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EXISTENCE AND UNIQUENESS RESULTS FOR A TWO-POINT NONLINEAR BOUNDARY VALUE PROBLEM OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS OF VARIABLE ORDER

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Abstract. In this article, we study the existence and uniqueness of solutions for a two-point boundary value problem of Caputo fractional differential equation of variable order. The results are obtained by means of Banach's and Krasnoselskii's fixed point theorems. In addition, the obtained results are illustrated with the aid of a numerical example.

Keywords: Caputo derivatives and integrals of variable-order, Boundary value problems, Existence and uniqueness of solutions, Piecewise constant functions, Green's function.

AMS Subject Classification: 26A33, 34A08, 34B15.

1. Introduction

The topic of fractional calculus generalizes the integer order integration and differentiation concepts to an arbitrary (real or complex) order. The essential objective of fractional order differential conditions is to develop mathematical models that give exact portrayals of the solutions based on the information of their dynamical behaviors. It is observed that mathematical models based on fractional order derivatives are more efficient than classical integer order ones. This approach had extensive applications in the mathematical modeling of real world phenomena occurring in scientific and engineering disciplines. For example, electromagnetics, fluid mechanics, signals processing, diffusion processes, control processing, fractional stochastic systems, etc. See ([3], [4], [5], [9], and [18], [23]) and the references therein. Many studies in fractional calculus are essentially based on Riemann-Liouville and Caputo's approaches. One can refer to ([1], [21], [8], [11], and [28]).

On the other hand, the operators of variable-order, which are the derivatives and integrals whose order is a function of certain variables, attract attention due to their applied significance in various research areas where physical processes appear to exhibit fractional

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order behavior that may vary with time or space. The continuum of order in fractional calculus allows the order of the fractional operator to be considered as a variable. Some practical examples of variable order fractional differential operators are: anomalous diffusion modeling [12], mechanical applications [13], multifractional Gaussian noises [14], FIR filters [15]. In additon, a comparative study of constant-order and variable-order fractional models has been considered in [26] and [27]. For more information, see ([16], [19], and [20]).

In the following, we present some related works that were recently done.

In [18], the authors considered the initial value problem of fractional differential equationwith the Riemann-Liouvile fractional derivative. The author obtained the basic theory of the above fractional differential equations by the classical approach.

In [30] and [31] using the monotone iterative method, the author established the existence and uniqueness of solutions to initial value problems for fractional differential equation of variable-orderwith the Riemann-Liouvile fractional derivative of variable order.

In [2], by using the Banach contraction mapping principle, the authors considered the existence and uniqueness of solution for the initial value problem for fractional differential equation of variable orderith the Riemann-Liouvile fractional derivative of variable-order.

In [24], the author considered a Caputo type variable order fractional differential equation and he was able to obtain the existence–uniqueness and the Ulam–Hyers stability of a solution of the considered problemwith the caputo derivative of variable-order.

In [32], the authors introduced the concept of approximate solution to an initial value problem for differential equations of variable order involving the derivative argument on half-axiswith Riemann-Liouvile fractional derivative of variable order.

In [17], the authors studied an existence and stability criteria for a boundary value problem for Hadamard fractional differential equations of variable order, where the results are obtained based on the Kuratowski measure of noncompactness.with Hadamard fractional derivative of variable order $p(t)$.

Motivated by the recent works, we study in this paper the existence and uniqueness of solutions for the following caputo fractional nonlinear two-point boundary value problems with variable order (VOCFBVP).

$$\begin{cases} {}^{c}D_{0^+}^{\alpha(t)}x(t) = f(t, x(t), {}^{c}D_{0^+}^{\beta(t)}x(t)), \\ x(t)|_{t=0} = u_0, \\ x\left(t\right)|_{t=T} = u_T. \end{cases} \tag{1}$$

where $t \in I = [0, T]$, $0 < T < +\infty$, $D_{0^+}^{\alpha(t)}$and $D_{0^+}^{\beta(t)}$ denote Caputo fractional derivative of variable order $\alpha(t)$ and $\beta(t)$, f is a continuous function such that $f : I \times R \times R \to R$, $1 < \beta\left(t\right) < \alpha\left(t\right) < 2$ are the respective variable orders of the derivatives. The semigroup properties of the Riemann-Liouville fractional integral have played a key role in dealing with the existence of solutions to differential equations of fractional order. Based on some results of some experts, we know that the Riemann-Liouville variable order fractional integral does not have semigroup property, thus the transform between the variable order fractional integral and derivative is not clear. These judgments bring us extreme difficulties in considering the existence of solutions of variable order fractional differential equations. For more details, see [29].

The article is organized as follows: Section (1) is an introduction. In section(2), we state some notations, definitions, lemmas, and theorems that will be used throughout our work. In section(3), we prove the existence and uniqueness of mild solution for the VOCFBVP

(1) by using Banach's and Krasnoselskii's fixed point theorems. While, in section(4), a numerical example is given to to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce some notations, definitions, lemmas and theorems that are considered prerequisites for our work

Definition 2.1. [21] *Let $\alpha : R \to (0, +\infty)$, the left Riemann-Liouville fractional integral of order $\alpha(t)$ for function $x(t)$ is defined as*

$$I_{0^+}^{\alpha(t)} x(t) = \int_0^t \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))} x(s)\, ds, \text{ with } 0 < t < +\infty.$$

Definition 2.2. [21] *Let $\alpha : R \to (n-1, n]$, where n is a natural number, the left Caputo fractional derivative of order $\alpha(t)$ for function $x(t)$ is defined as*

$${}^cD_{0^+}^{\alpha(t)} x(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} x(s)\, ds, \text{ with } 0 < t < +\infty.$$

Remark 2.1. [21] *The variable-order fractional derivatives and integrals are considered as extensions of the constant order fractional derivatives and integers. That is, if $\alpha(t) = \alpha$, where α is finite positive constant real number, then $I_{0^+}^{\alpha(t)}$ and ${}^cD_{0^+}^{\alpha(t)}$ are the usual Riemann–Liouville fractional integrals and derivatives (see* [21]*). In addition, as usual, in order to study the existence of solutions of a fractional differential equation, we transform it into an equivalent integral equation using some fundamental properties of $I_{0^+}^{\alpha}$ and ${}^cD_{0^+}^{\alpha}$.*

Lemma 2.1. [21] *The fractional integral $I_{0^+}^{\alpha} x(t), 0 \le t \le +\infty$ exists almost everywhere.*

Lemma 2.2. [21] *If $1 < \beta < \alpha < 2$, and $x \in L(0, b)$ with $0 < b < +\infty$, then the semigroup property for the Riemann–Liouville fractional integrals hold, i.e., $I_{0^+}^{\alpha} I_{0^+}^{\beta} x(t) = I_{0^+}^{\beta} I_{0^+}^{\alpha} x(t) = I_{0^+}^{\alpha+\beta} x(t), 0 \le t \le +\infty$.*

