TWMS J. App. and Eng. Math. V.14, N.4, 2024, pp. 1466-1472

DISCRETE LINEAR QUADRATIC OPTIMIZATION PROBLEM WITH CONSTRAINTS IN THE FORM OF EQUALITIES ON CONTROL ACTION

F.A. ALIEV[1,2*], N.S. HAJIYEVA[1], §

ABSTRACT. In the paper the discrete linear quadratic optimization problem, where, over a certain part of the time interval, some coordinates of the control actions are known constants. These equalities in the form of a penalty function with a certain weight are added to the quadratic functional and the corresponding discrete Euler-Lagrange equation is constructed, the solution of which is constructed using a discrete fundamental matrix. Then, an explicit expression of control actions over the entire time interval is given. The results are illustrated using the example of the vertical motion of a flying vehicle.

Keywords: linear quadratic optimization problem, discrete Euler-Lagrange equation, control action, optimal program trajectory, the system of linear algebraic equation, fundamental matrix, flying object.

AMS Subject Classification: 49M25, 49N20.

1. INTRODUCTION

Typically, the linear quadratic optimization problems of the standard [2, 4, 5, 6, 11, 13, 18] form are solved by different methods, for example, sweep methods [3, 4, 14, 15], methods of reducing solutions to linear algebraic equations with dimensions greater than the size of the original system [16], the Moszynski method [17], etc. However, when the linear quadratic optimization problems with equality constraints on some control coordinates for part of the time interval are considered, the situation changes; solutions to the optimization problem are reduced to solving a system of linear algebraic equations for the missing boundary conditions. Next, the control and trajectories are found from the corresponding Euler-Lagrange equations.

In this paper, a discrete optimization problem with a quadratic functional is formulated, where there are restrictions in the form of equalities on some coordinates of control actions.

[1] Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan.
[2] Institute of Information Technology, Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan.
e-mail: f_aliev@yahoo.com; ORCID: https://orcid.org/0000-0001-5402-8920.
[*] Corresponding author.
e-mail: nazile.m@mail.ru; ORCID: https://orcid.org/0000-0002-9227-9007.
§ Manuscript received: September 22, 2022; accepted: January 15, 2023.

Using the Euler-Lagrange method, this constraint is added to the functional in the form of a penalty with a certain weight. Further, an extended quadratic functional is constructed [3, 4, 11, 12] and the Euler-Lagrange equation for the discrete case is described [19, 20, 21]. To find trajectories and controls, concrete formulas are given. The results are illustrated using the example of the vertical movement of a flying vehicle [1].

2. Problem Statement of Continuous Case

As is known, the similar linear quadratic optimization problem in the continuous case, i.e. the motion of an object on an interval $[0,\, T+\Delta]$ is described by the system of linear differential equations

$$\dot{x}(t) = Fx(t) + Gu(t) + V \tag{1}$$

with initial condition

$$Hx(0) = q, \tag{2}$$

where $x^{'}(t) = \left[\begin{array}{cccc} z_1(t) & z_2(t) & z_3(t) & z_4(t) \end{array}\right]-$ phase coordinates of n-dimensional object, $z_i(t)$ are k_i- dimensional vectors,

$(\overline{i=1,4})$, $(\sum_{i=1}^{4} k_i = n)$, F, G and H

are constant matrices with dimensions $n\times n$, $n\times m$, $m\times n$, respectively. The sign $'$ means the transpose operation. V is an external disturbance with dimension $n\times 1$, $u^{'}(t) = \left[\begin{array}{cc} u_1(t) & u_2(t) \end{array}\right]$- m-dimensional vector of control actions, $u_1(t)$, $u_2(t)$ are (m-p) and p-dimensional vectors, correspondingly. Let $G = \left[\begin{array}{cc} G_1 & G_2 \end{array}\right]$,where G_1, G_2 are $n\times(m-p)$ and $n\times p$ matrices, correspondingly.

Assume that there is a condition at the time instant T

$$x(T+0) = F_\delta x(T-0) + G_1 u_1(T-0), \tag{3}$$

where F_δ is a known matrix with corresponding dimension.

