ for $0 \leq i \leq k-1$.
We have,

$$\begin{aligned} |v_h(i) - v_h(j)| &= |v_f(i) + v_g(i) - v_f(j) - v_g(j)| \\ &\leq |v_f(i) - v_f(j)| + |v_g(i) - v_g(j)| \\ &\leq 0 + 1 \\ |v_h(i) - v_h(j)| &\leq 1 \ ; \ 0 \leq i,j \leq k-1. \end{aligned}$$

Clearly, $e_h(i) = e_f(i) + tn$ for $0 \leq i \leq k-1$.
Also we have,

$$\begin{aligned} |e_h(i) - e_h(j)| &= |e_f(i) + tn - e_f(j) - tn| \\ &= |e_f(i) - e_f(j)| \\ &\leq 1 \ ; \ 0 \leq i,j \leq k-1. \end{aligned}$$

Therefore, h is a k-product cordial labeling of $G+\overline{K_t}$. Hence, $G+\overline{K_t}$ is a k-product cordial graph. □

An example of 5-product cordial labeling of $P_5+\overline{K_3}$ is shown in Figure 1.

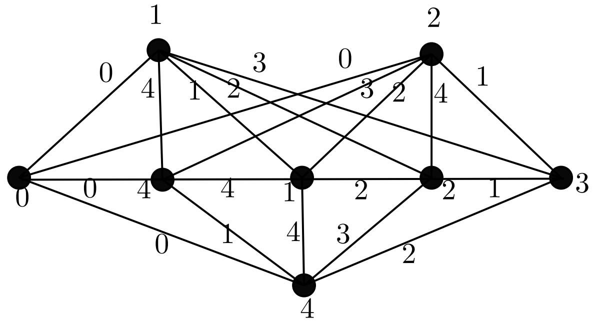


Figure 1 : $5-$ *product cordial labeling of* $P_5+\overline{K_3}$

In the following theorems, we establish an upper bound of the size of connected k-product cordial graphs.

Theorem 3.2. *If $G(p,q)$ is a 4-product cordial graph with $p \equiv 0(mod\ 4)$ then* $q \leq \frac{p(p-4)}{4} + 3$.

Proof. Let f be a 4-product cordial labeling of $G(p,q)$. Since $p \equiv 0(mod\ 4)$ then $p = 4n$. Clearly, $v_f(i) = n$ for $0 \leq i \leq 3$. Hence, $e_f(1) \leq \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n(n-1)$. Since f is a 4-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1\ for\ 0 \leq i,j \leq k-1$.
Therefore,

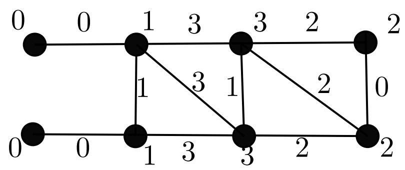
$$\begin{aligned} e_f(0) &\leq n(n-1)+1, \\ e_f(2) &\leq n(n-1)+1, \\ e_f(3) &\leq n(n-1)+1. \end{aligned}$$

We have,

$$\begin{aligned} q &= e_f(0)+e_f(1)+e_f(2)+e_f(3) \\ &\leq n(n-1)+1+n(n-1)+n(n-1)+1+n(n-1)+1 \\ &= 4n(n-1)+3 \\ &= \frac{p(p-4)}{4}+3. \end{aligned}$$

Hence, $q \leq \frac{p(p-4)}{4} + 3$. □

An example of 4-product cordial graph with $p = 8$ and $q = 11$ is shown in Figure 2.



$Figure\ 2:\ 4-product\ cordial\ labeling\ of\ G(8,11)$

Theorem 3.3. *If $G(p,q)$ is a 4-product cordial graph with $p \equiv 1(mod\ 4)$ then* $q \leq \frac{(p-1)^2}{4} + 3$.

Proof. Let f be a 4-product cordial labeling of $G(p,q)$. Since $p \equiv 1(mod\ 4)$ then $p = 4n+1$. Clearly, $v_f(i) = n\ or\ n+1$ for $0 \leq i \leq 3$. Hence, $e_f(1) \leq n^2$. Since f is a 4-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1\ for\ 0 \leq i,j \leq 3$.
Therefore,

$$\begin{aligned} e_f(0) &\leq n^2+1, \\ e_f(2) &\leq n^2+1, \\ e_f(3) &\leq n^2+1. \end{aligned}$$

We have,

$$\begin{aligned} q &= e_f(0)+e_f(1)+e_f(2)+e_f(3) \\ &\leq 4n^2+3 \\ &= \frac{(p-1)^2}{4}+3. \end{aligned}$$

Hence, $q \leq \frac{(p-1)^2}{4} + 3$. □

An example of 4-product cordial graph with $p = 5$ and $q = 7$ is shown in Figure 3.