Lemma 2.3. [21] *If $1 < \alpha < 2$, and $x \in L(0, b)$ with $0 < b < +\infty$, then ${}^cD_{0^+}^{\alpha} I_{0^+}^{\alpha} x(t) = x(t), 0 \le t \le +\infty$.*

Lemma 2.4. [21] *The differential equation ${}^cD_{0^+}^{\alpha} x(t) = 0$ has unique solution*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $n - 1 < \alpha \le n$, $c_i \in R, i = 0, 1, 2, ..., n-1$.
In addition,

$$I_{0^+}^{\alpha}\, {}^cD_{0^+}^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}.$$

Remark 2.2. [31] *The Riemann–Liouville variable order fractional integrals don't satisfy the semigroup property, i.e.,*

$$I_{0^+}^{\alpha(t)} I_{0^+}^{\beta(t)} x(t) \ne I_{0^+}^{\alpha(t)+\beta(t)} x(t), 0 \le t \le +\infty.$$

Thus, we obtain that there are extreme difficulties to consider the existence of solutions of solutions of differential equations with fractional variable order derivatives as in those of fixed order derivatives by means of nonlinear fractional analysis. This implies that we can not transform a differential equation with variable order fractional derivatives into an equivalent integral equation without using the above lemmas.

Definition 2.3. [31] *A generalized interval is a subset I of R which is either an interval , a point, or the empty set.*

Definition 2.4. [31] *If I is a generalized interval, then set P is a partition of I if P is a finite of generalized intervals contained in I , such that every x in I lies in exactly one of the generalized intervals J in P.*

Definition 2.5. [31] *Let I be a generalized interval, let $f : I \to R$ be a function, and let P be a partition of I. Then, f is said to be piecewise constant with respect to P if for every $J \in P$, f is constant on J.*

Definition 2.6. [31] *Let I be a generalized interval. The function $f : I \to R$ is called piecewise constant on I , if there exists a partition P of I such that f is piecewise constant with respect to P.*

Definition 2.7. [7] *Let $(X, \|.\|)$ be a Banach space. A mapping $\wp : X \to X$ is called a contraction on X if there exists a positive constant $K < 1$ such that*

$$\|\wp(x) - \wp(y)\| \leqslant K\|x - y\|, \qquad \text{for all } x,\, y \in X.$$

Theorem 2.1. [7]*(Banach's Fixed Point Theorem) Let $(X, \|.\|)$ be a Banach space and let $\wp : X \to X$ be a contraction on X. Then $\wp$ has a unique fixed point $x \in X$ (i.e. $\wp(x) = x$).*

Definition 2.8. *Denote by $C(J, R)$ the Banach space of continuous functions $\wp : J \to R$ with the norm $\|\wp\| = \sup\{|\wp(t)|;\ t \in J\}$.*

3. Main Results

Throughout this paper, we consider the following assumptions:

(A_1) $f : I \times R^2 \to R$ is continuous and there exists $\psi \in C(I, R^+)$, with norm $\|\psi\|$, such that:

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le \psi(t)(|u_1 - v_1| + |u_2 - v_2|),$$

$\forall\, t \in I,\ u_i, v_i \in R,\ (i = 1, 2).$

(A_2) If $\alpha : [0, T] \to (1, 2]$ and $\beta : [0, T] \to (1, 2]$ are piecewise constant functions with partition $P = \{[0, T_1], [T_1, T_2], [T_2, T_3], ..., [T_{N^*-1}, T_{N^*}]\}$ (N^* is a given natural number) of the finite interval $[0, T]$, then

$$\alpha\left(t\right) = \sum_{k=1}^{N^*} \alpha_k I_k\left(t\right), t \in \left[0, T\right],$$

$$\beta\left(t\right) = \sum_{k=1}^{N^*} \beta_k I_k\left(t\right), t \in \left[0, T\right],$$

where $1 < \alpha_k; \beta_k < 2, k = 1, 2, 3, ..., N^*$, $I_k\left(t\right)$ is the indicator of the interval $[T_{k-1}, T_k]$ for $k = 1, 2, 3, ..., N^*$, where $T_0 = 0$ and $T_{N^*} = T$, i.e.,

$$I_k\left(t\right) = \begin{cases} 1 \text{ for } t \in \left[T_{k-1}, T_k\right], \\ 0 \qquad\quad \text{elsewhere.} \end{cases}$$

(A_3) For $0 \le r \le \alpha_i, i = 1, 2, ..., N^*$, let $t^r f : [0, T] \times R^2 \to R$ be a continuous function, and there exists a positive constant L such that

$$\frac{\left(\frac{T_i^{\alpha_i}}{T_i\Gamma(\alpha_i+1)} + G_o T_i\right)\|\psi\|}{\left(1 - \frac{\|\psi\| T_1^{\alpha_i-\beta_i}}{\Gamma(\alpha_i-\beta_i+1)}\right)} < 1,$$

and

$$t^r|f(t, x(t), {}^c D_{0^+}^{\beta(t)} x(t)) - f(t, y(t), {}^c D_{0^+}^{\beta(t)} y(t))| \le L|x\left(t\right) - y\left(t\right)|, 0 \le t \le T, x, y \in R.$$

Remark 3.1. *From assumption* (A_1), *we have*

$$|f(t,u_1,u_2)| - |f(t,0,0)| \leq |f(t,u_1,u_2) - f(t,0,0)| \leq \psi(t)(|u_1| + |u_2|),$$

and

$$|f(t,u_1,u_2)| \leq \|\psi\|(|u_1| + |u_2|) + F, \text{ where } F = \sup_{t\in I} |f(t,0,0)|.$$

Now, in order to study the existence of solutions for (VOCFBVP) (1), we have to make the following essential analysis.

By assumption (A_2), we have

$$\int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} x(s)\, ds = \sum_{k=1}^{N^*} I_k(t) \int_0^t \frac{(t-s)^{-\alpha_k}}{\Gamma(1-\alpha_k)} x(s)\, ds, \; t \in [0,T].$$

Hence, (VOCFBVP) (1) is equivalent to

$$\frac{d}{dt}\sum_{k=1}^{N^*} I_k(t) \int_0^t \frac{(t-s)^{-\alpha_k}}{\Gamma(1-\alpha_k)} x(s)\, ds = f(t, x(t), \frac{d}{dt}\sum_{k=1}^{N^*} I_k(t) \int_0^t \frac{(t-s)^{-\beta_k}}{\Gamma(1-\beta_k)} x(s)\, ds), \; t \in [0,T]. \tag{2}$$

Now, Equation (2) in the interval $[0,T_1]$ is written as:

$$\frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha_1}}{\Gamma(1-\alpha_1)} x(s)\, ds = {}^cD_{0^+}^{\alpha_1} x(t) = f(t, x(t), {}^cD_{0^+}^{\beta_1} x(t)), \; t \in [0,T_1]. \tag{3}$$

