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BIPOLAR INTUITIONISTIC FUZZY MATRICES AND ITS DETERMINANT

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Abstract. The theory of fuzzy sets has earned a lot of applications in science and engineering fields. Fuzzy sets include extensions such as intuitionistic and bipolar fuzzy sets. These concepts have now become significant in recent research projects. The union of these sets can be defined as a bipolar intuitionistic fuzzy set that offers more flexibility in analyzing real-life problems. It effectively analyses the systems by examining the involvement and non-involvement grades of the element in a bipolar view. A matrix is a collection of crisp numbers arranged in a rectangular array with rows and columns. The concept of the fuzzy matrix can be defined when a precise solution cannot be obtained from the crisp matrix. The theory of fuzzy matrix plays a vital role in the field of decision-making systems. The fuzzy matrix offers a clear outcome, when examined with a bipolar intuitionistic fuzzy environment. Therefore, the present study coined the notion of bipolar intuitionistic fuzzy matrix and its determinant. To depict the flexibility, its three-dimensional representations are visualized. Also, the fundamental operations such as addition, multiplication, max-min and min-max compositions are defined with illustrations. Moreover, some examples and properties are provided to support the proposed study.

Keywords: Bipolar intuitionistic fuzzy matrix, Bipolar intuitionistic fuzzy determinant, Max-min, Min-max

AMS Subject Classification: 03E72, 15B15

1. Introduction

A collection of numbers arranged in a rectangular array with rows and columns is called a matrix. It has earned numerous applications in various domains, such as computer science, economic sciences, game theory, graph theory, etc. The determinant of a matrix is a scalar value that permits characterizing some properties of the matrix and the linear map represented by the matrix. Also, the matrix and its determinant can be utilized to solve the system of linear equations. When the crisp matrices failed to provide precise information, the concept of fuzzy matrices was defined. The elements of a fuzzy matrix can

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be represented as the membership value of the row with respect to the column. When the involvement and the negative view of the elements of a row with respect to the column were considered, the notions of intuitionistic fuzzy matrices (IFM) and bipolar fuzzy matrices (BFM) were innovated. To extend the flexibility of these matrices, the concept of bipolar intuitionistic fuzzy matrix (BIFM) can be defined. It offers a clear view of the elements by investigating their involvement and the non-involvement grades in a bipolar view.

In 1965, Zadeh [30] coined the theory of fuzzy sets to handle uncertain situations in mathematical problems. It has earned numerous applications in different fields like social sciences, environmental sciences, engineering and economics, etc., which cannot be dealt with the classical mathematical methods. A fuzzy set was represented by the collection of ordered pairs of elements and their belongingness (μ) degree that lies in the unit interval. The theory of intuitionistic fuzzy sets (IFS) [1] was introduced when the grade of non-belongingness (γ) of an element is considered, where (γ) and the sum $\mu + \gamma$ also lies between the unit interval.

Human perception always has positive and negative sides. Zhang [31] considered this logic and designed the view of bipolar fuzzy sets (BFS). It is defined as the collection of elements and their positive (μ^+), negative (μ^-) belongingness degree, which lie in $[0, 1]$ and $[-1, 0]$, respectively. The BFS helps to handle real-world problems with positive and negative perceptions, which has an extra possibility in analyzing the elements. The BFS are frequently applied in various disciplines such as artificial intelligence, computer science, psychology, etc. Especially the BFS have utilized in all decision-making systems as it gives a flexible solution by bipolar analysis. Recently, Deva [3] applied the BFS to design a novel decision-making technique, namely bipolar fuzzy DEMATEL method. Jana [10] used the BFS under soft environment to develop an aggregation technique for a decision-making system. To study both the non-belonging and negative perspective of the element, Ezhilmaran [8] defined the concept of bipolar intuitionistic fuzzy set (BIFS) by integrating the ideas of IFS and BFS. Moreover, he defined the bipolar intuitionistic fuzzy graph to study their relations. The BIFS can be represented as the collection of elements and their positive (μ^+), negative (μ^-) belongingness grades and positive (γ^+), negative (γ^-) non-belongingness grades. Here, the positive grades (μ^+) and (γ^+) lie in the interval $[0, 1]$, the negative degrees (μ^-), (γ^-) lie in the interval $[-1, 0]$. Devi [6] offered the decision support system with a remarkable application for COVID-19 lockdown relaxation to support the result under the system of IFS. As an extension of the system of IFS, Jana [12] provided the Pythagorean fuzzy system-based multiple-attribute decision-making system. To display the flexible outcome of the BIFS, Jana [11] discussed an application under the soft environment. With the context of BIFS, Deva [5] designed the competition graph of BIFG to investigate the competition among the vertices with flexible information. When the flow between the vertices exceeds the maximum value in BIFG, the view of a bipolar intuitionistic anti fuzzy graph is offered by Deva [4].

The theory of matrices is a robust tool in many sciences and engineering domains. Sometimes, crisp data cannot give precise information while dealing with a real-world problem. Therefore, Thomson [29] delivered the concept of fuzzy matrices to deal the uncertain issues and examine the convergence power of fuzzy matrices. After that, Kim [14] developed the idea of the fuzzy matrix as an extension of the Boolean matrix and discussed some results on the fuzzy matrix. In 1989, the first theory of fuzzy square matrix (FSM) and its determinant was provided by Kim [13]. Ragab [25] defined the notion of circular fuzzy matrix and showed few properties of the fuzzy square matrix. Hemasinha [9] derived the theory of determinants in a fuzzy matrix based on t-norms and t-conorms instead of multiplication and addition. Shyamal [28] showed the values of the fuzzy matrix in a three-dimensional view and defined new operations in the fuzzy matrix. Otadi [18] solved the fuzzy matrix equation with the aid of an embedding system and discussed some exciting results. The view of determinant and adjoint of the FSM is provided by Dhar [7]. Also, he studied the notion of fuzzy matrices using reference functions. In the fuzzy matrix, the rows and columns are considered accurate values. Often, it may be vague naturally. Therefore, the rows and columns were considered by the fuzzy terms by Pal [22]. Raich [26] discussed the theory of fuzzy matrix and its application to recognize an effective teacher for the development of educational institutions.

The non-belongingness degrees were considered by Pal [21] to define the concept of IFM. Also, the notion of the determinant of IFM was defined by Pal [23]. Some results on IFM and circulant IFM were discussed by Bhowmik [2]. Mondal [16] coined the notion of similarity relations and eigenvalues to analyze certain results on IFM. Pradhan [24] analyzed the idea of Atanassov's IFM to define the concept of generalized inverse on it and provided the idea of intuitionistic fuzzy vectors. The concept of incline IFM and its determinant are examined by Mondal [15]. To analyze the theory of determinants in IFM, Padder [19] attempted the determinant concept for IFM. Shyamal [27] stretched the fuzzy matrices into interval-valued fuzzy matrices by extending the max-min operation on fuzzy algebra. Mondal [16] defined eigenvectors and similarity relations of BFM to construct the theory of bipolar fuzzy linear space. Recently, the concept of the fuzzy matrix was extended to the BFM to analyze the attribute and alternatives with positive and negative degrees by Pal [20].

