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ABSOLUTE CONVERGENCE WITH SPEED AND MATRIX TRANSFORMS

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ABSTRACT. We define the notion of absolute convergence with speed, where the speed is defined by a monotonically increasing positive sequence λ. Also we present the notion of absolute λ-conservativity of a matrix, and the notion of improvement of λ-convergence by a matrix. Let X, Y be two sequence spaces defined by speeds of convergence. In this paper, we give necessary and sufficient conditions for a matrix A (with real or complex entries) to map X into Y, if X or Y is the set of absolutely λ-convergent sequences. We also present some examples of matrices being absolutely λ-conservative or improving the absolute λ-convergence, and consider these problems in the special cases if A is the Riesz matrix (R, p_n) or the Zweier matrix $Z_{1/2}$.

Keywords: Matrix transforms, convergence and absolute convergence with speed, absolute λ-conservativity, improvement of λ-convergence.

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1. INTRODUCTION

Let X, Y be two sequence spaces and $A = (a_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$A_n x := \sum_k a_{nk} x_k$$

converge and the sequence $Ax = (A_n x)$ belongs to Y, we say that A transforms X into Y. By (X, Y) we denote the set of all matrices, which transform X into Y. Let ω be the set of all real or complex valued sequences. Further we need the following well-known

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subspaces of ω: c - the space of all convergent sequences, c_0 - the space of all sequences converging to zero, l_∞ - the space of all bounded sequences, and

$$l_1 := \{x = (x_n) \,:\, \sum_n |x_n| < \infty\}.$$

Let $\lambda := (\lambda_k)$ be a positive (i.e.; $\lambda_k > 0$ for every k) monotonically increasing sequence. Following Kangro ([12], [11]), a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := s \text{ and } v_k = \lambda_k (x_k - s) \tag{1.1}$$

is called bounded with the speed λ (shortly, λ-bounded) if $v_k = O(1)$ (or $(v_k) \in l_\infty$), and convergent with the speed λ (shortly, λ-convergent) if the finite limit

$$\lim_k v_k := b$$

exists (or $(v_k) \in c$). In the following we define the notion of absolute convergence with speed λ.

Definition 1.1. *We say that a convergent sequence $x = (x_k)$ with the finite limit s is absolutely convergent with the speed λ (shortly, absolutely λ-convergent) if $(v_k) \in l_1$.*

We denote the set of all λ-bounded sequences by l_∞^λ, the set of all λ-convergent sequences by c^λ, and the set of all absolutely λ-convergent sequences by l_1^λ. Moreover, let

$$c_0^\lambda := \{x = (x_k) \,:\, x \in c^\lambda \text{ and } \lim_k \lambda_k(x_k - s) = 0\}$$

and

$$l_{\infty,0}^\lambda = \{x = (x_k) \,:\, x \in l_\infty^\lambda \cap c_0\}.$$

It is not difficult to see that

$$l_1^\lambda \subset c_0^\lambda \subset c^\lambda \subset l_\infty^\lambda \subset c, \ l_{\infty,0}^\lambda \subset l_\infty^\lambda \subset c.$$

In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$, we get $c^\lambda = l_\infty^\lambda = c$.

Let $e = (1, 1, ...)$, $e^k = (0, ..., 0, 1, 0, ...)$, where 1 is in the k-th position, and $\lambda^{-1} = (1/\lambda_k)$. We note that

$$e, e^k, \lambda^{-1} \in c^\lambda; \quad e, e^k \in l_1^\lambda.$$

A matrix A is said to be conservative if $A \in (c, c)$, and regular if $A \in (c, c)$ with $\lim_n A_n x = \lim_n x_n$ for every sequence $x = (x_n) \in c$. Let $\mu := (\mu_n)$ be another speed of convergence; i.e. a monotonically increasing positive sequence. Following Kangro ([12]), a matrix A is said to be λ-conservative if $A \in (c^\lambda, c^\lambda)$, and improves the λ-convergence if $A \in (c^\lambda, c^\mu)$ with $\mu_n/\lambda_n \neq O(1)$. We define the notions of the absolute λ-conservativity and the improvement of absolute λ-convergence.