Again, Equation (2) in the interval $(T_1,T_2]$ is written as:

$$\frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} x(s)\, ds = {}^cD_{0^+}^{\alpha_2} x(t) = f(t, x(t), {}^cD_{0^+}^{\beta_2} x(t)), \; t \in (T_1,T_2]. \tag{4}$$

If we complete in the following manner, we obtain that Equation (2) in the interval $(T_{i-1},T_i], i = 1,2,3,...,N^*$ $(T_0 = 0, T_{N^*} = T)$ can be written as:

$$\frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} x(s)\, ds = {}^cD_{0^+}^{\alpha_i} x(t) = f(t, x(t), {}^cD_{0^+}^{\beta_i} x(t)), \; t \in (T_{i-1},T_i]. \tag{5}$$

Remark 3.2. *By the above argument, we can say that (VOCFBVP) (1) has a (unique) solution if there exist (unique) functions* $x_i(t), i = 1,2,3,...,N^*$, *such that* $x_1 \in C[0,T_1]$ *that satisfies Equation (3) with* $x_1(t)|_{t=0} = u_0$ *and* $x_1(T)|_{t=T_1} = u_T$, $x_2 \in C[0,T_2]$ *that satisfies Equation (4) with* $x_2(t)|_{t=0} = u_0$ *and* $x_2(T)|_{t=T_2} = u_T$, ..., *and* $x_i \in C[0,T_i]$ *that satisfies Equation (5) with* $x_i(t)|_{t=0} = u_0$ *and* $x_i(T)|_{t=T_i} = u_T$ *for all* $i = 3,4,...,N^*$ *with* $T_{N^*} = T$.

Lemma 3.1. *The solution of Equation (3) with* $x(t)|_{t=0} = u_0$ *and* $x(T)|_{t=T_1} = u_T$ *is the solution of the integral equation*

$$x(t) = h(t, x(t)) + \int_0^{T_1} G_1(t,s) u(s) ds, \tag{6}$$

where u *is the solution of the fractional order integral equation*

$$u(t) = f(t, h(t,x(t)) + \int_0^{T_1} G_1(t,s)u(s)ds, I^{\alpha_1-\beta_1}u(t)), \tag{7}$$

and $G(t,s)$ *is the Green's function described by*

$$G_1(t,s) = \begin{cases} \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} - \frac{t(T_1-s)^{\alpha_1-1}}{T_1\Gamma(\alpha_1)}, & 0 \leq s \leq t \leq T_1, \\ -\frac{t(T_1-s)^{\alpha_1-1}}{T_1\Gamma(\alpha_1)}, & 0 \leq t \leq s \leq T_1, \end{cases} \tag{8}$$

such that

$$G_\circ := \max\{|G_1(t,s)|, (t,s) \in I_1 \times I_1\}, \ \textit{with } I_1 = [0, T_1]\,,$$

and

$$h(t) = u_0 + (u_T - u_0)\,\frac{t}{T_1}. \tag{9}$$

Proof. Let $x\,(t)$ be a solution of equation (3), then by applying the property that

$${}^{c}D_{0^+}^{\beta_1} x\,(t) = I^{\alpha_1-\beta_1}\ {}^{c}D_{0^+}^{\alpha_1} x\,(t) \quad \text{for } t \in [0, T_1]\,,$$

we obtain that

$${}^{c}D_{0^+}^{\alpha_1} x\,(t) = f(t,\ x(t),\ I^{\alpha_1-\beta_1}\ {}^{c}D_{0^+}^{\alpha_1} u\,(t)),\ t \in [0, T_1]\,,$$

where $u\,(t) = {}^{c}D_{0^+}^{\alpha_1} x\,(t)$. Hence,

$$u\,(t) = f(t,\ x(t),\ I^{\alpha_1-\beta_1}\ {}^{c}D_{0^+}^{\alpha_1} u\,(t)),\ t \in [0, T_1]\,,$$

But, by Lemma (2.4), we get

$$x\,(t) = c_0 + c_1 t + \frac{1}{\Gamma\,(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1}\, u(s) ds.$$

Substituting the boundary conditions $x(t)|_{t=0} = u_0$ and $x\,(T)\,|_{t=T_1} = u_T$, we get

$$c_0 = u_0,$$

and

$$c_1 = \frac{(u_T - u_0)}{T_1} - \frac{1}{T_1 \Gamma\,(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1-1}\, u(s) ds.$$

Hence, the solution of equation (3) can be written as:

$$x\,(t) = u_0 + (u_T - u_0)\,\frac{t}{T_1} - \frac{t}{T_1 \Gamma\,(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1-1}\, u(s) ds + \frac{1}{\Gamma\,(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1}\, u(s) ds.$$

Consequently, we obtain equation (7) from the fact that $\int_0^{T_1} = \int_0^t + \int_t^{T_1}$. ☐

The following result is based on Banach's fixed point Theorem to obtain the existence of a unique solution of the (VOCFBVP) (1).

Theorem 3.1. *Suppose that assumptions $(A_1) - (A_3)$ hold, then (VOCFBVP) (1) has a unique solution.*

Proof. According to the above argument, (VOCFBVP) (1) can be written as equation (2). Now, by Lemma (3.1), equation (2) can be written in the interval $[0, T_1]$ as:

$$x\,(t) = h_1\,(t, x\,(t)) + \int_0^{T_1} G_1(t,s) u(s) ds,\ 0 \le t \le T_1.$$

Define operator $T : C\,[0, T_1] \to C\,[0, T_1]$ by

$$Tx\,(t) = h_1\,(t, x\,(t)) + \int_0^{T_1} G_1(t,s) u(s) ds,\ 0 \le t \le T_1.$$

In fact, $Tx\,(t) \in C\,[0, T_1]$, since $x\,(t) \in C\,[0, T_1]$. Let

$$g(t,\ x(t),\ {}^{c}D_{0^+}^{\beta_1} x\,(t)) = t^r f(t,\ x(t),\ {}^{c}D_{0^+}^{\beta_1} x\,(t)),$$

then by assumption (A_3), we have function $g : [0, T_1] \times R^2 \to R$ is continuous. Thus, for $t, t_0 \in [0, T_1]$ we have