This review shows that the concept of fuzzy matrix and its extensions are widely applied in various domains. Also, it helps in dealing with the matrix approach when the matrix elements are vague. It is observed that the concepts of IFM and BFM are essential topics of current research aspects. Moreover, they provide more flexibility than other existing approaches in analyzing the elements when examined from intuitionistic and bipolar fuzzy perspectives. Therefore, the present study revealed the notion of BIFM and its properties. Also, an illustration discussed the view of the determinant of the BIFM. In addition, the 3D representations of the BIFM are depicted in this study.

1.1. **Motivation of the study.** The motivations of the proposed study are listed below

(i) The concept of fuzzy matrices was defined when the element involved in the unit interval [0, 1]. It has earned a wide variety of applications in this modern world.
(ii) The notions of IFM and BFM were defined as an extension of the fuzzy matrix. They provide effective outcomes with their different kind of grades.
(iii) The concept of BIFS was defined as a combination of IFS and BFS. It gives precise results in analyzing the problems with a positive and negative perception of the membership and non-membership degrees.
(iv) The matrices can be offered clear information when it is viewed by a bipolar intuitionistic fuzzy perspective.
(v) The BIFM has more flexibility in examining the elements of the matrix than other approaches.

These hints are motivated in designing the study of the BIFM.

1.2. **Objective of the study.** The main objectives of the present work are provided below

(i) To extend the view of fuzzy matrices by introducing the notion of bipolar intuitionistic fuzzy matrices

(ii) To display the extra possibilities of BIFM by depicting the diagrammatic representation of BIFM in a three-dimensional view.
(iii) To support the theoretical approaches of the BIFM by discussing some fundamental operations of the BIFM
(iv) To examine the convergence power of the BIFM by defining the concept of the determinant of the BIFM
(v) To strengthen the properties of BIFM by providing some results on the BIFM and its determinant.
(vi) To expose the flexibility of the BIFM by discussing some examples numerically and diagrammatically.

1.3. **Novelty of the present work.** The novelties of the proposed work are listed below
(i) The notion of BIFM is defined to expose the extra chances of BIFM in analyzing the elements more than other matrices.
(ii) The structures of BIFM are portrayed in a three-dimensional view. It shows the extra possibilities of the BIFM by investigating both belongingness and non-belongingness degrees with a bipolar perspective.
(iii) The matrix operations such as addition, multiplication, max-min and min-max compositions are derived from developing the theoretical approaches of the BIFM.
(iv) Illustrated the numerical and diagrammatical examples to display the flexibility of BIFM.
(v) The concept of the matrix determinant for BIFM is discussed to reveal the convergence power of the BIFM.
(vi) Some of the matrix properties are provided that satisfy the conditions for BIFM. It shows the applicability of the proposed study.

These are revealed to be the novelties of the present study.

The setting of the paper is designed as follows: In section 2, the fundamental notions are discussed. In section 3, the idea of BIFM and its numerical and diagrammatical illustrations are offered. In section 4, the concept of the determinant for the BIFM with illustrations is examined, and some properties are also derived. Finally, the conclusion of the work is provided in section 5.

2. Preliminaries

In this part, some basic notions are discussed.

Definition 2.1. [1] *An IFS $\tilde{A}$ is defined as $\tilde{A} = \{(m, \mu(m), \gamma(m)) | m \in X\}$, where $\mu(m)$, $\gamma(m) : X \to [0,1]$, $\mu(m)$ and $\gamma(m)$ are the function of belongingness and non-belongingness such that $0 \le \mu(m) + \gamma(m) \le 1$.*

Definition 2.2. [21] *Let $\tilde{A}_1 = \left\{\mu_{\tilde{A}_1}(m_1), \gamma_{\tilde{A}_1}(m_1)\right\}$ and $\tilde{A}_2 = \left\{\mu_{\tilde{A}_2}(m_2), \gamma_{\tilde{A}_2}(m_2)\right\}$ be two IFS. Then $c_1 \vee c_2$ and $c_1 \wedge c_2$ are expressed as follows*

$$c_1 \vee c_2 = \left\{max\left(\mu_{\tilde{A}_1}(m_1), \mu_{\tilde{A}_2}(m_2)\right), min\left(\gamma_{\tilde{A}_1}(m_1), \gamma_{\tilde{A}_2}(m_2)\right)\right\}$$
$$c_1 \wedge c_2 = \left\{min\left(\mu_{\tilde{A}_1}(m_1), \mu_{\tilde{A}_2}(m_2)\right), max\left(\gamma_{\tilde{A}_1}(m_1), \gamma_{\tilde{A}_2}(m_2)\right)\right\}.$$

Definition 2.3. [21] *Let $X = \{s_1, s_2, \ldots, s_m\}$ be a set of alternatives and $Y = \{t_1, t_2, \ldots, t_n\}$ be a set of attributes of each elements in X. An IFM of order $m \times n$ is defined as $\tilde{A} = \left\{(s_i, t_j), \mu_{\tilde{A}}(s_i, t_j), \gamma_{\tilde{A}}(s_i, t_j) \,|\, (s_i, t_j) \in X \times Y\right\}$ for $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$, where $\mu_{\tilde{A}}(s_i, t_j), \gamma_{\tilde{A}}(s_i, t_j) : X \times Y \to [0,1]$ such that $0 \le \mu_{\tilde{A}}(s_i, t_j) + \gamma_{\tilde{A}}(s_i, t_j) \le 1$.*

Definition 2.4. [19] *Let $\tilde{A}$ be an IFM of order $n \times n$. Then determinant of $\tilde{A}$ is defined as*

$$|\tilde{A}|_{n\times n} = \left[\left(\bigvee_{\sigma\in S_n}\left(s_{1\sigma(1)} \wedge s_{2\sigma(2)} \wedge \ldots \wedge s_{n\sigma(n)}\right), \bigwedge_{\sigma\in S_n}\left(t_{1\sigma(1)} \vee t_{2\sigma(2)} \vee \ldots \vee t_{n\sigma(n)}\right)\right)\right]$$

where S_n represents the set of all permutations of the indices $\{1, 2, \ldots n\}$.