Definition 1.2. *We say that a matrix A is absolutely λ-conservative if $A \in (l_1^\lambda, l_1^\lambda)$.*

Definition 1.3. *We say that a matrix A improves the absolute λ-convergence if $A \in (l_1^\lambda, l_1^\mu)$ with $\mu_n/\lambda_n \neq O(1)$.*

The sets $(l_\infty^\lambda, l_\infty^\mu)$, (c^λ, c^μ) and $(c^\lambda, l_\infty^\mu)$ have been described in [2] and in [10] - [13]. The sets $(l_\infty^\lambda, c^\mu)$, $\left(l_\infty^\lambda, l_{\infty,0}^\mu\right)$, $(l_\infty^\lambda, c_0^\mu)$, $\left(c^\lambda, l_{\infty,0}^\mu\right)$, (c^λ, c_0^μ), $(l_{\infty,0}^\lambda, l_\infty^\mu)$, $\left(l_{\infty,0}^\lambda, l_{\infty,0}^\mu\right)$, $(l_{\infty,0}^\lambda, c^\mu)$, $(l_{\infty,0}^\lambda, c_0^\mu)$, $(c_0^\lambda, l_\infty^\mu)$, $\left(c_0^\lambda, l_{\infty,0}^\mu\right)$, (c_0^λ, c^μ) and (c_0^λ, c_0^μ) have been characterized in [1]. Necessary and sufficient conditions for the λ-conservativity and the improvement of λ-convergence have been found in [12]. A short overview on the convergence with speed has been presented in [3] and [13].

We note that the results connected with convergence, absolute convergence and boundedness with speed can be used in several applications, for example in the approximation theory. Besides, Aasma used such results for the estimation of the order of approximation of Fourier expansions in Banach spaces ([4] - [7]).

In this paper we continue the studies started in [2], [1], [10], [12] and [11]. We describe the matrix transforms related to the absolute λ-convergence, giving the characterization of the sets (l_1^λ, l_1^μ), (l_1^λ, c_0^μ), (l_1^λ, c^μ), $(l_1^\lambda, l_\infty^\mu)$, $(l_\infty^\lambda, l_1^\mu)$, (c^λ, l_1^μ) and (c_0^λ, l_1^μ). Also we present some examples of absolute λ-conservative matrices, and matrices, which improve the absolute λ-convergence. We consider the absolute λ-conservativity and the improvement of the absolute λ-convergence in the special cases when A is the Riesz matrix (R, p_n) or the Zweier matrix $Z_{1/2}$.

2. Auxiliary results

For the proof of main results we need some auxiliary results.

Lemma 2.1 ([9], p. 44, see also [18], Proposition 12)**.** *A matrix $A = (a_{nk}) \in (c_0, c)$ if and only if conditions*

$$\lim_n a_{nk} := a_k \ for\ all\ k, \tag{2.1}$$

$$\sum_k |a_{nk}| = O(1) \tag{2.2}$$

are satisfied. Moreover,

$$\lim_n A_n x = \sum_k a_k x_k \tag{2.3}$$

for every $x = (x_k) \in c_0$.

Lemma 2.2 ([9], p. 46-47, see also [8], p. 17-19 or [18], Proposition 11)**.** *A matrix $A = (a_{nk}) \in (c, c)$ if and only if conditions* (2.1), (2.2) *are satisfied and*

$$there\ exists\ a\ number\ \tau\ such\ that\ limit\ \lim_n \sum_k a_{nk} := \tau. \tag{2.4}$$

Moreover, if $\lim_k x_k = s$ for $x = (x_k) \in c$, then

$$\lim_n A_n x = s\tau + \sum_k (x_k - s) a_k.$$

A matrix A is regular if and only if conditions (2.1), (2.2) *and* (2.4) *are satisfied with $a_k = 0$ and $\tau = 1$.*

Lemma 2.3 ([9], p. 51, see also [16], p. 187 or [17], p. 8 or [18], Proposition 10))**.** *The following statements are equivalent:*

(a) $A = (a_{nk}) \in (l_\infty, c)$.

