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ON SCHAUDER CONJECTURE FOR SET-VALUED MAPPINGS AND QUASI-EQUILIBRIUM PROBLEM

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ABSTRACT. In this paper, the Schauder conjecture for set-valued mappings is studied by using suitable conditions. Also, an existence theorem for a solution of quasi-equilibrium problem in the setting of topological vector spaces is provided.

Keywords: Equilibrium problem, Lower semicontinuity, Quasi-equilibrium problem, Topological vector space.

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1. INTRODUCTION

Generalizing the classical Brouwer theorem for Banach spaces is due to Schauder [10] in 1930. As an effect, the question of whether every compact convex subset of a Hausdorff topological vector space has the fixed point property was put, in August 1935, [[7], Problem 54]; became known as Schauder conjecture. Since then, many partial results have been obtained, but the general case went unsettled. In 2001, Cauty[6] provided a proof for the Schauder conjecture but its proof contained some gaps. In this paper, we study the Schauder conjecture, by using suitable conditions, for set-valued mappings. Then, an existence theorem for a solution of quasi-equilibrium problem in the setting of topological vector spaces is provided. Given a real topological vector space X, a subset C of X, a bifunction $f : C \times C \to \mathbb{R}$ and a set-valued map $K : C \to 2^C$, the quasi-equilibrium problem (QEP) consists in finding $x \in K(x)$ such that $f(x, y) \geq 0$, for all $y \in K(x)$.

When $K(x) = C$ for any $x \in C$, the QEP reduces to the classical equilibrium problem which was introduced by Oettli and Blum in[5].

In scientific contexts the term equilibrium has been widely used at least in Physics, Chemistry, Engineering and Economics within different frameworks, relying on different mathematical models such as optimization, variational inequalities and noncooperative games among others (see, for more details, [5]).

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The main tool for proving an existence theorem for a solution of the QEP is using of fixed point theorems. The general form of fixed point for set-valued mappings is Kakutani theorem [10] and its new version of it given in [9]. Hence, the authors studied QEP in Banach spaces ([1],[4]) or more general in locally convex spaces (see, section 4 of [3]).

In this article, by establishing a fixed point theorem in topological vector spaces for set-valued mapping, we state an existence theorem of a solution for QEP in topological vector spaces without using general monotonicity and general continuity.

2. Preliminaries

Let X be a real topological vector space, a subset C of X, a bifunction $T : C \times C \to \mathbb{R}$ and a set-valued map $K : C \to 2^C$, the quasi-equilibrium problem (QEP) consists in finding $x \in C$ such that

$$x \in K(x) \text{ such that } T(x, y) \geq 0, \ \forall\, y \in K(x).$$

It is easy to see that the solution set of QEP equals to the fixed points set of the set-valued mapping $S : C \to 2^C$ is defined by

$$S(x) = \{y \in K(x) : T(y, z) \geq 0, \ \forall z \in K(x)\}.$$

The following definition is needed in the next result.

Definition 2.1. *Let X and Y be sets and $\varphi : X \to 2^Y$ a set-valued mapping. A function $f : X \to Y$ is said to be a selection of φ, if*

$$f(x) \in \varphi(x), \ \forall x \in X.$$

The following theorem provides, under suitable conditions, the existence of a continuous selection for the set-value mapping which it will play a crucial role in the next section.

Theorem 2.1. [12] *Let X be a compact Hausdorff space and Y be a topological vector space. Suppose $\varphi : X \to 2^Y$ is a set-valued mapping such that*

(a) *For each $x \in X$, $\varphi(x)$ is nonempty and convex,*
(b) *For each $y \in Y$, $\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}$ is open in X.*

Then φ has continuous selection, that is there exists a continuous function $f : X \to Y$ such that $f(x) \in \varphi(x)$ for all $x \in X$.

Theorem 2.2. [11] *Let X be a Hausdorff topological vector space and let C be a nonempty, closed and convex subset of X. Suppose that T is a continuous mapping from C into a compact subset Z of C. Assume, moreover that the following condition is satisfied:*

For every x in Z and every neighbourhood W of x there exists a neighbourhood V of x such that

$$co(V \cap Z) \subseteq W.$$

Then T has a fixed point in C.