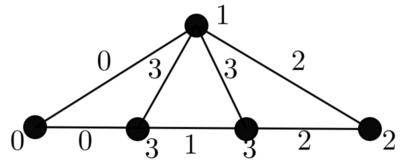


Figure 3 : $4-$*product cordial labeling of* $G(5, 7)$

Theorem 3.4. *If $G(p, q)$ is a 4-product cordial graph with $p \equiv 2(mod\ 4)$ then $q \leq \frac{p^2+8}{4}$.*

Proof. Let f be a 4-product cordial labeling of $G(p, q)$. Since $p \equiv 2(mod\ 4)$ then $p = 4n+2$. Clearly, $v_f(i) = n\ or\ n + 1$ for $0 \leq i \leq 3$. Hence, $e_f(1) \leq n(n + 1)$. Since f is a 4-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1\ for\ 0 \leq i, j \leq 3$.
Therefore,

$$\begin{aligned} e_f(0) &\leq n(n + 1) + 1, \\ e_f(2) &\leq n(n + 1) + 1, \\ e_f(3) &\leq n(n + 1) + 1. \end{aligned}$$

We have,

$$\begin{aligned} q &= e_f(0) + e_f(1) + e_f(2) + e_f(3) \\ &\leq 4n(n + 1) + 3 \\ &= \frac{p^2 + 8}{4}. \end{aligned}$$

Hence, $q \leq \frac{p^2+8}{4}$. □

An example of 4-product cordial graph with $p = 6$ and $q = 11$ is shown in Figure 4.

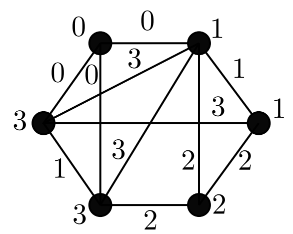


Figure 4 : $4-$*product cordial labeling of* $G(6, 11)$

Theorem 3.5. *If $G(p, q)$ is a 4-product cordial graph with $p \equiv 3(mod\ 4)$ then $q \leq \frac{(p+1)(p-3)}{4} + 3$.*

Proof. Let f be a 4-product cordial labeling of $G(p, q)$. Since $p \equiv 3(mod\ 4)$ then $p = 4n+3$. Obviously, $v_f(i) = n\ or\ n + 1$ for $0 \leq i \leq 3$. Therefore, $e_f(1) \leq n(n + 1)$. Since f is a 4-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1\ for\ 0 \leq i, j \leq 3$.
Hence,

$$\begin{aligned} e_f(0) &\leq n(n + 1) + 1, \\ e_f(2) &\leq n(n + 1) + 1, \\ e_f(3) &\leq n(n + 1) + 1. \end{aligned}$$

We have,

$$\begin{aligned} q &= e_f(0)+e_f(1)+e_f(2)+e_f(3) \\ &\leq 4n(n+1)+3 \\ &= \frac{(p+1)(p-3)}{4}+3. \end{aligned}$$

Hence, $q \leq \frac{(p+1)(p-3)}{4}+3$. $\square$

An example of 4-product cordial graph with $p=7$ and $q=11$ is shown in Figure 5.

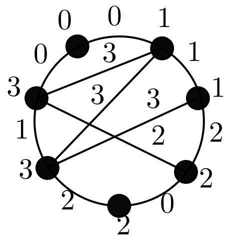


Figure $5:\ 4-product\ cordial\ labeling\ of\ G(7,11)$

The unification of an upper bound of the edges of the connected 4-product cordial graph is given in Corollary 3.1 from the results obtained in Theorems 3.2 - 3.5.

Corollary 3.1. *If $G(p,q)$ is a 4-product cordial graph then $q \leq 4\lceil\frac{p-1}{4}\rceil\lfloor\frac{p-1}{4}\rfloor+3$.*

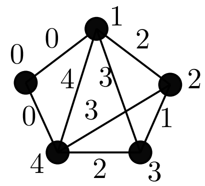
Theorem 3.6. *If $G(p,q)$ is a k-product cordial graph with $p \equiv 0(mod\ k)$ and k is a prime number then $q \leq \frac{p[(\frac{k-1}{2})p-k]}{k}+(\frac{k+1}{2})$ if $k \geq 3$ and $q \leq \frac{p(p-2)}{4}+1$ if $k=2$.*