(b) *The conditions* (2.1), (2.2) *are satisfied and*

$$\lim_n \sum_k |a_{nk} - a_k| = 0. \tag{2.5}$$

(c) *The condition* (2.1) *holds and*

$$the\ series\ \sum_k |a_{nk}|\ converges\ uniformly\ in\ n. \tag{2.6}$$

Moreover, if one of statements (a)-(c) *is satisfied, then the equation* (2.3) *holds for every $x = (x_k) \in l_\infty$.*

Lemma 2.4 ([8], p. 38-40 or [18], Proposition 72)). *A matrix $A = (a_{nk}) \in (c_0, l_1) = (l_\infty, l_1)$ if and only if*

$$\sum_{n \in I} \sum_{k \in J} a_{nk} = O\left(1\right)$$

for all finite subsets I and J of $\mathbf{N}\{0, 1, 2, ...\}$.

Lemma 2.5 ([8], p. 30 or [18], Proposition 6). *A matrix $A = (a_{nk}) \in (l_1, l_\infty)$ if and only if*

$$a_{nk} = O\left(1\right). \tag{2.7}$$

Lemma 2.6 ([17], p. 19 or[8], p. 25-26 or [18], Proposition 17). *A matrix $A = (a_{nk}) \in (l_1, c)$ if and only if conditions* (2.1) *and* (2.7) *are satisfied. Moreover, the equation* (2.3) *holds for every $x = (x_k) \in l_1$.*

Lemma 2.7 ([9], p. 50 or [18], Proposition 77). *A matrix $A = (a_{nk}) \in (l_1, c_0)$ if and only if condition* (2.1) *with $a_k = 0$, and condition* (2.7) *are satisfied.*

Lemma 2.8 ([17], p. 31, see also [8], p. 34-35 or [18], Proposition 77). *A matrix $A = (a_{nk}) \in (l_1, l_1)$ if and only if*

$$\sum_{n} |a_{nk}| = O\left(1\right).$$

Moreover, the equation (2.3) *holds for every $x = (x_k) \in l_1$.*

3. Main results

First we prove

Theorem 3.1. *A matrix $A = (a_{nk}) \in \left(l_1^\lambda, l_1^\mu\right)$ if and only if*

$$Ae = (A_n e) \in l_1^\mu, \ A_n e = \sum_k a_{nk} \tag{3.1}$$

$$\frac{a_{nk}}{\lambda_k} = O(1), \tag{3.2}$$

$$there\, exist\, the\, finite\, limits\, \lim_n a_{nk} := a_k\, for\, all\, k, \tag{3.3}$$

$$\frac{1}{\lambda_k} \sum_n \mu_n \left|a_{nk} - a_k\right| = O(1). \tag{3.4}$$

Proof. Necessity. Let $A \in \left(l_1^\lambda, l_1^\mu\right)$. It is easy to see that $e \in l_1^\lambda$ and $e^k \in l_1^\lambda$. Hence conditions (3.1) and (3.3) hold. Since, from (1.1) we have

$$x_k = \frac{v_k}{\lambda_k} + s; \ s := \lim_k x_k, \ (v_k) \in l_1$$

for every $x := (x_k) \in l_1^\lambda$, it follows that

$$A_n x = \sum_k \frac{a_{nk}}{\lambda_k} v_k + s\mathfrak{A}_n; \ \mathfrak{A}_n := \sum_k a_{nk}. \tag{3.5}$$

As $(\mathfrak{A}_n) \in l_1^\mu$ by (3.1), then, from (3.5) we obtain that the matrix

$$A_\lambda := \left(\frac{a_{nk}}{\lambda_k}\right)$$

transforms this sequence $(v_k) \in l_1$ into c. In addition, for every sequence $(v_k) \in l_1$, the sequence $(v_k/\lambda_k) \in c_0$. But, for (v_k/λ_k), there exists a convergent sequence $x := (x_k)$ with $s := \lim_k x_k$, such that $v_k/\lambda_k = x_k - s$. So we have proved that, for every sequence