$$\begin{aligned}
& \left|Tx(t) - Tx(t_0)\right| \\
= \ & |h_1(t, x(t)) + \int_0^{T_1} G_1(t,s)u(s)ds - h(t_0, x(t_0)) - \int_0^{T_1} G_1(t_0,s)u(s)ds|, \\
\leq \ & |h_1(t, x(t)) - h_1(t_0, x(t_0))| + |\int_0^{T_1} |G_1(t,s) - G_1(t_0,s)|\ u(s)ds, \\
\leq \ & |\frac{(u_T - u_0)}{T_1} - \frac{1}{T_1\Gamma(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1 - 1} u(s)ds|\ |t - t_0| \\
& + |\frac{1}{\Gamma(\alpha_1)} \int_0^{t} (t - s)^{\alpha_1 - 1} f(s, x(s), {}^c D_{0^+}^{\beta_1} x(s))ds \\
& - \frac{1}{\Gamma(\alpha_1)} \int_0^{t_0} (t_0 - s)^{\alpha_1 - 1} f(s, x(s), {}^c D_{0^+}^{\beta_1} x(s))ds|, \\
\leq \ & |\frac{(u_T - u_0)}{T_1} - \frac{1}{T_1\Gamma(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1 - 1} u(s)ds|\ |t - t_0| \\
& + |\frac{t^{\alpha_1 - r}}{\Gamma(\alpha_1)} \int_0^{1} (1 - \tau)^{\alpha_1 - 1} \tau^{-r} g(t\tau, x(t\tau), {}^c D_{0^+}^{\beta_1} x(t\tau))d\tau \\
& - \frac{t_0^{\alpha_1 - r}}{\Gamma(\alpha_1)} \int_0^{1} (1 - \tau)^{\alpha_1 - 1} \tau^{-r} g(t_0\tau, x(t_0\tau), {}^c D_{0^+}^{\beta_1} x(t_0\tau))d\tau|, \\
\leq \ & |\frac{(u_T - u_0)}{T_1} - \frac{1}{T_1\Gamma(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1 - 1} u(s)ds|\ |t - t_0| \\
& + \frac{|t^{\alpha_1 - r} - t_0^{\alpha_1 - r}|}{\Gamma(\alpha_1)} \int_0^{1} (1 - \tau)^{\alpha_1 - 1} \tau^{-r}\ |g(t\tau, x(t\tau), {}^c D_{0^+}^{\beta_1} x(t\tau))|\ d\tau + \frac{t_0^{\alpha_1 - r}}{\Gamma(\alpha_1)} \\
& \int_0^{1} (1 - \tau)^{\alpha_1 - 1} \tau^{-r}\ |g(t\tau, x(t\tau), {}^c D_{0^+}^{\beta_1} x(t\tau)) - g(t_0\tau, x(t_0\tau), {}^c D_{0^+}^{\beta_1} x(t_0\tau))|\ d\tau.
\end{aligned}$$

By the continuity of $t^{\alpha_1 - r}$ and g, we obtain that $Tx(t) \in C[0, T_1]$.

In addition, if $x(t), y(t) \in C[0, T_1]$, we have

$$\begin{aligned}
& |Tx(t) - Ty(t)| \\
& = |h_1(t, x(t)) + \int_0^{T_1} G_1(t,s)u(s)ds - h_1(t, y(t)) - \int_0^{T_1} G_1(t,s)v(s)ds|, \\
& \leq |h_1(t, y(t)) - h_1(t, x(t))| + \int_0^{T_1} |G_1(t,s)|\ |u(s) - v(s)|ds, \\
& \leq |\frac{1}{T_1\Gamma(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1 - 1} v(s)ds - \frac{1}{T_1\Gamma(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1 - 1} u(s)ds| \\
& + \int_0^{T_1} |G_1(t,s)|\ |u(s) - v(s)|ds, \\
& \leq \frac{1}{T_1\Gamma(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1 - 1} |v(s) - u(s)|ds + \int_0^{T_1} |G_1(t,s)|\ |u(s) - v(s)|ds.
\end{aligned}$$

But, by assumption (A_1) and if we take supremum for $t \in [0, T_1]$, we get

$$\begin{aligned} |u(s) - v(s)| &= |f(t, x(t), {}^c D_{0^+}^{\beta_1} x(t)) - f(t, y(t), {}^c D_{0^+}^{\beta_1} y(t))|, \\ &= |f(t, x(t), I_{0^+}^{\alpha_1-\beta_1} u(t)) - f(t, y(t), I_{0^+}^{\alpha_1-\beta_1} v(t))|, \\ &\leq \psi(t) \left(|x(t) - y(t)| + \int_0^t \frac{(t-s)^{\alpha_1-\beta_1-1}}{\Gamma(\alpha_1-\beta_1)} |u(s) - v(s)| ds \right), \\ &\leq \|\psi\| \left(\|x - y\| + \frac{T_1^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)} \|u - v\| \right). \end{aligned}$$

Hence,

$$\|u - v\| \leq \frac{\|\psi\|}{\left(1 - \frac{\|\psi\| T_1^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)} \|x - y\|.$$

Thus,

$$|Tx(t) - Ty(t)| \leq \frac{\left(\frac{T_1^{\alpha_1}}{T_1\Gamma(\alpha_1+1)} + G_o T_1\right)\|\psi\|}{\left(1 - \frac{\|\psi\| T_1^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)} \|x - y\|. \tag{10}$$

Since, $\frac{\left(\frac{T_1^{\alpha_1}}{T_1\Gamma(\alpha_1+1)} + G_o T_1\right)\|\psi\|}{\left(1 - \frac{\|\psi\| T_1^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)} < 1$, then by Banach's contraction principle, we obtain that operator T has unique fixed point $x_1(t) \in C[0, T_1]$ such that $x_1(0) = u_0$ and $x_1(T_1) = u_T$. Therefore, $x_1(t)$ is a unique solution of equation (3) with the boundary conditions $x_1(t)|_{t=0} = u_0$ and $x_1(t)|_{t=T_1} = u_T$.

In addition, equation (2) in the interval $(T_1, T_2]$ can be written as equation (4). So, in order to consider the existence result of solution to equation (4), we have to discuss its existence in the $(0, T_2]$.

Consider the following equation

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} x(s)\, ds = {}^c D_{0^+}^{\alpha_1} x(t) = f(t, x(t), {}^c D_{0^+}^{\beta_2} x(t)),\ t \in (0, T_2]. \tag{11}$$

It is clear that if $x \in C[0, T_2]$ satisfies equation (11), then it also satisfies equation (4). Hence, let $x^* \in C[0, T_2]$ be a solution of equation (11) such that $x^*(t)|_{t=0} = u_0$ and $x^*(t)|_{t=T_2} = u_T$. That is,

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} x^*(s)\, ds = f(t, x^*(t), {}^c D_{0^+}^{\beta_2} x^*(t)),\ t \in (T_1, T_2].$$

Hence, we deduce that if $x^* \in C[0, T_2]$ is a solution of equation (11) then $x^* \in C(T_1, T_2]$ is also a solution of equation (4).

Based on this result, we will consider the existence of solution of equation (11) instead of equation (4).