It is required to find such control action $u(t)$ and the corresponding trajectory $x(t)$ with boundary conditions (2), (3) such that the equation (1) is satisfied and the following functional received the minimum value

$$\begin{array}{l} J = \frac{1}{2}(x(T-0)-x_d)'N(x(T-0)-x_d) + (u(T-0)-C)'\delta(u(T-0)-C) + \\ +\frac{1}{2}\int_0^T (x'(t)Qx(t) + 2x'(t)Ku(t) + u'(t)Ru(t))dt + \frac{1}{2}\int_T^{T+\Delta}(u(t)-C)'\gamma(u(t)-C)dt, \end{array} \tag{4}$$

where $N\geq 0$, $\delta=\delta'>0$, $R=R'>0$, $Q=Q'\geq 0$, $\gamma=\gamma'>0$, K and x_d, C are known matrices and vectors with corresponding dimensions. Note that $C = \left[\begin{array}{c} C_1 \\ 0 \end{array}\right]$, C_1 is a known constant parameter.

An algorithm for solving problem (1)-(4) has been given [22]. However, when the function is piecewise constant, it is required to consider the discrete analogue of problem (1)-(4).

3. Discrete case

If we discretize the problem (1)-(4), we'll receive the following discrete problem analogously (1)-(4)

$$x_{i+1} = \Phi x_i + \Gamma u_i + \omega,\ \ 0\leq i\leq N-1,\ \ N+1\leq i\leq N+\Delta-1, \tag{5}$$

$$Hx_0 = q, \tag{6}$$

$$x_{N+1} = F_\delta x_N + G_1 u_N, \tag{7}$$

$$\begin{aligned} J = \tfrac{1}{2}(x_N - x_d)' N (x_N - x_d) + (u_N - C)'\delta(u_N - C) + \\ \tfrac{1}{2}h \textstyle\sum_{i=0}^{N-1}(x_i' Q x_i + 2x_i' K u_i + u_i' R u_i) + \tfrac{1}{2}h \sum_{i=M+1}^{N+\Delta-1}(u_i - C)'\gamma(u_i - C), \end{aligned} \tag{8}$$

where
$\Phi = E + hF,\ \Gamma = hG,\ \omega = hV,$ h is a discretization step, E is an identity matrix with corresponding dimension.

4. Euler-Lagrange equation for the problem (5)-(8) and construction solution on the interval $0 \le i \le N + \Delta$.

Using the results of [7, 8, 9, 10], we get the discrete Euler-Lagrange equations on the intervals $0 \le i \le N-1,\ N+1 \le i \le N+\Delta-1$ for the problem (5)-(8) in the following form, correspondingly

$$\begin{bmatrix} x_{i+1} \\ \lambda_{i+1} \end{bmatrix} = \Phi_1 \begin{bmatrix} x_i \\ \lambda_i \end{bmatrix} + \Phi_2,\ 0 \le i \le N-1, \tag{9}$$

$$\begin{bmatrix} x_{i+1} \\ \lambda_{i+1} \end{bmatrix} = \Phi_3 \begin{bmatrix} x_i \\ \lambda_i \end{bmatrix} + \Phi_4,\ N+1 \le i \le N+\Delta-1 \tag{10}$$

with boundary conditions

$$\lambda_{N+1} = (F_\delta')^{-1} h \lambda_N - (F_\delta')^{-1} N'(x_N - x_d), \tag{11}$$

$$u_N = C - \delta^{-1} G_1' \lambda_{N+1}, \tag{12}$$

$$\lambda_{N+\Delta} = 0, \tag{13}$$

$$H'\nu + \lambda_0 = 0, \tag{14}$$

where

$$\Phi_1 = \begin{bmatrix} \Phi - \Gamma R^{-1}K' - (\Phi' - KR^{-1}\Gamma')^{-1}\Gamma R^{-1}\Gamma'(-Q' + KR^{-1}K') & -(\Phi' - KR^{-1}\Gamma')^{-1}\Gamma R^{-1}\Gamma' \\ (\Phi - \Gamma R^{-1}K')(-Q' + KR^{-1}K') & (\Phi' - KR^{-1}\Gamma')^{-1} \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} E & -(\Phi' - KR^{-1}\Gamma')^{-1}\Gamma R^{-1}\Gamma' \\ 0 & (\Phi' - KR^{-1}\Gamma')^{-1} \end{bmatrix} \begin{bmatrix} \omega \\ 0 \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} \Phi & -\Gamma\gamma^{-1}\Gamma'(\Phi^{-1})' \\ 0 & (\Phi^{-1})' \end{bmatrix},$$