$(v_k) \in l_1$ there exists a sequence $(x_k) \in l_1^\lambda$ such that $v_k = \lambda_k (x_k - s)$. Hence $A_\lambda \in (l_1, c)$. This implies, by Lemma 2.6, that condition (3.2) is satisfied and the finite limit

$$\phi := \lim_n A_n x = \sum_k \frac{a_k}{\lambda_k} v_k + s \lim_n \mathfrak{A}_n$$

exists for every $x \in l_1^\lambda$. Writing

$$\mu_n(A_n x - \phi) = \mu_n \sum_k \frac{a_{nk} - a_k}{\lambda_k} v_k + s\mu_n(\mathfrak{A}_n - \lim_n \mathfrak{A}_n), \tag{3.6}$$

we conclude, by (3.1) that the matrix $A_{\lambda,\mu} \in (l_1, l_1)$, where

$$A_{\lambda,\mu} := \left(\mu_n \frac{a_{nk} - a_k}{\lambda_k}\right).$$

Hence condition (3.4) is satisfied by Lemma 2.8.

Sufficiency. Let conditions (3.1) - (3.4) be fulfilled. Then relation (3.5) also holds for every $x \in l_1^\lambda$ and $(\mathfrak{A}_n) \in l_1^\mu$ by (3.1). In addition, $A_\lambda \in (l_1, c)$ and the finite limit ϕ exists for every $x \in l_1^\lambda$ by Lemma 2.6, since (3.2) and (3.3) hold. Hence relation (3.6) holds for every $x \in l_1^\lambda$. As (3.4) is valid, then $A_{\lambda,\mu} \in (l_1, l_1)$ by Lemma 2.8. Therefore, due to (3.1), $A \in \left(l_1^\lambda, l_1^\mu\right)$. □

Remark 3.1. *If* $\lambda_k = O(1)$*, then condition* (3.2) *can be replaced by condition* (2.7)*, and condition* (3.4) *by condition*

$$\sum_n \mu_n |a_{nk} - a_k| = O(1)$$

in Theorem 3.1.

Next we present the following theorems.

Theorem 3.2. *A matrix* $A = (a_{nk}) \in \left(l_1^\lambda, c^\mu\right)$ *if and only if conditions* (3.2) *and* (3.3) *hold,* $Ae \in c^\mu$*, and*

$$there\, exist\, the\, finite\, limits\ \lim_n \mu_n(a_{nk} - a_k) := a_k^\mu\, for\, all\, k, \tag{3.7}$$

$$\mu_n \frac{a_{nk} - a_k}{\lambda_k} = O(1). \tag{3.8}$$

Theorem 3.3. *A matrix* $A = (a_{nk}) \in \left(l_1^\lambda, c_0^\mu\right)$ *if and only if* $Ae \in c_0^\mu$ *and conditions* (3.2), (3.3), (3.7) *with* $a_k^\mu = 0$*, and* (3.8) *hold.*

Theorem 3.4. *A matrix* $A = (a_{nk}) \in \left(l_1^\lambda, l_\infty^\mu\right)$ *if and only if* $Ae \in l_\infty^\mu$ *and conditions* (3.2), (3.3) *and* (3.8) *hold.*

Theorem 3.5. *A matrix* $A = (a_{nk}) \in \left(l_\infty^\lambda, l_1^\mu\right)$ *if and only if conditions* (3.1) *and* (3.3) *hold, and*

$$\sum_k \frac{|a_{nk}|}{\lambda_k} = O(1), \tag{3.9}$$

$$\lim_n \sum_k \frac{|a_{nk} - a_k|}{\lambda_k} = 0, \tag{3.10}$$

$$\sum_{n \in I} \sum_{k \in J} \mu_n \frac{a_{nk} - a_k}{\lambda_k} = O\left(1\right) \tag{3.11}$$

for all finite subsets I and J of $\mathbf{N}$.