In a similar manner as above, if we take the operator $I_{0^+}^{\alpha_2}$ on both sides of equation (11) and use Lemma (2.4), we get

$$x(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} f(t, x(t), {}^c D_{0^+}^{\beta_2} x(t)) ds,\ 0 \leq t \leq T_2.$$

Substituting the boundary conditions $x(t)|_{t=0} = u_0$ and $x(T)|_{t=T_2} = u_T$, we get $c_0 = u_0$, and $c_1 = \frac{(u_T - u_0)}{T_2} - \frac{1}{T_2\Gamma(\alpha_1)} \int_0^{T_2} (T_2 - s)^{\alpha_2-1} u(s) ds$.Hence, the solution of equation (11) can be written as:

$$\begin{aligned} x\left(t\right)= & \ u_0+\left(u_T-u_0\right)\tfrac{t}{T_2}-\tfrac{t}{T_2\Gamma(\alpha_2)}\int_0^{T_2}\left(T_2-s\right)^{\alpha_2-1}u(s)ds \\ & +\tfrac{1}{\Gamma(\alpha_2)}\int_0^{t}\left(t-s\right)^{\alpha_2-1}u(s)ds \\ & =h_2\left(t,x\left(t\right)\right)+\int_0^{T_2}G_2(t,s)u(s)ds. \end{aligned}$$

Define operator $T:C\left[0,T_2\right]\to C\left[0,T_2\right]$ by

$$Tx\left(t\right)=h_2\left(t,x\left(t\right)\right)+\int_0^{T_2}G_2(t,s)u(s)ds,\ 0\leq t\leq T_2.$$

In a similar argument as above, it follows from the continuity of the function

$$g(t,\ x(t),\ {}^cD_{0^+}^{\beta_2}x\left(t\right))=t^rf(t,\ x(t),\ {}^cD_{0^+}^{\beta_2}x\left(t\right))$$

that the operator $T:C\left[0,T_2\right]\to C\left[0,T_2\right]$ is continuous and well defined. In addition, if $u(t),v(t)\in C[0,T_2]$, we have

$$\begin{aligned} &|Tu(t)-Tv\left(t\right)|=|h_2\left(t,y\left(t\right)\right)-h_2\left(t,x\left(t\right)\right)|+|\int_0^{T_2}G_2(t,s)u(s)ds-\int_0^{T_2}G_2(t,s)v(s)ds|, \\ &\leq\frac{1}{T_2\Gamma\left(\alpha_2\right)}\int_0^{T_2}\left(T_2-s\right)^{\alpha_2-1}|v(s)-u\left(s\right)|ds+|\int_0^{t}\frac{(t-s)^{\alpha_2-\beta_2-1}}{\Gamma(\alpha_2-\beta_2)}|u\left(s\right)-v\left(s\right)|ds)|, \\ &\leq\frac{\left(\frac{T_2^{\alpha_2}}{T_2\Gamma(\alpha_2+1)}+G_oT_2\right)\|\psi\|}{\left(1-\frac{\|\psi\|T_2^{\alpha_2-\beta_2}}{\Gamma(\alpha_2-\beta_2+1)}\right)}\left\|u-v\right\|. \end{aligned}$$

Hence, by assumption (A_3), we obtain that Banach's contraction principle assures that the operator T has unique fixed point $x_2\left(t\right)\in C\left[0,T_2\right]$ such that $x_2\left(0\right)=u_0$ and $x_2\left(T_2\right)=u_T$. Therefore, $x_2\left(t\right)$ is a unique solution of equation (4) with the boundary conditions $x_2\left(t\right)|_{t=0}=u_0$ and $x_1\left(t\right)|_{t=T_1}=u_T$.

In a similar manner, we can prove that equation (2) defined on $(T_{i-1},T_i]$, for all $i=3,4,...,N^*$ with $T_{N^*}=T$, has one unique solution $x_i\left(t\right)\in C\left[0,T_i\right]$ such that $x_i\left(t\right)|_{t=0}=u_0$ and $x_i\left(t\right)|_{t=T_i}=u_T$.

Therefore, we already proved that (VOCFBVP) (1) has one unique solution. Thus, the proof is completed. □

Now, we present our second existence result for the (VOCFBVP) (1) which is based on Krasnoselskii's fixed point Theorem (see [9]).

Definition 3.1. *By a mild solution of (VOCFBVP) (1), we refer to a function* $u\in C(I_1,R)$, $I_1=[0,T_1]$, *that satisfies the integral equation (7), with* u *the solution of the following integral equation*

$$u(t)=f(t,\ h\left(t,x\left(t\right)\right)+\int_0^{T_1}G_1(t,s)u(s)ds,\ I^{\alpha_1-\beta_1}u(t)),$$

for all $t\in[0,T_1]$.

Lemma 3.2. *Let For arbitrary* $x\left(t\right),y\left(t\right)\in C\left[0,T_1\right]$, *and let* $1<\beta_1<2$, *then*

$$|{}^cD_{0^+}^{\beta_1}y\left(t\right)-\ {}^cD_{0^+}^{\beta_1}x\left(t\right)|\leq\frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|}\left\|y-x\right\|$$

Proof. It is clear that if we take supremum for all $t \in [0, T_1]$ that

$$\begin{aligned} |{}^cD^{\beta_1}_{0^+} y\,(t) - {}^cD^{\beta_1}_{0^+} x\,(t)\,| &= |\frac{d}{dt}\int_0^t \frac{(t-s)^{-\beta_1}}{\Gamma(1-\beta_1)} y\,(s)\,ds - \frac{d}{dt}\int_0^t \frac{(t-s)^{-\beta_1}}{\Gamma(1-\beta_1)} x\,(s)\,ds| \\ &\le \frac{d}{dt}\int_0^t \frac{(t-s)^{-\beta_1}}{\Gamma(1-\beta_1)} \,|y\,(s) - x\,(s)|\,ds \\ &\le \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|}\,\|y - x\| \end{aligned}$$

□

Lemma 3.3. *The function $h : I \times R \to R$, is Lipschitzian function with a Lipschitz constant c such that*

$$\|h(t, x\,(t)) - h(t, y\,(t))\| \le c\,\|x - y\|.$$

Proof. Let $x\,(t)\,, y\,(t) \in C\,[0, T_1]$. By applying assumption (A_1) and taking supremum for all $t \in [0, T_1]$, we get

$$\begin{aligned} &|h(t, x(t)) - h(t, y(t))| \\ &= \Big|\frac{1}{T_1\Gamma\,(\alpha_1)}\int_0^{T_1} (T_1 - s)^{\alpha_1 - 1}\,v(s)ds - \frac{1}{T_1\Gamma\,(\alpha_1)}\int_0^{T_1} (T_1 - s)^{\alpha_1 - 1}\,u(s)ds\Big| \\ &\le \frac{1}{T_1\Gamma\,(\alpha_1)}\int_0^{T_1} (T_1 - s)^{\alpha_1 - 1}\,|v(s) - u\,(s)\,|ds \\ &\le \frac{1}{T_1\Gamma\,(\alpha_1)}\int_0^{T_1} (T_1 - s)^{\alpha_1 - 1}\,|f(t, y(t), {}^cD^{\beta_1}_{0^+} y\,(t)) - f(t, x(t), {}^cD^{\beta_1}_{0^+} x\,(t))|ds \\ &\le \frac{1}{T_1\Gamma\,(\alpha_1)}\int_0^{T_1} (T_1 - s)^{\alpha_1 - 1}\,|\psi(t)|(|y\,(t) - x\,(t)\,| + |{}^cD^{\beta_1}_{0^+} y\,(t) - {}^cD^{\beta_1}_{0^+} x\,(t)\,|)ds \\ &\le \frac{\|\psi\|\,T_1^{\alpha_1}}{T_1\Gamma\,(\alpha_1+1)}\left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|}\right)\|x - y\|. \end{aligned}$$

Thus

$$\|h(t, x) - h(t, y)\| \le c\,\|x - y\|,$$

where $c = \frac{\|\psi\| T_1^{\alpha_1}}{T_1\Gamma(\alpha_1+1)}\left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|}\right)$. □

Theorem 3.2. *Suppose that assumptions (A_1) holds. If*

$$\frac{\|\psi\|\,T_1^{\alpha_1}}{T_1\Gamma\,(\alpha_1+1)}\left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|}\right) < 1 \tag{12}$$

then the (VOCFBVP) (1) has at least one mild solution in $C\,[0, T_1]$.