Proof. Let f be a k-product cordial labeling of $G(p,q)$. Since $p \equiv 0(mod\ k)$ then $p=kn$. Then, $v_f(i)=n$ for $0 \leq i \leq k-1$. Let $S=\{x : j^2 \equiv x(mod\ k)\ for\ 1 \leq j \leq k-1\}$. Clearly S is a subset of $\{0,1,2,...,k-1\}$. For $k \geq 3$, $e_f(i) \leq n(n-1)+(\frac{k-3}{2})n^2$ if $i \in S$. Since f is a k-product cordial labeling then $|e_f(i)-e_f(j)| \leq 1\ for\ 0 \leq i,j \leq k-1$. Therefore, $e_f(i) \leq n(n-1)+(\frac{k-3}{2})n^2+1$ for $i \in \{0,1,2,...,k-1\} \setminus S$.
We have,

$$\begin{aligned} q &= \sum_{i=0}^{k-1} e_f(i) \\ &\leq kn(n-1)+k\left(\frac{k-3}{2}\right)n^2+\left(\frac{k+1}{2}\right) \\ &= \frac{p\left[\left(\frac{k-1}{2}\right)p-k\right]}{k}+\left(\frac{k+1}{2}\right). \end{aligned}$$

Hence, $q \leq \frac{p\left[\left(\frac{k-1}{2}\right)p-k\right]}{k}+\left(\frac{k+1}{2}\right)$.
For $k=2$, $e_f(1) \leq \frac{n(n-1)}{2}$. Since f is a 2-product cordial labeling then $e_f(0) \leq \frac{n(n-1)}{2}+1$.
Thus, $q=e_f(0)+e_f(1) \leq \frac{2n(n-1)}{2}+1=\frac{p(p-2)}{4}+1$. $\square$

An example of 4-product cordial graph with $p=5$ and $q=8$ is shown in Figure 6.



$Figure\ 6:\ 4-product\ cordial\ labeling\ of\ G(5,8)$

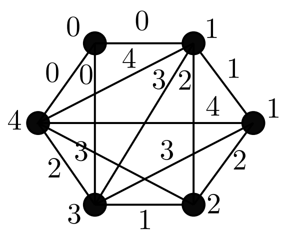
Theorem 3.7. *If $G(p,q)$ is a k-product cordial graph with $p \equiv 1(mod\ k)$ and k is a prime number then $q \leq \left(\frac{k-1}{2}\right)\frac{(p-1)^2}{k} + \left(\frac{k+1}{2}\right)$ if $k \geq 3$ and $q \leq \frac{p^2-1}{4} + 1$ if $k = 2$.*

Proof. Let f be a k-product cordial labeling of $G(p,q)$. Since $p \equiv 1(mod\ k)$ then $p = kn+1$. Clearly, $v_f(i) = n\ or\ n+1$ for $0 \leq i \leq k-1$.
Let $S = \left\{x : j^2 \equiv x(mod\ k)\ for\ 1 \leq j \leq k-1\right\}$. Clearly S is a subset of $\{0, 1, 2, ..., k-1\}$. For $k \geq 3$, $e_f(i) \leq \left(\frac{k-1}{2}\right)n^2$ if $i \in S$. Since f is a k-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1\ for\ 0 \leq i, j \leq k-1$. Therefore, $e_f(i) \leq (\frac{k-1}{2})n^2+1$ for $i \in \{0, 1, 2, ..., k-1\}\backslash S$. We have,

$$\begin{aligned} q &= \sum_{i=0}^{k-1} e_f(i) \\ &\leq k\left(\frac{k-1}{2}\right)n^2 + \left(\frac{k+1}{2}\right) \\ &= \left(\frac{k-1}{2}\right)\frac{(p-1)^2}{k} + \left(\frac{k+1}{2}\right). \end{aligned}$$

Hence, $q \leq \left(\frac{k-1}{2}\right)\frac{(p-1)^2}{k} + \left(\frac{k+1}{2}\right)$.
For $k = 2$, $e_f(1) \leq \frac{n(n+1)}{2}$. Since f is a 2-product cordial labeling then $e_f(0) \leq \frac{n(n+1)}{2} + 1$. Thus, $q = e_f(0) + e_f(1) \leq \frac{2n(n+1)}{2} + 1 = \frac{p^2-1}{4} + 1$. □

An example of 5-product cordial graph with $p = 6$ and $q = 13$ is shown in Figure 7.



$Figure\ 7:\ 5-product\ cordial\ labeling\ of\ G(6,13)$

Theorem 3.8. *If $G(p,q)$ is a k-product cordial graph with $p \equiv 2(mod\ k)$ and k is a prime number then $q \leq (p-2)\left[\left(\frac{k-1}{2}\right)\frac{(p-2)}{k} + 1\right] + \left(\frac{k+3}{2}\right)$ if $k \geq 5$ and $q \leq \frac{(p+1)(p-2)}{3} + 2$ if $k = 3$.*