As the proofs of Theorems 3.2 - 3.5 are similar to the proof of Theorem 3.1, we only give a short description of the proofs. As $e \in l_1^\lambda$ and $e \in l_\infty^\lambda$, then instead of condition (3.1), for Theorems 3.2 - 3.5 we correspondingly obtain $Ae \in c^\mu$, $Ae \in c_0^\mu$, $Ae \in l_\infty^\mu$, and $Ae \in l_1^\mu$. Also, the matrix transform $A_n x$ for $x := (x_k) \in l_1^\lambda$ or for $x \in l_\infty^\lambda$ may be presented in the form (3.5). In addition, in the proof of Theorems 3.2-3.4 (similarly to the proof of Theorem 3.1) $A_\lambda \in (l_1, c)$, and in the proof of Theorem 3.5 $A_\lambda \in (l_\infty, c)$. Hence the finite limit ϕ exists for every $x \in l_1^\lambda$ by Lemma 2.6, and for every $x \in l_\infty^\lambda$ by Lemma 2.3. Hence relation (3.6) also holds for every $x \in l_1^\lambda$ and for every $x \in l_\infty^\lambda$. The role of the matrix $A_{\lambda,\mu}$ is different in the proof of each theorem: in the proof of Theorem 3.2, $A_{\lambda,\mu} \in (l_1, c)$, in the proof of Theorem 3.3, $A_{\lambda,\mu} \in (l_1, c_0)$, in the proof of Theorem 3.4, $A_{\lambda,\mu} \in (l_1, l_\infty)$, and in the proof of Theorem 3.5, $A_{\lambda,\mu} \in (l_\infty, l_1)$. Therefore, for completing the proof of Theorem 3.2 it is necessary to use Lemma 2.6, for completing the proof of Theorem 3.3 - Lemmas 2.6 and 2.7, for completing the proof of Theorem 3.4 - Lemmas 2.6 and 2.5, and for completing the proof of Theorem 3.5 - Lemmas 2.3 and 2.4.

Remark 3.2. *Using Lemma* 2.3 (c) *we obtain that conditions* (3.9) *and* (3.10) *we can replace by the condition*

$$\text{the series } \sum_k \frac{|a_{nk}|}{\lambda_k} \text{ converges uniformly in } n$$

in Theorem 3.5.

Remark 3.3. *If* $\lambda_k = O(1)$, *then condition* (3.2) *can be replaced by condition* (2.7), *and condition* (3.8) *by condition*

$$\mu_n |a_{nk} - a_k| = O(1)$$

in Theorems 3.2 - 3.4.

Remark 3.4. *If* $\mu_k = O(1)$, *then conditions* (3.7) *and* (3.8) *are redundant in Theorems* 3.2 - 3.4, *and condition* (3.8) *is redundant in Theorem* 3.5.

Corollary 3.1. *Condition* (3.2) *can be replaced by condition*

$$\frac{a_k}{\lambda_k} = O(1) \tag{3.12}$$

in Theorems 3.1 - 3.4.

Proof. It is easy to see that condition (3.12) follows from (3.2) and (3.3). From the other side, conditions (3.3), (3.4) and (3.12) imply the validity of (3.2). Indeed, first from condition (3.4) we obtain that

$$\frac{a_{nk} - a_k}{\lambda_k} = O(1), \tag{3.13}$$

since (μ_n) is bounded from below due to $\mu_n \geq \mu_0 > 0$ for every n. As

$$\frac{a_{nk}}{\lambda_k} = \frac{a_{nk} - a_k}{\lambda_k} + \frac{a_k}{\lambda_k},$$

then

$$\frac{|a_{nk}|}{\lambda_k} \leq \frac{|a_{nk} - a_k|}{\lambda_k} + \frac{|a_k|}{\lambda_k}.$$

Moreover, the finite limits a_k exist by (3.3). Hence condition (3.2) is satisfied by (3.12) and (3.13). □

In the following we characterize the set $\left(c^\lambda, l_1^\mu\right)$.

Theorem 3.6. *A matrix* $A = (a_{nk}) \in \left(c^\lambda, l_1^\mu\right)$ *if and only if conditions* (3.9) *and* (3.11) *are satisfied, and*

$$Ae \in l_1^\mu, Ae^k \in l_1^\mu, A\lambda^{-1} \in l_1^\mu. \tag{3.14}$$

Proof. Necessity. Assume that $A \in \left(c^\lambda, l_1^\mu\right)$. It is easy to see that $e^k \in c^\lambda$, $e \in c^\lambda$ and $\lambda^{-1} \in c^\lambda$. Hence condition (3.14) holds. As equality (3.5) holds for every $x := (x_k) \in c^\lambda$, and the finite limit