Proof. By converting (VOCFBVP) (1) into a fixed point problem. Define the operator $A : C(I_1, R) \to C(I_1, R)$, $I_1 = [0, T_1]$ by:

$$Ax(t) = h(t, x(t)) + \int_0^{T_1} G_1(t, s)u(s)ds,\ t \in [0, T_1]\,, \tag{13}$$

with

$$u(t) = f(t,\ x(t),\ I^{\alpha_1 - \beta_1}\ u\,(t)).$$

Let $B_{\varrho_1} = \{x \in C(I_1, R) : \|x\| \leq \varrho_1\}$ be a closed subset of $C[0, T_1]$, where ϱ_1 is a positive constant satisfying

$$\varrho_1 \geq \max_{t\in[0,T_1]} \left(\frac{\Im + \Re F}{1 - \aleph}, \frac{F G_o T_1}{1 - \|\psi\| G_0 T_1 \left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|}\right)} \right),$$

where

$$\Im = 2u_0 + u_T,$$

$$\aleph = \|\psi\| \left(\frac{T_1^{\alpha_1}}{T_1 \Gamma(\alpha_1 + 1)} + G_o T_1 \right) \left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1 - \beta_1)|} \right),$$

and

$$\Re = \left(\frac{T_1^{\alpha_1}}{T_1 \Gamma(\alpha_1 + 1)} + G_o T_1 \right).$$

Obviously, $B_{\varrho_1}$is a Banach space with metric in $C[0, T_1]$.

Now, consider the operators A_1 and A_2 on B_{ϱ_1} as

$$A_1 x(t) = h(t, x(t))$$

$$A_2 x(t) = \int_0^{T_1} G_1(t, s) u(s) ds.$$

Then, for any $x \in C([0, T_1], R)$ we have

$$Ax(t) = A_1 x(t) + A_2 x(t), \ t \in [0, T_1].$$

The proof is divided into three steps.

Step 1: $A_1 x_1 + A_2 x_2 \in B_{\varrho_1}$ for every $x_1, x_2 \in B_{\varrho_1}$

Let $x_1, x_2 \in B_{\varrho_1}$ and $t \in I$, we have

$$\begin{aligned} &|A_1 x_1(t) + A_2 x_2(t)| \\ &\leq |A_1 x_1(t)| + |A_2 x_2(t)| \\ &\leq |h(t, x_1(t))| + \int_0^{T_1} |G_1(t, s)| |f(t,\ x(t),\ {}^c D_{0^+}^{\beta_1} x(t))| ds \\ &\leq |u_0| + |u_T - u_0| \frac{t}{T_1} + \frac{1}{T_1 \Gamma(\alpha_1)} \int_0^{T_1} (T_1 - s)^{\alpha_1 - 1} |u(s)| ds + \int_0^T |G_1(t, s)| |u(s)| ds. \end{aligned} \tag{14}$$

By Lemma 3.2 and taking supremum for $t \in [0, T_1]$, we have

$$\begin{aligned} |u(t)| &= |f(t,\ x(t),\ {}^c D_{0^+}^{\beta_1} x(t))| \\ &\leq \|\psi\| (|x(t)| + |{}^c D_{0^+}^{\beta_1} x(t))|) + F, \text{ where } F = \sup_{t\in I} |f(t, 0, 0)|. \\ &\leq \|\psi\| \left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1 - \beta_1)|} \right) \|x\| + F. \end{aligned}$$

Thus, for each $t \in [0, T_1]$

$$\begin{aligned} |A_1 x_1(t) + A_2 x_2(t)| &\leq 2u_0 + u_T \\ &+ \left(\frac{T_1^{\alpha_1}}{T_1 \Gamma(\alpha_1 + 1)} + G_o T_1 \right) \left(\|\psi\| \left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1 - \beta_1)|} \right) \|x\| + F \right). \end{aligned}$$

Taking supremum over $t \in [0, T_1]$, we have

$$\|A_1x_1 + A_2x_2\| \leq \ \varrho_1$$

for $\varrho_1 \geq \frac{\Im+\Re F}{1-\aleph}$, where$\Im = 2u_0 + u_T$, $\aleph = \|\psi\| \left(\frac{T_1^{\alpha_1}}{T_1\Gamma(\alpha_1+1)} + G_oT_1 \right) \left(1 + \frac{T_1^{-\beta_1}}{\Gamma(1-\beta_1)} \right)$, and $\Re = \left(\frac{T_1^{\alpha_1}}{T_1\Gamma(\alpha_1+1)} + G_oT_1 \right)$. This proves that $A_1x_1 + A_2x_2 \in B_{\varrho_1}$ for every $x_1, x_2 \in B_{\varrho_1}$.

Step 2: The operator A_1 is a contraction mapping on B_{ϱ_1}.
It is clear that from Lemma 3.3, A_1 is a contraction mapping for

$$c = \frac{\|\psi\| \, T_1^{\alpha_1}}{T_1\Gamma\left(\alpha_1 + 1\right)} \left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|} \right) < 1.$$

Step 3: The operator A_2 is completely continuous (compact and continuous) on B_{ϱ_1}.
First, we prove that operator A_2 is continuous.
Let $\{x_n\}_{n\in N}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $C([0, T_1]\,, R)$.
To show that A_2 is continuous, we have to prove that

$$\|A_2x_n - A_2x\| \to 0 \text{ as } n \to \infty.$$

Then for each $t \in [0, T_1]$, we have

$$|A_2x_n - A_2x| \leq \int_0^T |G_1(t, s)||u_n(s) - u(s)|ds,$$

where $u_n, u \in C([0, T_1]\,, R)$, such that

$$\begin{aligned} u_n(t) &= f(t,\ x_n(t),\ {}^cD_{0^+}^{\beta_1} x_n\left(t\right)) = f(t, x_n(t), I^{\alpha_1-\beta_1}\ u_n(t)), \\ u(t) &= f(t,\ x(t),\ {}^cD_{0^+}^{\beta_1} x\left(t\right)) = f(t, x(t), I^{\alpha_1-\beta_1}\ u(t)), \end{aligned}$$

and by (A_1), we have

$$\begin{aligned} |u_n(t) - u(t)| &= |f(t, x_n(t), {}^c D_{0^+}^{\beta_1} x_n\left(t\right)) - f(t, x(t), {}^c D_{0^+}^{\beta_1} x\left(t\right))| \\ &\leq |\psi(t)|(|x_n(t) - x(t)| + |\ {}^cD_{0^+}^{\beta_1} x_n\left(t\right) -\ {}^cD_{0^+}^{\beta_1} x\left(t\right)|) \\ &\leq \|\psi\|(\|x_n - x\| + \frac{T_1^{-\beta_1}}{\Gamma(1-\beta_1)}\|x_n - x\|). \\ &\leq \|\psi\|(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|})\|x_n - x\|. \end{aligned}$$