$$\Phi_4 = \begin{bmatrix} E & \Gamma\gamma^{-1}\Gamma' \\ 0 & E \end{bmatrix}^{-1} \begin{bmatrix} \Gamma C + \omega \\ 0 \end{bmatrix},$$

$$\lambda_0 = hQx_0 + hKu_0 + h\Phi'\lambda_1, \tag{15}$$

We assume that $\det(\Phi) \ne 0,\ \det(\Phi' - KR^{-1}\Gamma') \ne 0.$

The control u_i on the intervals $1 \le i \le N-1,\ N+1 \le i \le N+\Delta-1$ is defined as, correspondingly

$$u_i = -R^{-1}K'x_i - R^{-1}\Gamma'\lambda_{i+1}, \tag{16}$$

$$u_i = C - \gamma^{-1}\Gamma'\lambda_{i+1}. \tag{17}$$

Substituting the expression of (11) into (12), writing the obtained expression of u_N into (7), then we have

$$x_{N+1} = F_\delta x_N + G_1 C - G_1\delta^{-1}G_1'\lambda_{N+1}. \tag{18}$$

Combining (17) with (11), we get

$$\begin{bmatrix} x_{N+1} \\ \lambda_{N+1} \end{bmatrix} = \Phi_5 \begin{bmatrix} x_N \\ \lambda_N \end{bmatrix} + \Phi_6, \tag{19}$$

where

$$\Phi_5 = \begin{bmatrix} F_\delta + G_1\delta^{-1}G'(F_\delta')^{-1}N' & -G_1\delta^{-1}G'(F_\delta')^{-1}h \\ -(F_\delta')^{-1}N' & (F_\delta')^{-1}h \end{bmatrix},$$

$$\Phi_6 = \begin{bmatrix} G_1C - G_1\delta^{-1}G'(F_\delta')^{-1}N'x_d \\ (F_\delta')^{-1}N'x_d \end{bmatrix}.$$

We assume that $\det(F_\delta') \neq 0$.
Combining (6), (13), (14) we have

$$\begin{bmatrix} H & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & H' \\ 0 & 0 & 0 & E & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda_0 \\ x_{N+\Delta} \\ \lambda_{N+\Delta} \\ \nu \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix}. \tag{20}$$

Let us introduce the fundamental solutions of the systems (9) and (10) as follows

$$\begin{bmatrix} x_i \\ \lambda_i \end{bmatrix} = \Phi_1^i \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} + \sum_{j=i-1}^{0} \Phi_1^j \Phi_2, \quad 1 \le i \le N, \tag{21}$$

$$\begin{bmatrix} x_i \\ \lambda_i \end{bmatrix} = \Phi_3^{i-N-1} \begin{bmatrix} x_{N+1} \\ \lambda_{N+1} \end{bmatrix} + \sum_{j=i-1}^{N+1} \Phi_3^{i-1-j}\Phi_4, \quad N+2 \le i \le N+\Delta. \tag{22}$$

Thus, by means of (19), (21), (22) we get

$$\begin{gathered} \begin{bmatrix} x_{N+\Delta} \\ \lambda_{N+\Delta} \end{bmatrix} = \Phi_3^{\Delta-1}\Phi_5\Phi_1^N \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} + \Phi_3^{\Delta-1}\Phi_5 \textstyle\sum_{j=N-1}^{0} \Phi_1^j\Phi_2 + \Phi_3^{\Delta-1}\Phi_6 + \\ + \textstyle\sum_{j=N+\Delta-1}^{N+1} \Phi_3^{N+\Delta-1-j}\Phi_4. \end{gathered} \tag{23}$$

Further, from (20), (21) we get the following system of linear algebraic equations as follows

$$\begin{bmatrix} -\Phi_3^{\Delta-1}\Phi_5\Phi_1^N & E & 0 \\ H_1 & H_2 & H_3 \\ H_4 & H_5 & H_6 \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda_0 \\ x_{N+\Delta} \\ \lambda_{N+\Delta} \\ \nu \end{bmatrix} =$$

$$= \begin{bmatrix} \Phi_3^{\Delta-1}\Phi_5 \sum_{j=N-1}^{0} \Phi_1^j\Phi_2 + \Phi_3^{\Delta-1}\Phi_6 + \sum_{j=N+\Delta-1}^{N+1} \Phi_3^{N+\Delta-1-j}\Phi_4 \\ q \\ 0 \end{bmatrix}, \tag{24}$$

where

$$H_1 = \begin{bmatrix} H & 0 \\ 0 & E \end{bmatrix}, \; H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; H_3 = \begin{bmatrix} 0 \\ H' \end{bmatrix}, \; H_4 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; H_5 = \begin{bmatrix} 0 & E \end{bmatrix}, \; H_6 = 0.$$