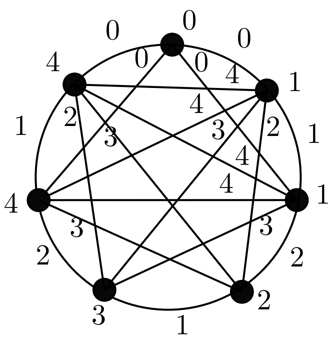
Proof. Let f be a k-product cordial labeling of $G(p,q)$. Since $p \equiv 2(mod\ k)$ then $p = kn+2$. Clearly, $v_f(i) = n\ or\ n+1$ for $0 \leq i \leq k-1$.
Let $S = \left\{x : j^2 \equiv x(mod\ k)\ for\ 1 \leq j \leq \frac{k-3}{2}\right\}$. Clearly, S is a subset of $\{0, 1, 2, ..., k-1\}$. For $k \geq 4$, $e_f(i) \leq n(n+1) + \left(\frac{k-3}{2}\right)n^2$ if $i \in S$. Since f is a k-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1\ for\ 0 \leq i, j \leq k-1$. Therefore, $e_f(i) \leq n(n+1) + (\frac{k-3}{2})n^2 + 1$

for $i \in \{0,1,2,...,k-1\} \setminus S$.
We have,

$$\begin{aligned} q &= \sum_{i=0}^{k-1} e_f(i) \\ &\leq kn(n+1) + k\left(\frac{k-3}{2}\right)n^2 + \left(\frac{k+3}{2}\right) \\ &= (p-2)\left[\left(\frac{k-1}{2}\right)\frac{(p-2)}{k} + 1\right] + \left(\frac{k+3}{2}\right). \end{aligned}$$

Hence, $q \leq (p-2)\left[\left(\frac{k-1}{2}\right)\frac{(p-1)}{k} + 1\right] + \left(\frac{k+3}{2}\right)$.
For $k = 3$, $e_f(1) \leq n(n+1)$. Since f is a 3-product cordial labeling then $e_f(0) \leq n(n+1)+1$ and $e_f(2) \leq n(n+1)+1$.
Thus, $q = e_f(0) + e_f(1) + e_f(2) \leq 3n(n+1) + 2 = \frac{(p+1)(p-2)}{3} + 2$. □

An example of 5-product cordial graph with $p = 7$ and $q = 19$ is shown in Figure 8.



$Figure\ 8:\ 5-product\ cordial\ labeling\ of\ G(7,19)$

Theorem 3.9. *If $G(p,q)$ is a k-product cordial graph with $p \equiv k-1(mod\ k)$ and k is a prime number then* $q \leq \begin{cases} \frac{p+1}{k}\left[\left(\frac{k-1}{2}\right)(p+1) - k\right] + \left(\frac{k+1}{2}\right) & \text{if } k \geq 3 \\ \frac{p^2-1}{4} + 1 & \text{if } k = 2. \end{cases}$

Proof. Let f be a k-product cordial labeling of $G(p,q)$. Since $p \equiv k-1(mod\ k)$ then $p = kn + k - 1$. Clearly, $v_f(i) = n\ or\ n+1$ for $0 \leq i \leq k-1$.
Let $S = \left\{x : j^2 \equiv x(mod\ k)\ for\ 1 \leq j \leq k-1\right\}$. Clearly S is a subset of $\{0,1,2,...,k-1\}$.
For $k \geq 3$, $e_f(i) \leq n(n+1) + \left(\frac{k-3}{2}\right)(n+1)^2$ if $i \in S$. Since f is a k-product cordial labeling then $|e_f(i) - e_f(j)| \leq 1\ for\ 0 \leq i,j \leq k-1$.
Therefore, $e_f(i) \leq n(n+1) + (\frac{k-3}{2})(n+1)^2 + 1$ for $i \in \{0,1,2,...,k-1\} \setminus S$.
We have,

$$\begin{aligned} q &= \sum_{i=0}^{k-1} e_f(i) \\ &\leq kn(n+1) + k\left(\frac{k-3}{2}\right)(n+1)^2 + \left(\frac{k+1}{2}\right) \\ &= \frac{p+1}{k}\left[\left(\frac{k-1}{2}\right)(p+1) - k\right] + \left(\frac{k+1}{2}\right). \end{aligned}$$

Hence, $q \leq \frac{p+1}{k}\left[\binom{k-1}{2}(p+1)-k\right]+\binom{k+1}{2}$.
For $k=2$, we showed that [Theorem 3.8] $q \leq \frac{p^2-1}{4}+1$. $\square$

An example of 3-product cordial graph with $p=5$ and $q=8$ is shown in Figure 9.

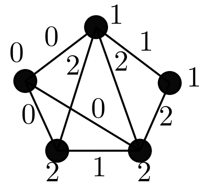


Figure 9 : $3-$*product cordial labeling of* $G(5,8)$

4. Conclusions

In this paper, we study the k-product cordial behaviour of $G+\overline{K_t}$. Further, we find an upper bound of the size of connected k-product cordial graphs. In future, we propose to find an upper bound of the size of disconnected k-product cordial graphs.

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