Definition 2.5. [8] *A BIFS is defined as $\widetilde{D} = \{(m, \mu^+_{\widetilde{D}}(m), \mu^-_{\widetilde{D}}(m), \gamma^+_{\widetilde{D}}(m), \gamma^-_{\widetilde{D}}(m))|m \in X\}$, where $\mu^+_{\widetilde{D}}(m), \gamma^+_{\widetilde{D}}(m) : X \to [0,1], \mu^-_{\widetilde{D}}(m), \gamma^-_{\widetilde{D}}(m) : X \to [-1,0]$ such that $0 \leq \mu^+_{\widetilde{D}}(m) + \gamma^+_{\widetilde{D}}(m) \leq$ 1 and $-1 \leq \mu^-_{\widetilde{D}}(m) + \gamma^-_{\widetilde{D}}(m) \leq 0$.*

3. Bipolar Intuitionistic Fuzzy Matrices

This part provides the notion of BIFM. Also, some operations and properties of BIFM is examined.

Definition 3.1. *Let $X = \{s_1, s_2, \ldots, s_m\}$ and $Y = \{t_1, t_2, \ldots, t_n\}$ be the set of alternatives (row) and attributes (column) of each elements in X. A BIFM is defined by $\widetilde{B} = \left\{(s_i, t_j), \mu^+_{\widetilde{B}}(s_i, t_j), \mu^-_{\widetilde{B}}(s_i, t_j), \gamma^+_{\widetilde{B}}(s_i, t_j), \gamma^-_{\widetilde{B}}(s_i, t_j)\right\}$ for $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$, where $\mu^+_{\widetilde{B}}(s_i, t_j), \gamma^+_{\widetilde{B}}(s_i, t_j) : X \times Y \to [0,1]$ and $\mu^-_{\widetilde{B}}(s_i, t_j), \gamma^-_{\widetilde{B}}(s_i, t_j) : X \times Y \to [-1,0]$ such that $0 \leq \mu^+_{\widetilde{B}}(s_i, t_j) + \gamma^+_{\widetilde{B}}(s_i, t_j) \leq 1$ and $-1 \leq \mu^-_{\widetilde{B}}(s_i, t_j) + \gamma^-_{\widetilde{B}}(s_i, t_j) \leq 0$. Generally, we denote the notions of $\mu^+_{\widetilde{B_1}}(s_i, t_j), \mu^-_{\widetilde{B}}(s_i, t_j), \gamma^+_{\widetilde{B}}(s_i, t_j)$ and $\gamma^-_{\widetilde{B}}(s_i, t_j)$ to be c^+_{ij}, c^-_{ij}, d^+_{ij} and d^-_{ij} respectively.*

$$\widetilde{B} = \left[\left(c^+_{ij}, c^-_{ij}, d^+_{ij}, d^-_{ij}\right)\right]_{m\times n}$$

$$= \begin{bmatrix} \left(c^+_{11}, c^-_{11}, d^+_{11}, d^-_{11}\right) & \left(c^+_{12}, c^-_{12}, d^+_{12}, d^-_{12}\right) & \cdots & \left(c^+_{1n}, c^-_{1n}, d^+_{1n}, d^-_{1n}\right) \\ \left(c^+_{21}, c^-_{21}, d^+_{21}, d^-_{21}\right) & \left(c^+_{22}, c^-_{22}, d^+_{22}, d^-_{22}\right) & \cdots & \left(c^+_{2n}, c^-_{2n}, d^+_{2n}, d^-_{2n}\right) \\ \vdots & \vdots & \ddots & \\ \left(c^+_{m1}, c^-_{m1}, d^+_{m1}, d^-_{m1}\right) & \left(c^+_{m2}, c^-_{m2}, d^+_{m2}, d^-_{m2}\right) & \cdots & \left(c^+_{mn}, c^-_{mn}, d^+_{mn}, d^-_{mn}\right) \end{bmatrix}$$

Definition 3.2. *Let $\widetilde{B_1} = \left\{c^+_{ij}, c^-_{ij}, d^+_{ij}, d^-_{ij}\right\}$ and $\widetilde{B_2} = \left\{f^+_{ij}, f^-_{ij}, g^+_{ij}, g^-_{ij}\right\}$ are any two BIFM. Then maximum $(\widetilde{B_1} \vee \widetilde{B_2})$ and minimum $(\widetilde{B_1} \wedge \widetilde{B_2})$ operations are defined as follows*

$$\widetilde{B_1} \vee \widetilde{B_2} = \left\{c^+_{ij} \vee f^+_{ij}, c^-_{ij} \wedge f^-_{ij}, d^+_{ij} \wedge g^+_{ij}, d^-_{ij} \vee g^-_{ij}\right\}$$

$$\widetilde{B_1} \wedge \widetilde{B_2} = \left\{c^+_{ij} \wedge f^+_{ij}, c^-_{ij} \vee f^-_{ij}, d^+_{ij} \vee g^+_{ij}, d^-_{ij} \wedge g^-_{ij}\right\}$$

Example 3.1. *Consider two BIFM $\widetilde{B_1}$ and $\widetilde{B_2}$ that are given below*

$$\widetilde{B_1} = \begin{bmatrix} (0.5,\ -0.6,\ 0.5,\ -0.4) & (0.3,\ -0.7,\ 0.7,\ -0.3) & (0.8,\ -0.5,\ 0.2,\ -0.5) \\ (0.3,\ -0.3,\ 0.7,\ -0.7) & (0.7,\ -0.2,\ 0.3,\ -0.8) & (0.6,\ -0.4,\ 0.4,\ -0.6) \\ (0.5,\ -0.5,\ 0.5,\ -0.5) & (0.8,\ -0.4,\ 0.2,\ -0.6) & (0.5,\ -0.3,\ 0.5,\ -0.7) \end{bmatrix}$$

$$\widetilde{B_2} = \begin{bmatrix} (0.8,\ -0.3,\ 0.2,\ -0.7) & (0.7,\ -0.3,\ 0.3,\ -0.7) & (0.8,\ -0.3,\ 0.2,\ -0.7) \\ (0.7,\ -0.5,\ 0.3,\ -0.5) & (0.6,\ -0.7,\ 0.4,\ -0.3) & (0.6,\ -0.5,\ 0.4,\ -0.5) \\ (0.6,\ -0.7,\ 0.4,\ -0.3) & (0.5,\ -0.3,\ 0.5,\ -0.7) & (0.4,\ -0.4,\ 0.6,\ -0.6) \end{bmatrix}$$