Lemma 2.1. [8] *Let D be a nonempty subset of a topological vector space X and $F : D \longrightarrow 2^X$ be a mapping with closed values in K and for every finite subset $\{x_1, x_2, ..., x_n\}$ of D, $conv\{x_1, x_2, ..., x_n\}$ is contained in $\bigcup_{i=1}^n F(x_i)$, (where conv denotes the convex hull). Assume that there exists a nonempty compact convex subset B of D such that $\bigcap_{x \in B} F(x)$ is compact. Then $\bigcap_{x \in D} F(x) \neq \emptyset$.*

3. Main Result

The following result considers the Schauder conjecture for set-valued mappings. Moreover, it is an extension of Kakutani fixed point theorem from Hausdorff locally convex spaces to Hausdorff topological vector spaces for set-valued mappings which are not necessarily upper semi-continuous and do not have convex.

Theorem 3.1. *Let C be a nonempty compact convex subset of a Hausdorff topological vector space X, Z a nonempty closed subset of C and $S : C \to 2^Z$ be a set-valued mapping such that the values of S are non-empty and convex, with open fibers, that is $S^{-1}(y) = \{x \in C : y \in S(x)\}$ is open in C, for all $y \in C$. If for every $x \in Z \subset C$ and every neighbourhood W of x there exists a neighbourhood V of x such that*

$$co(V \cap Z) \subseteq W,$$

then S admits a fixed point.

Proof. It is easy to see that the set-valued mapping S satisfies all the conditions of Theorem 2.1. Thus there exists a continuous function $f : C \to Z$ such that $f(x) \in S(x)$ for all $x \in C$. Hence it follows from Theorem 2.2 that there exists $x \in C$ such that $x = f(x) \in S(x) \subset C$. This completes the proof. □

The next result extends the corresponding results given in [1, 2, 3, 4] from Banach spaces (even Hausdorff locally convex spaces) to Hausdorff topological spaces without using general monotonicity and general continuity.

Theorem 3.2. *Let C be a non-empty compact and convex subset of a Hausdorff topological vector space of X, $F : C \times C \to \mathbb{R}$ a function and $K : C \to 2^C$ a set-valued mapping with open fibers. If the following conditions are satisfied:*

(a) *For each $x \in C$, the set $\{u \in K(x) : F(u, z) \geq 0, \forall z \in K(x)\}$ is nonempty and convex;*
(b) *For each $(x, u) \in C \times C$, the $\{z \in K(x) : F(u, z) < 0\}$ is convex and open ;*
(c) *For each $x \in C$, $K(x)$ is nonempty, closed and convex set and $F(x, x) \geq 0$.*
(d) *for every $x \in C$ and every neighbourhood W of x there exists a neighbourhood v of x such that*

$$co(V \cap Z) \subseteq W,$$

then the solution set of QEP is non-empty.

Proof. We define the set-valued map $S : C \to 2^C$ by

$$S(x) = \{y \in K(x) : F(y, z) \geq 0, \ \forall z \in K(x)\}.$$

It is easy to verify that the solution set of QEP equals to the fixed point set of S. Hence it is enough to show that S has a fixed point. We show that S fulfils all the assumptions of Theorem 3.1. It is easy to see that the values of S are nonempty, that is, $S(x) \neq \emptyset$, for all $x \in C$. Indeed, for given $x \in C$, we define the set-valued mapping $g_x : K(x) \to 2^{K(x)}$, where