Thus, if we take the supremum for $t \in [0, T_1]$, we get

$$\|u_n - u\| \leq \|\psi\|(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|})\|x_n - x\|.$$

Since $x_n \to x$, then we get $u_n(t) \to u(t)$ as $n \to \infty$ for each $t \in [0, T_1]$. And let $\varepsilon > 0$ be such that, for each $t \in [0, T_1]$, we have $|u_n(t)| \leq \varepsilon/2$, and $|u(t)| \leq \varepsilon/2$. Then, we have

$$\begin{aligned} |G_1(t, s)||u_n(s) - u(s)| &\leq |G_1(t, s)|(|u_n(s)| + |u(s)|) \\ &\leq \varepsilon|G_1(t, s)| \end{aligned}$$

For each $t \in [0, T_1]$, the function $s \to \varepsilon|G_1(t, s)|$ is integrable on I. Then applying Lebesgue Dominated Convergence Theorem, it implies that

$$\|A_2x_n - A_2x\| \to 0 \ as \ n \to \infty.$$

Consequently, A_2 is continuous.
In addition, we have

$$\|A_2 x\| \leq G_o T_1 \left[\|\psi\| \left(1 + \frac{T_1^{-\beta_1}}{|\Gamma(1-\beta_1)|} \right) \varrho_1 + F \right] \leq \varrho_1$$

due to definitions of ϱ_1. This proves that A_2 is uniformly bounded on B_{ϱ_1}.

Finally, we prove that A_2 maps bounded sets into equicontinuous sets of $C(I, R)$, i.e., B_{ϱ_1} is equicontinuous.

Now, Let $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ and $t_1, t_2 \in I,\ t_1 < t_2,\ |t_2 - t_1| < \delta$. Then we have

$$\begin{aligned}
|A_2 x(t_2) - A_2 x(t_1)| &\leq \int_0^{T_1} |G_1(t_2, s) - G_1(t_1, s)|\ |u(s)| ds \\
&\leq \|u\| \int_0^{T_1} |G_1(t_2, s) - G_1(t_1, s)|\ ds \\
&\leq \left[\|\psi\| \left(1 + \frac{T_1^{-\beta_1}}{\Gamma(1-\beta_1)} \right) \varrho_1 + F \right] \int_0^{T} |G(t_2, s) - G(t_1, s)|\ ds.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality is not dependent on x and tends to zero. Consequently,

$$|A_2 x(t_2) - A_2 x(t_1)| \to 0,\ \forall\ |t_2 - t_1| \to 0.$$

Thus, $\{Ax\}$ is equi-continuous on B_{ϱ_1}. and A is compact operator by the Arzela-Ascoli Theorem [10], we conclude that $A : C([0, T_1], R) \to C([0, T_1], R)$ is continuous and compact. Hence, all the hypotheses of Krasnoselskiiâ's fixed point theorem are satisfied and hence the operator $A = A_1 + A_2$ has a fixed point $x_1(t) \in C[0, T_1]$ on B_{ϱ_1} with $x_1(0) = u_0$ and $x_1(T_1) = u_T$. Therefore, $x_1(t)$ is a mild solution of equation (3) with the boundary conditions $x_1(t)|_{t=0} = u_0$ and $x_1(t)|_{t=T_1} = u_T$.

Now, if we make the same argument done in Theorem (3.1), we have equation (2) in the interval $(T_1, T_2]$ is equivalent equation (4). So, considering the existence results of solution for equation (4) is equivalent to discussing its existence in the $(0, T_2]$.

Consider the following equation. In addition, it is clear that if $x_2 \in C[0, T_2]$ satisfies equation (11), then it also satisfies equation (4) such that $x_2(t)|_{t=0} = u_0$ and $x_2(t)|_{t=T_2} = u_T$.

By the similar way, for $i = 3, ..., N^*$, we could get that equation (4) defined on $(T_i - 1, T_i]$ has at least one mild solution $x_i(t) \in C[0, T_i]$ with $x_i(t)|_{t=0} = u_0$ and $x_i(t)|_{t=Ti} = u_T$ ($T_{N^*} = T$). Therefore, by applying Krasnoselskii's fixed point Theorem, we obtain that (VOCFBVP) (1) has at least one mild solution in $C[0, T]$. The proof is completed. □

Remark 3.3. *We studied the existence and uniqueness of the proposed model. A natural (but involved, see, e.g.,* [24]*) problem would be to study the stability of our proposed model in the sense of the Ulam-Hyers stability (see e.g.* [6]*). This will be investigated elsewhere.*

4. Numerical Example

Given the following VOCFBVP:

$$\begin{cases} {}^c D^{\alpha(t)} y(t) = \frac{\sqrt{2t+1}}{69 e^{2t+1}} \left[\frac{5 + y(t) + {}^c D^{\beta(t)} y(t)}{1 + y(t) + {}^c D^{\beta(t)} y(t)} \right] \text{ for all } t \in [0, 3], \\ y(0) = 1, \quad \text{and } y(3) = 1, \end{cases} \tag{15}$$

where

$$\alpha(t) = \begin{cases} \frac{7}{5} \text{ if } 0 \le t \le 1 \\ \frac{6}{5} \text{ if } 1 < t \le 2 \\ \frac{9}{5} \text{ if } 2 < t \le 3 \end{cases}, \tag{16}$$

and

$$\beta(t) = \begin{cases} \frac{6}{5} \text{ if } 0 \le t \le 1 \\ \frac{11}{10} \text{ if } 1 < t \le 2 \\ \frac{8}{5} \text{ if } 2 < t \le 3 \end{cases}, \tag{17a}$$

with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$, and $T_3 = 3$.