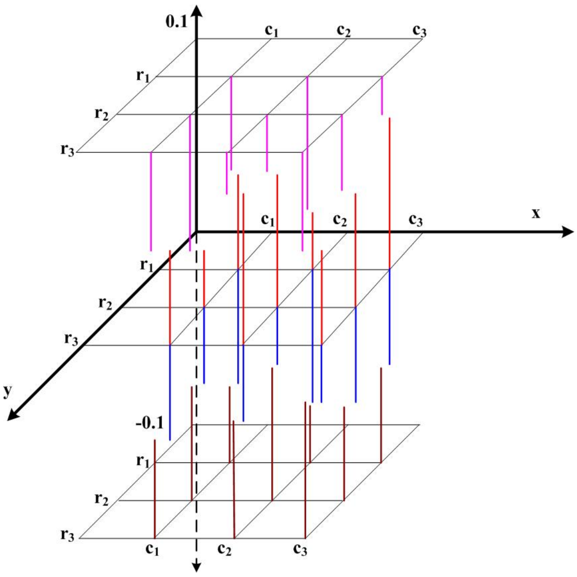

FIGURE 1. Representation of the BIFM $\widetilde{B_1}$

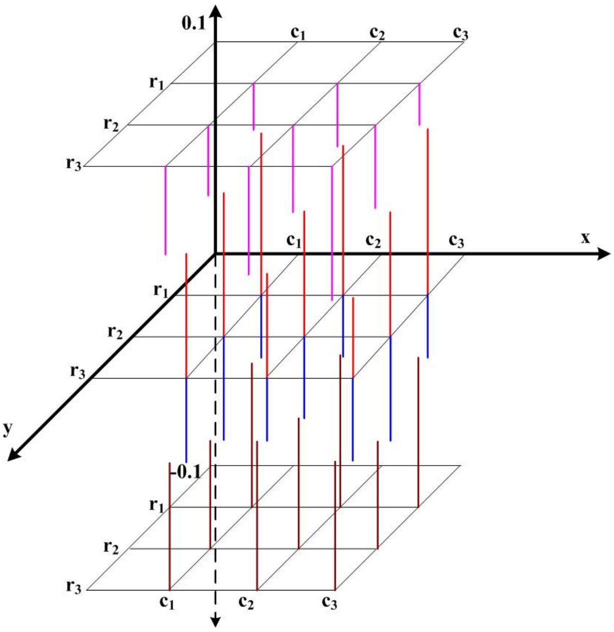

FIGURE 2. Representation of the BIFM $\widetilde{B_2}$

The $\widetilde{B_1} \vee \widetilde{B_2}$ *is defined as*

$$\widetilde{B_1} \vee \widetilde{B_2} = \begin{bmatrix} (0.8,\ -0.6,\ 0.2,\ -0.4) & (0.7,\ -0.7,\ 0.3,\ -0.3) & (0.8,\ -0.5,\ 0.2,\ -0.5) \\ (0.7,\ -0.5,\ 0.3,\ -0.5) & (0.7,\ -0.7,\ 0.3,\ -0.3) & (0.6,\ -0.5,\ 0.4,\ -0.5) \\ (0.6,\ -0.7,\ 0.4,\ -0.3) & (0.8,\ -0.4,\ 0.2,\ -0.6) & (0.5,\ -0.4,\ 0.5,\ -0.6) \end{bmatrix}$$

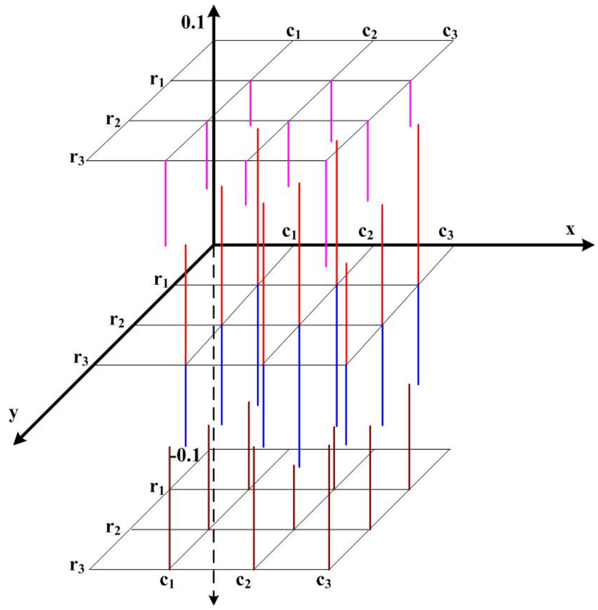

FIGURE 3. Representation of the BIFM $\widetilde{B_1} \vee \widetilde{B_2}$

The $\widetilde{B_1} \wedge \widetilde{B_2}$ *is defined as*

$$\widetilde{B_1} \wedge \widetilde{B_2} = \begin{bmatrix} (0.5,\ -0.3,\ 0.5,\ -0.7) & (0.3,\ -0.3,\ 0.7,\ -0.7) & (0.8,\ -0.3,\ 0.2,\ -0.7) \\ (0.3,\ -0.3,\ 0.7,\ -0.7) & (0.6,\ -0.2,\ 0.4,\ -0.8) & (0.6,\ -0.4,\ 0.4,\ -0.6) \\ (0.5,\ -0.5,\ 0.5,\ -0.5) & (0.5\ -0.4,\ 0.5,\ -0.6) & (0.5,\ -0.3,\ 0.5,\ -0.7) \end{bmatrix}$$

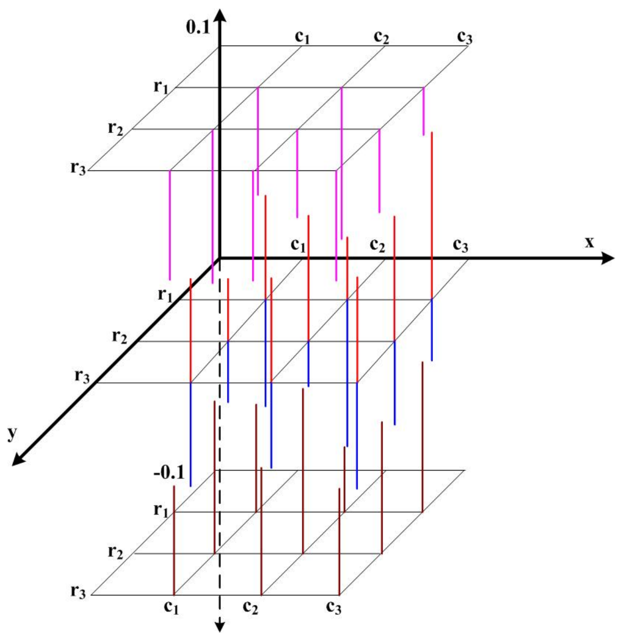

FIGURE 4. Representation of the BIFM $\widetilde{B_1} \wedge \widetilde{B_2}$

Definition 3.3. *Let $\widetilde{B}$ be a BIFM. The complement of $\widetilde{B}$ is defined as* $\widetilde{B}^c = \left\{1 - c_{ij}^+,\ -1 - c_{ij}^-,\ 1 - d_{ij}^+,\ -1 - d_{ij}^-\right\}$.