$$\tau := \lim_n \mathfrak{A}_n$$

exists due to $Ae \in l_1^\mu$, then the method A_λ transforms this convergent sequence (v_k) into c. Similar to the proof of the necessity of Theorem 3.1, it is possible to show that, for every sequence $(v_k) \in c$, there exists a sequence $(x_k) \in c^\lambda$ such that $v_k = \lambda_k (x_k - s)$. Hence $A_\lambda \in (c, c)$. This implies by Lemma 2.2 that the finite limits a_k and

$$a^\lambda := \lim_n \sum_k \frac{a_{nk}}{\lambda_k}$$

exist, and that condition (3.9) is satisfied. With the help of (3.5), for every $x \in c^\lambda$, we can write by Lemma 2.2 that

$$\phi := \lim_n A_n x = a^\lambda b + \sum_k \frac{a_k}{\lambda_k} (v_k - b) + \tau s, \tag{3.15}$$

where $s := \lim_k x_k$ and $b := \lim_k v_k$. Now, using (3.5) and (3.15), we obtain

$$\mu_n(A_n x - \phi) = \mu_n \sum_k \frac{a_{nk} - a_k}{\lambda_k} (v_k - b) + \mu_n (\mathfrak{A}_n - \tau) s + \mu_n \left(\sum_k \frac{a_{nk}}{\lambda_k} - a^\lambda \right) b. \tag{3.16}$$

As $Ae \in l_1^\mu$ and $A\lambda^{-1} \in l_1^\mu$ by (3.14), then $A_{\lambda,\mu} \in (c_0, l_1)$. Therefore we can conclude by Lemma 2.4 that condition (3.11) holds.

Sufficiency. Assume that conditions (3.9), (3.11) and (3.14) are satisfied. First we note that relation (3.5) holds for every $x \in c^\lambda$ and the finite limits a_k, τ and a^λ exist by (3.14). As (3.9) also holds, then $A_\lambda \in (c, c)$ by Lemma 2.2, and therefore relations (3.15) and (3.16) hold for every $x \in c^\lambda$. As condition (3.11) holds, then $A_{\lambda,\mu} \in (c_0, l_1)$ by Lemma 2.4. In addition, $Ae \in l_1^\mu$ and $A\lambda^{-1} \in l_1^\mu$ by (3.14). Thus, $A \in \left(c^\lambda, l_1^\mu\right)$. □

Theorem 3.7. *A matrix* $A = (a_{nk}) \in \left(c_0^\lambda, l_1^\mu\right)$ *if and only if* $Ae \in l_1^\mu$, $Ae^k \in l_1^\mu$, *and conditions* (3.9) *and* (3.11) *are satisfied.*

Proof. The proof is similar to the proof of Theorem 3.6; we only note that in this case $A\lambda^{-1}$ does not belong into c_0^λ. □

Similarly to the proof of Corollary 3.1 it is possible to prove the following result.

Corollary 3.2. *Condition* (3.9) *can be replaced by condition*

$$\sum_k \frac{|a_k|}{\lambda_k} = O(1) \tag{3.17}$$

in Theorems 3.5 - 3.7.

4. Absolutely λ-conservative matrices and the improvement of absolute λ-convergence

In this section we study the absolute λ-conservativity of matrices and the improvement of absolute λ-convergence by matrices. First we present some examples on λ-conservative matrices, and on matrices, improving the absolute λ-convergence.

Example 4.1. *Let λ be defined by*

$$\lambda_k := (k+1)^{\beta},\ \beta \geq 1. \tag{4.1}$$

Then a lower triangular matrix $A = (a_{nk})$, defined by

$$a_{nk} := \frac{k+1}{(n+1)^{\alpha}};\ \alpha > 0, \tag{4.2}$$

is absolutely λ-conservative if $\alpha > \beta + 3$. For proving it, we show that all conditions of Theorem 3.1 for $\mu_n = \lambda_n$ are satisfied. It is easy to see that conditions (3.2) and (3.3) with $a_k = 0$ hold, and

$$T_k := \frac{1}{\lambda_k}\sum_n \mu_n \left|a_{nk} - a_k\right| = \frac{1}{(k+1)^{\beta-1}}\sum_{n=k}^{\infty}\frac{1}{(n+1)^{\alpha-\beta}} = O(1)$$