It is obvious that

$$f(t,u,v) = \frac{\sqrt{2t+1}}{69e^{2t+1}} \left[\frac{5+|u|+|v|}{1+|u|+|v|} \right].$$

is a mutually continuous function. Besides, for any $u_1, v_1, u_2, v_2 \in R$, and $t \in [0,T]$ we have

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le \frac{1}{69e} \left(|u_1 - u_2| + |v_1 - v_2| \right).$$

Thus,

$$|f(t,u,v)| = \frac{\sqrt{2t+1}}{69e^{2t+1}} \left(3 + |u| + |v|\right), \text{ with } F = \frac{5}{69e}, \text{ and } \|\psi\| = \frac{1}{69e}.$$

Hence, assumptions $(A_1) - (A_3)$ are satisfied with

$$\psi(t) = \frac{\sqrt{2t+1}}{69e^{2t+1}}, \text{ and } \|\psi\| = \frac{1}{69e}.$$

By (16) and (17a), we consider three BVPs for caputo fractional differential equations of constant order

$$\begin{cases} {}^cD^{\frac{7}{5}} y(t) = \frac{\sqrt{2t+1}}{69e^{2t+1}} \left[\frac{5+y(t)+\ {}^cD^{\frac{6}{5}} y(t)}{1+y(t)+\ {}^cD^{\frac{6}{5}} y(t)} \right] \text{ for all } t \in [0,1], \\ y(0) = 1, \quad \text{and } y(1) = 1, \end{cases} \tag{18a}$$

$$\begin{cases} {}^cD^{\frac{6}{5}} y(t) = \frac{\sqrt{2t+1}}{69e^{2t+1}} \left[\frac{5+y(t)+\ {}^cD^{\frac{11}{10}} y(t)}{1+y(t)+\ {}^cD^{\frac{11}{10}} y(t)} \right] \text{ for all } t \in (1,2], \\ y(1) = 1, \quad \text{and } y(2) = 1, \end{cases} \tag{19}$$

$$\begin{cases} {}^cD^{\frac{9}{5}} y(t) = \frac{\sqrt{2t+1}}{69e^{2t+1}} \left[\frac{5+y(t)+\ {}^cD^{\frac{8}{5}} y(t)}{1+y(t)+\ {}^cD^{\frac{8}{5}} y(t)} \right] \text{ for all } t \in (2,3], \\ y(2) = 1, \quad \text{and } y(3) = 1, \end{cases} \tag{20a}$$

Condition (12) is satisfied on $[0,1]$ since

$$\frac{\|\psi\| T_1^{\alpha_1}}{T_1 \Gamma(\alpha_1 + 1)} \left(1 + \frac{T_1^{-\beta_1}}{\Gamma(1-\beta_1)} \right) = \frac{\frac{1}{69e}}{\Gamma\left(\frac{12}{5}\right)} \left(1 + \frac{1}{\Gamma\left(\frac{-1}{5}\right)} \right) \approx 0.0408804 < 1,$$

which implies that BVP (18a) has at least one mild solution $x_1 \in C[0,1]$.

In addition, we have $G_o < 0.5$ and condition (10) is satisfied on $[0,1]$ since

$$\frac{\left(\frac{T_1^{\alpha_1 - 1}}{\Gamma(\alpha_1+1)} + G_o T_1 \right) \|\psi\|}{\left(1 - \frac{\|\psi\| T_1^{\alpha_1 - \beta_1}}{\Gamma(\alpha_1 - \beta_1 + 1)} \right)} = \frac{\frac{1}{69e} \left(\frac{1}{\Gamma\left(\frac{12}{5}\right)} + 0.5 \right)}{\left(1 - \frac{\frac{1}{69e}}{\Gamma\left(\frac{-42}{5}\right)} \right)} \approx 0.0857421 < 1,$$

which implies that BVP (18a) one unique solution $x_1 \in C[0,1]$.

Also, condition (12) is satisfied on $(1,2]$ since

$$\frac{\|\psi\|\, T_2^{\alpha_2}}{T_2\Gamma\left(\alpha_2+1\right)}\left(1+\frac{T_2^{-\beta_2}}{\Gamma(1-\beta_2)}\right)=\frac{\frac{1}{69e}2^{\frac{6}{5}}}{2\Gamma\left(\frac{11}{5}\right)}\left(1+\frac{2^{-\frac{11}{10}}}{\Gamma\left(\frac{-1}{10}\right)}\right)\approx 0.0611323<1,$$

which implies that BVP (19) has at least one mild solution $x_2\in C(1,2]$.

In addition, we have $G_o<0.5$ and condition (10) is satisfied on $(1,2]$ since

$$\frac{\left(\frac{T_2^{\alpha_2-1}}{\Gamma(\alpha_2+1)}+G_oT_2\right)\|\psi\|}{\left(1-\frac{\|\psi\|T_2^{\alpha_2-\beta_2}}{\Gamma(\alpha_2-\beta_2+1)}\right)}=\frac{\frac{1}{69e}\left(\frac{2^{\frac{1}{5}}}{\Gamma(\frac{11}{5})}+1\right)}{\left(1-\frac{\frac{1}{69e}2^{\frac{1}{10}}}{\Gamma(\frac{11}{10})}\right)}\approx 0.134529<1,$$

which implies that BVP (18a) one unique solution $x_2\in C(1,2]$.

Finally, condition (12) is satisfied on $(2,3]$ since

$$\frac{\|\psi\|\, T_3^{\alpha_3}}{T_3\Gamma\left(\alpha_3+1\right)}\left(1+\frac{T_3^{-\beta_3}}{\Gamma(1-\beta_3)}\right)=\frac{\frac{1}{69e}3^{\frac{9}{5}}}{3\Gamma\left(\frac{14}{5}\right)}\left(1+\frac{3^{-\frac{8}{5}}}{\Gamma\left(\frac{-3}{5}\right)}\right)\approx 0.0839666<1,$$

which implies that BVP (20a) has at least one mild solution $x_2\in C(2,3]$.

In addition, we have $G_o<1.5$ and condition (10) is satisfied on $(1,2]$ since

$$\frac{\left(\frac{T_3^{\alpha_3-1}}{\Gamma(\alpha_3+1)}+G_oT_3\right)\|\psi\|}{\left(1-\frac{\|\psi\|T_3^{\alpha_3-\beta_3}}{\Gamma(\alpha_3-\beta_3+1)}\right)}=\frac{\frac{1}{69e}\left(\frac{3^{\frac{4}{5}}}{\Gamma(\frac{14}{5})}+4.5\right)}{\left(1-\frac{\frac{1}{69e}3^{\frac{1}{5}}}{\Gamma(\frac{6}{5})}\right)}\approx 0.39701<1,$$

which implies that BVP (18a) one unique solution $x_3\in C(2,3]$.

It follows from Theorems (3.2) and (3.1) that problem (15) has a unique mild solution $x\left(t\right)\in C\left[0,3\right]$ such that

$$x\left(t\right)=\begin{cases} x_1\left(t\right),\ t\in[0,1],\\ x_2\left(t\right),\ t\in(1,2],\\ x_3\left(t\right),\ t\in(2,3].\end{cases}$$

5. Conclusion

In this work, we proved the existence and uniqueness of solutions for a two-point boundary value problem of Caputo fractional differential equation of variable order. These results are investigated by means of Banach's and Krasnoselskii's fixed point theorems. Furthermore, we gave a numerical example that confirm the obtained theoretical results. In the future, we will consider the existence and uniqueness of solutions for a two-point boundary value problem of singular fractional differential equation of variable order.

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