Definition 3.4. *A BIFM $\widetilde{B}$ is called a self-complementary if $\widetilde{B}^c = \widetilde{B}$*

Example 3.2. *Consider the BIFM $\widetilde{B}$. Then the complement of $\widetilde{B}$ is*

$$\widetilde{B}^c = \begin{bmatrix} (0.5,\ -0.4,\ 0.5,\ -0.4) & (0.7,\ -0.3,\ 0.3,\ -0.7) & (0.2,\ -0.5,\ 0.8,\ -0.5) \\ (0.7,\ -0.7,\ 0.3,\ -0.3) & (0.3,\ -0.8,\ 0.7,\ -0.2) & (0.4,\ -0.6,\ 0.6,\ -0.4) \\ (0.5,\ -0.5,\ 0.5,\ -0.5) & (0.2,\ -0.6,\ 0.8,\ -0.4) & (0.5,\ -0.7,\ 0.5,\ -0.3) \end{bmatrix}$$

Proposition 3.1. *(De Morgan's laws) Let $\widetilde{B_1} = \left[\left(c_{ij}^+, c_{ij}^-, d_{ij}^+, d_{ij}^-\right)\right]_{m\times n}$ and $\widetilde{B_2} = \left[\left(f_{ij}^+, f_{ij}^-, g_{ij}^+, g_{ij}^-\right)\right]_{m\times n}$ be two BIFM. Then*

(i) $\left(\widetilde{B_1} \vee \widetilde{B_2}\right)^c = \widetilde{B_1}^c \wedge \widetilde{B_2}^c$

(ii) $\left(\widetilde{B_1} \wedge \widetilde{B_2}\right)^c = \widetilde{B_1}^c \vee \widetilde{B_2}^c$

Proposition 3.2. *Let $\widetilde{B} = \left[\left(c_{ij}^+, c_{ij}^-, d_{ij}^+, d_{ij}^-\right)\right]_{m\times n}$ be a BIFM. Then $\left(\widetilde{B}^c\right)^c = \widetilde{B}$*

Definition 3.5. *The addition of two BIFM $\widetilde{B_1}$ and $\widetilde{B_2}$ is defined as equal to the maximum operator, i.e., $\widetilde{B_1} + \widetilde{B_2} = \left\{c_{ij}^+ \vee f_{ij}^+,\ c_{ij}^- \wedge f_{ij}^-,\ d_{ij}^+ \wedge g_{ij}^+,\ d_{ij}^- \vee g_{ij}^-\right\}$*

Suppose the matrix addition is considered as an ordinary matrix order, then the bipolar intuitionistic fuzzy values of the elements exceed 1. Therefore, the maximum operator is considered to be a perfect operator for the addition of the BIFM. An example (3.3) is given below to expose the benefit of this operation.

Example 3.3. *Consider two BIFM $\widetilde{B_1}$ and $\widetilde{B_2}$. Then the ordinary matrix addition of $\widetilde{B_1}$ and $\widetilde{B_2}$ is*

$$\widetilde{B_1} + \widetilde{B_2} = \begin{bmatrix} (1.3,\ -0.9,\ 0.7,\ -1.1) & (1.0,\ -1.0,\ 1.0,\ -1.0) & (1.6,\ -0.8,\ 0.4,\ -1.2) \\ (1.0,\ -0.8,\ 1.0,\ -1.2) & (1.3,\ -0.9,\ 0.7,\ -1.1) & (1.2,\ -0.9,\ 0.8,\ -1.1) \\ (1.1,\ -1.2,\ 0.9,\ -0.8) & (1.3,\ -0.7,\ 0.7,\ -1.3) & (0.9,\ -0.7,\ 1.1,\ -1.3) \end{bmatrix}$$

This is failed to provide the fuzzy values. Therefore, the maximum operator are considered for addition of BIFM. It is given in the example 3.4.

Example 3.4. *Consider two BIFM $\widetilde{B_1}$ and $\widetilde{B_2}$. Then the addition of $\widetilde{B_1}$ and $\widetilde{B_2}$ is*

$$\widetilde{B_1} + \widetilde{B_2} = \begin{bmatrix} (0.8,\ -0.6,\ 0.2,\ -0.4) & (0.7,\ -0.7,\ 0.3,\ -0.3) & (0.8,\ -0.5,\ 0.2,\ -0.5) \\ (0.7,\ -0.5,\ 0.3,\ -0.5) & (0.7,\ -0.7,\ 0.3,\ -0.3) & (0.6,\ -0.5,\ 0.4,\ -0.5) \\ (0.6,\ -0.7,\ 0.4,\ -0.3) & (0.8,\ -0.4,\ 0.2,\ -0.6) & (0.5,\ -0.4,\ 0.5,\ -0.6) \end{bmatrix}$$

Proposition 3.3. *If $\widetilde{B_1}, \widetilde{B_2}$ and $\widetilde{B_3}$ are the BIFM with the same order. Then the following properties are satisfied*

(i) $\widetilde{B_1} + \widetilde{B_1} = \widetilde{B_1}$

(ii) $\widetilde{B_1} + \widetilde{B_2} = \widetilde{B_2} + \widetilde{B_1}$

(iii) $\left(\widetilde{B_1} + \widetilde{B_2}\right) + \widetilde{B_3} = \widetilde{B_1} + \left(\widetilde{B_2} + \widetilde{B_3}\right)$

The logic of using the maximum operator for ordinary matrix addition is also considered for matrix multiplication. Sometimes, the ordinary matrix multiplications may provide values greater than 1. Therefore, the max-min (definition 3.6) can be utilized for the multiplication of BIFM instead of ordinary matrix multiplication. Based on the applications, the min-max (definition 3.7) operation can also be used for multiplication. Thus, these methods are derived in this study to obtain the perfect result. The examples (3.5 & 3.6) show the quality of the outcome by using these operators.

Definition 3.6. *Let $\widetilde{B_1}$ and $\widetilde{B_2}$ be the BIFM. Then the max-min composition is defined as follows*

$$\widetilde{B_1} \odot \widetilde{B_2} = \left\{ \bigvee_{k=1}^{n} \left(c_{ik}^{+} \wedge f_{kj}^{+}\right),\ \bigwedge_{k=1}^{n} \left(c_{ik}^{-} \vee f_{kj}^{-}\right),\ \bigwedge_{k=1}^{n} \left(d_{ik}^{+} \vee g_{kj}^{+}\right),\ \bigvee_{k=1}^{n} \left(d_{ik}^{-} \wedge g_{kj}^{-}\right) \right\}$$

Example 3.5. *The max-min composition of $\widetilde{B_1}$ and $\widetilde{B_2}$ of order 3×3 is given below*

$$\widetilde{B_1} \odot \widetilde{B_2} = \left\{ \bigvee_{k=1}^{3} (c_{1k}^{+} \wedge f_{k3}^{+}),\ \bigwedge_{k=1}^{3} (c_{1k}^{-} \vee f_{k3}^{-}),\ \bigwedge_{k=1}^{3} (d_{1k}^{+} \vee g_{k3}^{+}),\ \bigvee_{k=1}^{3} (d_{1k}^{-} \wedge g_{k3}^{-}) \right\}$$