for $\alpha - \beta > 1$ or $\alpha > \beta + 1$ (but we have $\alpha > \beta + 3$ by assumption). Thus, condition (3.4) is fulfilled. Finally we show that condition (3.1) holds. As

$$A_n e = \sum_{k=0}^{n}\frac{k+1}{(n+1)^{\alpha}} = \frac{1}{2}\frac{(n+2)(n+1)}{(n+1)^{\alpha}},$$

then

$$\lim_n A_n e = 0,$$

since $\alpha > 2$ by the assumption. Hence

$$S := \sum_n \lambda_n \left|A_n e\right| = \sum_n \frac{1}{(n+1)^{\alpha-\beta}}\left|\sum_{k=0}^{n}(k+1)\right| = \frac{1}{2}\sum_n \left(\frac{n+2}{n+1}\right)\frac{1}{(n+1)^{\alpha-\beta-2}} = O(1)$$

for $\alpha > \beta + 3$.

Example 4.2. *Let λ be defined by (3.5), and μ by*

$$\mu_n := (n+1)^{\gamma},\ \gamma \geq 1.$$

Then the matrix $A = (a_{nk})$, defined by (4.2), improves the absolute λ-convergence if $1 \leq \beta < \gamma < \alpha - 3.$. Indeed, as in Example 4.1, conditions (3.2) and (3.3) with $a_k = 0$ hold. In this case

$$T_k = \frac{1}{(k+1)^{\beta-1}}\sum_{n=k}^{\infty}\frac{1}{(n+1)^{\alpha-\gamma}} = O(1)$$

and

$$S = \frac{1}{2}\sum_n \left(\frac{n+2}{n+1}\right)\frac{1}{(n+1)^{\alpha-\gamma-2}} = O(1),$$

since $\beta \geq 1$ and $\alpha - \gamma > 3$. Hence all conditions of Theorem 3.1 are satisfied and $\mu_n/\lambda_n \neq O(1)$.

We note that, from the point of view of applications, regular matrices are still the most important. Therefore, let us look at some examples of them. Let (p_n) be a sequence of nonzero real numbers and $P_n = p_0 + ... + p_n \neq 0$. Then the Riesz matrix (R, p_n), defined by a lower triangular matrix $A = (a_{nk})$, is given by equalities ([3], p. 29 or p. 131)

$$a_{nk} = \frac{p_k}{P_n}, \ k \leq n.$$

Proposition 4.1. *The Riesz matrix (R, p_n) with $\lim_n P_n = \infty$ is absolutely λ-conservative if and only if*

$$\frac{p_k}{\lambda_k} \sum_{n=k}^{\infty} \frac{\lambda_n}{P_n} = O(1). \tag{4.3}$$

Proof. For the proof it is sufficient to show that all conditions of Theorem 3.1 are satisfied for $\mu_n = \lambda_n$ and $A = (R, p_n)$. It is easy to see that in this case $Ae = 1$ and $a_k = 0$; so conditions (3.1) and (3.3) are satisfied. Therefore the validity of condition (3.4) implies condition (3.2). In addition, condition (3.4) takes now the form (4.3). □

Corollary 4.1. *The Riesz matrix (R, p_n) with $\lim_n P_n = \infty$ and $p_n > 0$ does not absolutely λ-conservative for any unbounded λ.*

Proof. As $p_n > 0$ and $\lim_n P_n = \infty$, then the series

$$\sum_n \frac{p_n}{P_n}$$

diverges by Dini's test. Therefore there exist $M > 0$ such that

$$P_n < M p_n (n + 1) ln(n + 1).$$

Hence

$$L_k := \sum_{n=k}^{\infty} \frac{\lambda_n}{P_n} > \frac{1}{M} \sum_{n=k}^{\infty} \frac{\lambda_n}{p_n (n + 1) ln(n + 1)}.$$

For the boundedness of L_k it is necessary that $p_n/\lambda_n \neq O(1)$. But in this case condition (4.3) does not hold. Therefore, (R, p_n) does not absolutely λ-conservative for any unbounded λ. □

It is easy to see that (R, p_n) is regular for $\lim_n P_n = \infty$ and $p_n > 0$ by Lemma 2.2. Therefore, the question may arise as to whether it exists at all a regular absolutely λ-conservative matrix. The answer is yes. To confirm the statement, let's look at the Zweier matrix $Z_{1/2}$, defined by the lower triangular matrix $A = (a_{nk})$, where (see [9], p. 14) $a_{00} = 1/2$ and

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } k = n - 1 \text{ and } k = n; \\ 0, & \text{if } k < n - 1 \end{cases}$$

for $n \geq 1$. The method $A = Z_{1/2}$ is regular by Lemma 2.2.