$$g_x(z) = \{y \in K(x) : F(y, z) \geq 0\}.$$

By (b) the values of g_x are closed subsets of C. If $Z = \{z_1, z_2, ..., z_n\} \subset K(x)$ such that $co\{z_1, z_2, ..., z_n\} \not\subseteq \bigcup_{k=1}^{n} g_x(z_k)$, then there exists $v \in co\{z_1, z_2, ..., z_n\}$ such that $v \notin \bigcup_{n=1}^{k} g_x(z_x)$. Hence by (b), $F(v, v) < 0$ which is a contradiction. Then the family $\{g_x(z) : z \in K(x)\}$ satisfies all the conditions of Lemma 2.1 and then the set $\bigcap_{z \in K(x)} g_x(z)$ is nonempty. This means that $S(x)$ is nonempty. Also, by (a), for all $x \in C$, $S(x)$ is convex and it is straightforward to see , for each x in C, that $S^{-1}(x) = K^{-1}(x) \cap K^{-1}((\pi_2(\{x\} \times C) \cap T^{-1}([0, \infty)))$ which is open, note K has open fibers. Hence the set-valued mapping S verifies all the conditions of Theorem 3.1 and this completes the proof. □

The next result is topological vector space version of Theorem 1 given in [2] without using the lower semi-continuity conditions on the maps.

Theorem 3.3. *Let C be a nonempty compact convex set in a Hausdorff topological vector space X and $H, R : C \to 2^C$ be two set-valued mappings with closed convex values. If the following conditions are satisfied*

(a) *H and (RoH) have open fibers;*
(b) *if $x \in C$ and $y \in H(x)$ then $H(x) \cap R(y) \neq \emptyset$;*
(c) *for each $z \in C, (R^{-1}(z))^c$ is convex and $z \in R(z)$;*
(d) *for every $x \in C$ and every neighbourhood W of x there exists a neighbourhood V of x such that $co(V \cap C) \subseteq W$;*

then, there exists $x^* \in C$ such that $x^* \in H(x^*) \cap (\cap_{y \in H(x^*)} R(y))$.

Proof. For each $x \in C$, we define the set-valued mapping $Q_x : H(x) \to 2^{H(x)}$ by $Q_x(y) = H(x) \cap R(y)$. It follows from the hypotheses the values of Q_x are nonempty closed and convex. Moreover, if $co\{y_1, ..., y_n\} \not\subseteq \cup_{i=1}^n Q_x(y_i) = H(x) \cap (\cup_{i=1}^n R(y_i))$, then there exists

$$z = \sum_{i=1}^{n} \lambda_i y_i \in co\{y_1, ..., y_n\} : z \notin \cup_{i=1}^n R(y_i) \Longrightarrow z \notin R(y_i).$$

Then

$$y_i \notin R^{-1}(z) \Longrightarrow y_i \in (R^{-1}(z))^c \Longrightarrow z = \sum_{i=1}^{n} \lambda_i y_i \in (R^{-1}(z))^c$$

So $z \notin R(z)$ which is a contradiction. Hence, the set-valued mapping Q_x satisfies all the conditions of Theorem 2.1, and so

$$\emptyset \neq \bigcap_{y \in H(x)} Q_x(y) = H(x) \cap (\cap_{y \in H(x)} R(y)).$$

Now, we define the set-valued mapping $P : C \to 2^C$ by $P(x) = H(x) \cap (\cap_{y \in H(x)} R(y))$. The values of P are compact and convex and further

$$z \in P^{-1}(x) \Leftrightarrow x \in P(z) = H(z) \cap (\cap_{y \in H(z)} R(y)) \Leftrightarrow z \in H^{-1}(x) \wedge (x \in R(y), \forall y \in H(z)) \Leftrightarrow$$

$$z \in H^{-1}(x) \wedge (x \in R(H(z))) \Leftrightarrow z \in H^{-1}(x) \cap (RoH)^{-1}(x).$$

Consequently P fulfils all the conditions of Theorem 3.1. This completes the proof. □

Remark 3.1. *In Theorem 3.3, if X is locally convex space then we can relax conditions (a) and (d) when the set-valued mappings H and R are closed. Moreover, if in Theorem 3.3 we take, respectively, $H(x) = C$, for all $x \in C$, and $R(x) = C$, for all $x \in C$, then we obtain, respectively, new version of Theorem 2.1 and the Kakutani fixed point theorem.*

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