Here,
$\bigvee_{k=1}^{3} \left(c_{1k}^{+} \wedge f_{k1}^{+}\right) = 0.6,$
$\bigwedge_{k=1}^{3} \left(c_{1k}^{-} \vee f_{k1}^{-}\right) = -0.5,$
$\bigwedge_{k=1}^{3} \left(d_{1k}^{+} \vee g_{k1}^{+}\right) = 0.4,$
$\bigvee_{k=1}^{3} \left(d_{1k}^{-} \wedge g_{k1}^{-}\right) = -0.5.$

$\bigvee_{k=1}^{3} \left(c_{1k}^{+} \wedge f_{k2}^{+}\right) = 0.5,$
$\bigwedge_{k=1}^{3} \left(c_{1k}^{-} \vee f_{k2}^{-}\right) = -0.7,$
$\bigwedge_{k=1}^{3} \left(d_{1k}^{+} \vee g_{k2}^{+}\right) = 0.5,$
$\bigvee_{k=1}^{3} \left(d_{1k}^{-} \wedge g_{k2}^{-}\right) = -0.3.$

$\bigvee_{k=1}^{3} \left(c_{1k}^{+} \wedge f_{k3}^{+}\right) = 0.5,$
$\bigwedge_{k=1}^{3} \left(c_{1k}^{-} \vee f_{k3}^{-}\right) = -0.5,$
$\bigwedge_{k=1}^{3} \left(d_{1k}^{+} \vee g_{k3}^{+}\right) = 0.5,$
$\bigvee_{k=1}^{3} \left(d_{1k}^{-} \wedge g_{k3}^{-}\right) = -0.5.$

$x_{11} = (0.6, -0.5, 0.4, -0.5)\,,\ x_{12} = (0.5, -0.7, 0.5, -0.4)\,,\ x_{13} = (0.5, -0.5, 0.5, -0.5)$

Similarly, the values for the 2nd and 3rd row is found below,
$x_{21} = (0.7, -0.4, 0.3, -0.6)\,,\ x_{22} = (0.6, -0.5, 0.4, -0.5)\,,\ x_{23} = (0.6, -0.5, 0.4, -0.5)$
$x_{31} = (0.7, -0.4, 0.3, -0.6)\,,\ x_{32} = (0.6, -0.7, 0.4, -0.3)\,,\ x_{33} = (0.6, -0.4, 0.4, -0.6).$

$$\widetilde{B_1} \odot \widetilde{B_2} = \begin{bmatrix} (0.6,\ -0.5,\ 0.4,\ -0.5) & (0.5,\ -0.7,\ 0.5,\ -0.3) & (0.5,\ -0.5,\ 0.5,\ -0.5) \\ (0.7,\ -0.4,\ 0.3,\ -0.6) & (0.6,\ -0.5,\ 0.4,\ -0.5) & (0.6,\ -0.5,\ 0.4,\ -0.5) \\ (0.7,\ -0.4,\ 0.3,\ -0.6) & (0.6,\ -0.7,\ 0.4,\ -0.3) & (0.6,\ -0.4,\ 0.4,\ -0.6) \end{bmatrix}$$

Definition 3.7. *Let $\widetilde{B_1}$ and $\widetilde{B_2}$ be the BIFM. Then the min-max composition is defined as follows*

$$\widetilde{B_1} \otimes \widetilde{B_2} = \left\{ \bigwedge_{k=1}^{n} \left(c_{ik}^{+} \vee f_{kj}^{+}\right),\ \bigvee_{k=1}^{n} \left(c_{ik}^{-} \wedge f_{kj}^{-}\right),\ \bigvee_{k=1}^{n} \left(d_{ik}^{+} \wedge g_{kj}^{+}\right),\ \bigwedge_{k=1}^{n} \left(d_{ik}^{-} \vee g_{kj}^{-}\right) \right\}$$

Example 3.6. *The min-max composition of $\widetilde{B_1}$ and $\widetilde{B_2}$ is given below*

$$\widetilde{B_1} \otimes \widetilde{B_2} = \left\{ \bigwedge_{k=1}^{3} \left(c_{ik}^{+} \vee f_{kj}^{+}\right), \ \bigvee_{k=1}^{3} \left(c_{ik}^{-} \wedge f_{kj}^{-}\right), \bigvee_{k=1}^{3} \left(d_{ik}^{+} \wedge g_{kj}^{+}\right), \ \bigwedge_{k=1}^{3} \left(d_{ik}^{-} \vee g_{kj}^{-}\right) \right\}$$

Here,
$\bigwedge_{k=1}^{3} \left(c_{1k}^{+} \vee f_{k1}^{+}\right) = 0.7,$
$\bigvee_{k=1}^{3} \left(c_{1k}^{-} \wedge f_{k1}^{-}\right) = -0.6,$
$\bigvee_{k=1}^{3} \left(d_{1k}^{+} \wedge g_{k1}^{+}\right) = 0.3,$
$\bigwedge_{k=1}^{3} \left(d_{1k}^{-} \vee g_{k1}^{-}\right) = -0.4.$

$\bigwedge_{k=1}^{3} \left(c_{1k}^{+} \vee f_{k2}^{+}\right) = 0.6,$
$\bigvee_{k=1}^{3} \left(c_{1k}^{-} \wedge f_{k2}^{-}\right) = -0.5,$
$\bigvee_{k=1}^{3} \left(d_{1k}^{+} \wedge g_{k2}^{+}\right) = 0.4,$
$\bigwedge_{k=1}^{3} \left(d_{1k}^{-} \vee g_{k2}^{-}\right) = -0.5.$

$\bigwedge_{k=1}^{3} \left(c_{1k}^{+} \vee f_{k3}^{+}\right) = 0.6,$
$\bigvee_{k=1}^{3} \left(c_{1k}^{-} \wedge f_{k3}^{-}\right) = -0.5,$
$\bigvee_{k=1}^{3} \left(d_{1k}^{+} \wedge g_{k3}^{+}\right) = 0.4,$
$\bigwedge_{k=1}^{3} \left(d_{1k}^{-} \vee g_{k3}^{-}\right) = -0.5.$