Proposition 4.2. *The Zweier matrix $Z_{1/2}$ is absolutely λ-conservative if and only if $\lambda_{k+1}/\lambda_k = O(1)$.*

Proof. For the proof it is sufficient to show that all conditions of Theorem 3.1 are satisfied for $\mu_n = \lambda_n$ and $A = Z_{1/2}$. Similarly to the proof of Proposition 4.1, it is possible to show that conditions (3.1) and (3.3) hold, and condition (3.2) follows from (3.4) Condition (3.4) takes now the form

$$\frac{1}{2\lambda_k}(\lambda_k + \lambda_{k+1}) = \frac{1}{2}\left(1 + \frac{\lambda_{k+1}}{\lambda_k}\right) = O(1),$$

which is equivalent to $\lambda_{k+1}/\lambda_k = O(1)$. This completes the proof. □

Remark 4.1. *It is not difficult to see that* λ*, defined by* (4.1)*, satisfies the condition of Proposition* 4.2*.*

Next we prove the following result.

Lemma 4.1. *If a matrix* $A = (a_{nk})$ *improves the absolute* λ*-convergence, then*

$$\lim_r \sum_{n=k}^{r} |a_{nk} - a_k| = 0, \ r > k \tag{4.4}$$

for every k*.*

Proof. If a matrix $A = (a_{nk})$ improves the absolute λ-convergence, then $\mu_n/\lambda_n \neq O(1)$ by definition. Suppose (4.4) does not hold. Then, for every $\epsilon > 0$ and for every k, there exists a sequence of indexes $\left(i_r^k\right)$, such that

$$\sum_{n=k}^{i_r^k} |a_{nk} - a_k| \geq \epsilon$$

for every k. Hence

$$\frac{1}{\lambda_k} \sum_{n=k}^{i_r^k} \mu_n |a_{nk} - a_k| \geq \epsilon \frac{\mu_k}{\lambda_k}$$

(since μ is monotonically increasing). As a matrix A improves the absolute λ-convergence, then

$$\frac{1}{\lambda_k} \sum_{n=k}^{i_r^k} \mu_n |a_{nk} - a_k| = O(1) \tag{4.5}$$

by Theorem 3.1. But condition (4.5) can be satisfied only in the case if $\mu_n/\lambda_n = O(1)$, which contradicts $\mu_n/\lambda_n \neq O(1)$. Thus condition (4.4) holds. □

Using Lemmas 2.2 and 4.1, we immediately obtain the following corollary.

Corollary 4.2. *Any regular matrix cannot improve the absolute* λ*-convergence for any unbounded* λ*.*

5. Conclusions

In this paper we introduced the notions of absolute convergence with speed (where the speed is defined by a monotonically increasing positive sequence λ), absolute λ-conservativity of a matrix, and improvement of λ-convergence by a matrix. We characterized certain matrix classes involving some spaces with involvement of speeds. Also we studied the absolute λ-conservativity and the improvement of λ-convergence for regular matrices, and proved that the (regular) Zweier matrix is absolutely λ-conservative with respect to some λ, and any regular matrix cannot improve the absolute λ-convergence for any unbounded λ.

The findings of the present paper should inspire to investigate for several other matrix classes characterization by assigning speeds to different classes of participating spaces. For example, it is possible to consider the convergence and absolute convergence with speed in Hahn sequence space h (see [14], [15], [19] - [21]); so for different speeds it is possible to obtain several Hahn sequence spaces with speed and to study matrix transforms between them. The results of the present paper also can be interesting for approximation theory, for example, to compare the approximation orders of Fourier expansions.

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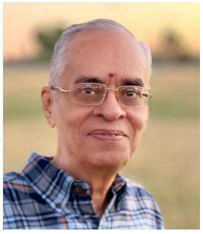

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