$x_{11} = (0.7, -0.6, 0.3, -0.4)\,,\ x_{12} = (0.6, -0.5, 0.4, -0.5)\,, x_{13} = (0.6, -0.5, 0.4, -0.5)$
Similarly, the values for the 2nd and 3rd row is found below,
$x_{21} = (0.6, -0.3, 0.4, -0.7)\,,\ x_{22} = (0.6, -0.3, 0.4, -0.7)\,,\ x_{23} = (0.6, -0.3, 0.4, -0.7)$
$x_{31} = (0.6, -0.5, 0.4, -0.5)\,,\ x_{32} = (0.5, -0.3, 0.5, -0.7)\,,\ x_{33} = (0.5, -0.4, 0.5, -0.6).$

$$\widetilde{B_1} \otimes \widetilde{B_2} = \begin{bmatrix} (0.7,\ -0.6,\ 0.3,\ -0.4) & (0.6,\ -0.5,\ 0.4,\ -0.5) & (0.6,\ -0.5,\ 0.4,\ -0.5) \\ (0.6,\ -0.3,\ 0.4,\ -0.7) & (0.6,\ -0.3,\ 0.4,\ -0.7) & (0.6,\ -0.3,\ 0.4,\ -0.7) \\ (0.6,\ -0.5,\ 0.4,\ -0.5) & (0.5,\ -0.3,\ 0.5,\ -0.7) & (0.5,\ -0.4,\ 0.5,\ -0.6) \end{bmatrix}$$

4. Determinant of the BIFM

The view of determinant of the BIFM and its properties are investigated in this section. The matrix determinant provides the characterization of the matrix. The ordinary matrix determinant may provide the output as greater than 1 or a negative value. Therefore, this study considers the bipolar intuitionistic fuzzy determinant to obtain the outcome, which shows the quality of the BIFM.

Definition 4.1. *Let $\widetilde{B_1} = \left\{ c_{ij}^{+},\ c_{ij}^{-},\ d_{ij}^{+},\ d_{ij}^{-} \right\}$ be the BIFSM. The bipolar intuitionistic fuzzy determinant of $\widetilde{B_1}$ is defined as*

$$\left|\widetilde{B_1}\right| = \begin{vmatrix} (c_{11}^{+},\ c_{11}^{-},\ d_{11}^{+},\ d_{11}^{-}) & (c_{12}^{+},\ c_{12}^{-},\ d_{12}^{+},\ d_{12}^{-}) & \cdots & (c_{1n}^{+},\ c_{1n}^{-},\ d_{1n}^{+},\ d_{1n}^{-}) \\ (c_{21}^{+},\ c_{21}^{-},\ d_{21}^{+},\ d_{21}^{-}) & (c_{22}^{+},\ c_{22}^{-},\ d_{22}^{+},\ d_{22}^{-}) & \cdots & (c_{2n}^{+},\ c_{2n}^{-},\ d_{2n}^{+},\ d_{2n}^{-}) \\ \vdots & \vdots & \ddots & \vdots \\ (c_{n1}^{+},\ c_{n1}^{-},\ d_{n1}^{+},\ d_{m1}^{-}) & (c_{n2}^{+},\ c_{n2}^{-},\ d_{n2}^{+},\ d_{n2}^{-}) & \cdots & (c_{nn}^{+},\ c_{nn}^{-},\ d_{nn}^{+},\ d_{nn}^{-}) \end{vmatrix}$$

$$= \Bigg(\bigvee_{\sigma \in S_n} \left(c^{+}_{1\sigma(1)} \wedge c^{+}_{2\sigma(2)} \wedge \ldots \wedge c^{+}_{n\sigma(n)} \right), \bigwedge_{\sigma \in S_n} \left(c^{-}_{1\sigma(1)} \vee c^{-}_{2\sigma(2)} \vee \ldots \vee c^{-}_{n\sigma(n)} \right),$$

$$\bigwedge_{\sigma \in S_n} \left(d^{+}_{1\sigma(1)} \vee d^{+}_{2\sigma(2)} \vee \ldots \vee d^{+}_{n\sigma(n)} \right), \bigvee_{\sigma \in S_n} \left(d^{-}_{1\sigma(1)} \wedge d^{-}_{2\sigma(2)} \wedge \ldots \wedge d^{-}_{n\sigma(n)} \right) \Bigg)$$

Here S_n represents the set of all permutations of the indices $\{1, 2, \ldots n\}$. Consider a 3×3 BIFM. Then the determinant value is as follows

$$|\widetilde{B_1}| = \begin{vmatrix} (c^{+}_{11}, c^{-}_{11}, d^{+}_{11}, d^{-}_{11}) & (c^{+}_{12}, c^{-}_{12}, d^{+}_{12}, d^{-}_{12}) & \cdots & (c^{+}_{11}, c^{-}_{1n}, d^{+}_{1n}, d^{-}_{1n}) \\ (c^{+}_{21}, c^{-}_{21}, d^{+}_{21}, d^{-}_{21}) & (c^{+}_{22}, c^{-}_{22}, d^{+}_{22}, d^{-}_{22}) & \cdots & (c^{+}_{2n}, c^{-}_{2n}, d^{+}_{2n}, d^{-}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (c^{+}_{n1}, c^{-}_{n1}, d^{+}_{n1}, d^{-}_{m1}) & (c^{+}_{n2}, c^{-}_{n2}, d^{+}_{n2}, d^{-}_{n2}) & \cdots & (c^{+}_{nn}, c^{-}_{nn}, d^{+}_{nn}, d^{-}_{nn}) \end{vmatrix}$$

$$= \begin{pmatrix} \bigvee_{\sigma \in S_n} \left(c^{+}_{1\sigma(1)} \wedge c^{+}_{2\sigma(2)} \wedge \ldots \wedge c^{+}_{n\sigma(n)} \right), \bigwedge_{\sigma \in S_n} \left(c^{-}_{1\sigma(1)} \vee c^{-}_{2\sigma(2)} \vee \ldots \vee c^{-}_{n\sigma(n)} \right), \\ \bigwedge_{\sigma \in S_n} \left(d^{+}_{1\sigma(1)} \vee d^{+}_{2\sigma(2)} \vee \ldots \vee d^{+}_{n\sigma(n)} \right), \bigvee_{\sigma \in S_n} \left(d^{-}_{1\sigma(1)} \wedge d^{-}_{2\sigma(2)} \wedge \ldots \wedge d^{-}_{n\sigma(n)} \right) \end{pmatrix}$$

Here S_n represents the set of all permutations of the indices $\{1, 2, \ldots n\}$. Consider a 3×3 BIFM. Then the determinant value is as follows

$$|\widetilde{B_1}| = \Bigg(\bigvee_{\sigma \in S_n} \left(c^{+}_{1\sigma(1)} \wedge c^{+}_{2\sigma(2)} \wedge c^{+}_{3\sigma(3)} \right), \bigwedge_{\sigma \in S_n} \left(c^{-}_{1\sigma(1)} \vee c^{-}_{2\sigma(2)} \vee c^{-}_{3\sigma(3)} \right), \bigwedge_{\sigma \in S_n} \Big(d^{+}_{1\sigma(1)} \vee$$

$$d^{+}_{2\sigma(2)} \vee d^{+}_{3\sigma(3)} \Big), \bigvee_{\sigma \in S_n} \left(d^{-}_{1\sigma(1)} \wedge d^{-}_{2\sigma(2)} \wedge d^{-}_{3\sigma(3)} \right) \Bigg)$$

$$= \Bigg(\bigvee \begin{pmatrix} (c^{+}_{11} \wedge c^{+}_{22} \wedge c^{+}_{33}), (c^{+}_{11} \wedge c^{+}_{23} \wedge c^{+}_{32}), (c^{+}_{12} \wedge c^{+}_{21} \wedge c^{+}_{33}), (c^{+}_{12} \wedge c^{+}_{23} \wedge c^{+}_{31}) \\ (c^{+}_{13} \wedge c^{+}_{21} \wedge c^{+}_{32}), (c^{+}_{13} \wedge c^{+}_{22} \wedge c^{+}_{31}) \end{pmatrix},$$

$$\bigwedge \begin{pmatrix} (c^{-}_{11} \vee c^{-}_{22} \vee c^{-}_{33}), (c^{-}_{11} \vee c^{-}_{23} \vee c^{-}_{32}), (c^{-}_{12} \vee c^{-}_{21} \vee c^{-}_{33}), (c^{-}_{12} \vee c^{-}_{23} \vee c^{-}_{31}), \\ (c^{-}_{13} \vee c^{-}_{21} \vee c^{-}_{32}), (c^{-}_{13} \vee c^{-}_{22} \vee c^{-}_{31}) \end{pmatrix},$$

$$\bigwedge \begin{pmatrix} (d^{+}_{11} \vee d^{+}_{22} \vee d^{+}_{33}), (d^{+}_{11} \vee d^{+}_{23} \vee d^{+}_{32}), (d^{+}_{12} \vee d^{+}_{21} \vee d^{+}_{33}), (d^{+}_{12} \vee d^{+}_{23} \vee d^{+}_{31}), \\ (d^{+}_{13} \vee d^{+}_{21} \vee d^{+}_{32}), (d^{+}_{13} \vee d^{+}_{22} \vee d^{+}_{31}) \end{pmatrix},$$

$$\bigvee \begin{pmatrix} (d^{-}_{11} \wedge d^{-}_{22} \wedge d^{-}_{33}), (d^{-}_{11} \wedge d^{-}_{23} \wedge d^{-}_{32}), (d^{-}_{12} \wedge d^{-}_{21} \wedge d^{-}_{33}), (d^{-}_{12} \wedge d^{-}_{23} \wedge d^{-}_{31}) \\ (d^{-}_{13} \wedge d^{-}_{21} \wedge d^{-}_{32}), (d^{-}_{13} \wedge d^{-}_{22} \wedge d^{-}_{31}) \end{pmatrix} \Bigg)$$

Example 4.1. *Consider the BIFSM $\widetilde{B_1}$ of order 3×3. The determinant of $\widetilde{B_1}$ is expressed as follows*

$$\widetilde{B_1} = \begin{bmatrix} (0.5,-0.6,0.5,-0.4) & (0.3,-0.7,0.7,-0.3) & (0.8,-0.5,0.2,-0.5) \\ (0.3,-0.3,0.7,-0.7) & (0.7,-0.2,0.3,-0.8) & (0.6,-0.4,0.4,-0.6) \\ (0.5,-0.5,0.5,-0.5) & (0.8,-0.4,0.2,-0.6) & (0.5,-0.3,0.5,-0.7) \end{bmatrix}$$

$$= \left(\vee \begin{pmatrix} (0.5\wedge0.7\wedge0.5),(0.5\wedge0.6\wedge0.8),(0.3\wedge0.3\wedge0.5), \\ (0.3\wedge0.6\wedge0.5),(0.8\wedge0.3\wedge0.8),(0.8\wedge0.7\wedge0.5) \end{pmatrix}, \right.$$

$$\wedge \begin{pmatrix} (-0.6\vee-0.2\vee-0.3),(-0.6\vee-0.4\vee-0.4),(-0.7\vee-0.3\vee-0.3), \\ (-0.7\vee-0.4\vee-0.5),(-0.5\vee-0.3\vee-0.4),(-0.5\vee-0.2\vee-0.3) \end{pmatrix},$$

$$\wedge \begin{pmatrix} (0.5\vee0.3\vee0.5),(0.5\vee0.4\vee0.2),(0.7\vee0.7\vee0.5), \\ (0.7\vee0.4\vee0.5),(0.5\vee0.7\vee0.2),(0.5\vee0.3\vee0.5) \end{pmatrix},$$

$$\left. \vee \begin{pmatrix} (-0.4\wedge-0.8\wedge-0.7),(-0.4\wedge-0.6\wedge-0.6),(-0.3\wedge-0.7\wedge-0.7), \\ (-0.3\wedge-0.6\wedge-0.5),(-0.5\wedge-0.7\wedge-0.6),(-0.5\wedge-0.8\wedge-0.5) \end{pmatrix} \right)$$

$$= \vee(0.5,0.5,0.3,0.3,0.3,0.5) = 0.5,$$

$$= \wedge(-0.2,-0.4,-0.3,-0.4,-0.3,-0.2) = -0.4,$$

$$= \wedge(0.5,0.5,0.7,0.7,0.7,0.5) = 0.5,$$

$$= \vee(-0.8,-0.6,-0.7,-0.6,-0.7,-0.8) = -0.6.$$

$$|\widetilde{B_1}| = (0.5,-0.4,0.5,-0.6).$$

Proposition 4.1. *If M be a BIFM of order $n\times n$. Then it satisfies the following properties*

(i) $a\cdot|\widetilde{B_1}| = |a\cdot\widetilde{B_1}|$, *where* $a\in\widetilde{D}$

(ii) $|\widetilde{B_1}\cdot\widetilde{B_2}| = |\widetilde{B_1}|\cdot|\widetilde{B_2}|$.

5. Conclusions

The theory of fuzzy sets has a wide variety of applications in many research domains. The extensions of fuzzy sets, such as IFS and BFS provide different views on the element's belongingness grades. These concepts were combined as BIFS to study the element's involvement and non-involvement grades from a bipolar perspective. When the matrix elements are vague, the theory of fuzzy matrices helps in dealing with real-time situations. Mainly the fuzzy matrices were utilized in various multi-criteria decision-making systems. Therefore, this study offers the concepts of the BIFM and its determinant. The three-dimensional view of the BIFM is portrayed to display the flexible outcome of the problems. Furthermore, some essential operations and results were discussed to strengthen the theoretical field of the BIFM.

Advantages: (i). The BIFM gives extra benefits when it is examined with a non-involvement and negative elements grade, (ii). The three-dimensional representation of the BIFM helps to understand the extra space of the BIFM, (iii). The discussed operations and properties depict the applicability of the BIFM, (iv). The determinant of BIFM and its examples provide a clear view of the solution of BIFM.

The present study further can be carried out in the following attempts

(i). Multi-criteria decision-making techniques using BIFM

(ii). Bipolar single-valued neutrosophic matrices

(iii). BIFG-based decision-making techniques using BIFM

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