After finding $\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}$ from (24) we solve the system of difference equation (21), i.e. we find the trajectory x_i on the interval $0 \le i \le N$. Then according (15) we find control u_0.Further, we obtain the control u_i from (16) on the interval $1 \le i \le N$. Thus we get the control u_i on the interval $0 \le i \le N$.Then from (19) we find $\begin{bmatrix} x_{N+1} \\ \lambda_{N+1} \end{bmatrix}$.Further, substituting the obtained expression of $\begin{bmatrix} x_{N+1} \\ \lambda_{N+1} \end{bmatrix}$ into (22), we solve the system of difference equation (22), i.e. we find the trajectory x_i on the interval $N+1 \le i \le N+\Delta$. The control u_i is defined from (17) on the interval $N+1 \le i \le N+\Delta$.

5. Example

Let us consider an example that describes the vertical movements of a flying object on the interval $0 \le i \le N+\Delta$.We assume that the object moves only vertically. In this case, from (5) and (6) the parameters have the form:

$x^{'}(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}$, $\Phi = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$, $\omega = \begin{bmatrix} 0 \\ -0.01g \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $q = z_{10}$ g is a gravitational acceleration.

Now introduce the following functional which is to be minimized

$$J = \frac{1}{2}(x_N - x_d)'N(x_N - x_d) + (u_N - C)'\delta(u_N - C) + \\ \frac{1}{2}h\sum_{i=0}^{N-1}(x_i'Qx_i + 2x_i'Ku_i + u_i'Ru_i) + \frac{1}{2}h\sum_{i=M+1}^{N+\Delta-1}(u_i - C)'\gamma(u_i - C),$$

where

$N = \begin{bmatrix} \chi & 0 \\ 0 & \alpha \end{bmatrix}$, $x_d = \begin{bmatrix} z_d \\ 0 \end{bmatrix}$, $Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$, $K = \begin{bmatrix} k_1 \\ 0 \end{bmatrix}$, $\chi, \alpha, z_d, \delta, c, R, q_1, q_2, k_1, \gamma$ are given positive numbers.

Using the procedure outlined in sections 3-4, we can find x_i solving on the interval $0 \le i \le N+\Delta$ and the control action u_i determining by formulas (16), (17) on the interval $0 \le i \le N+\Delta$. The results in the discrete case coincide with the results in the continuous case [20] of the order of 10^{-5}. Let us introduce the graphs of program trajectory $z_1(i)$ and control action u_i on the interval $0 \le i \le N+\Delta$:

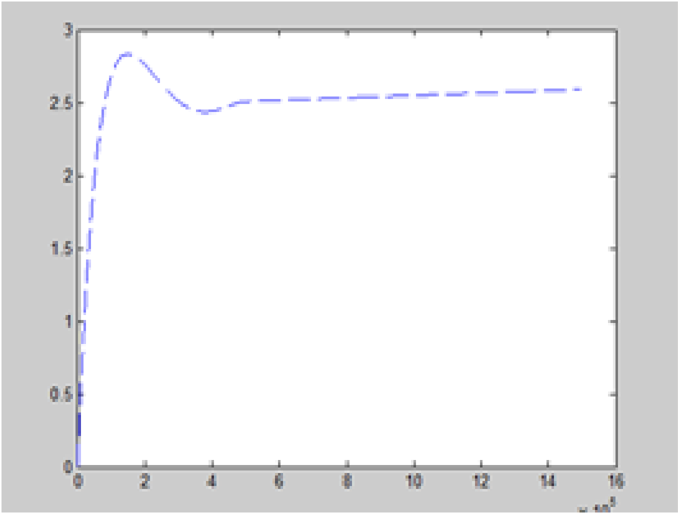

Figure 1. Changing $z_1(i)$ on the interval $0 \leq i \leq N + \Delta$.

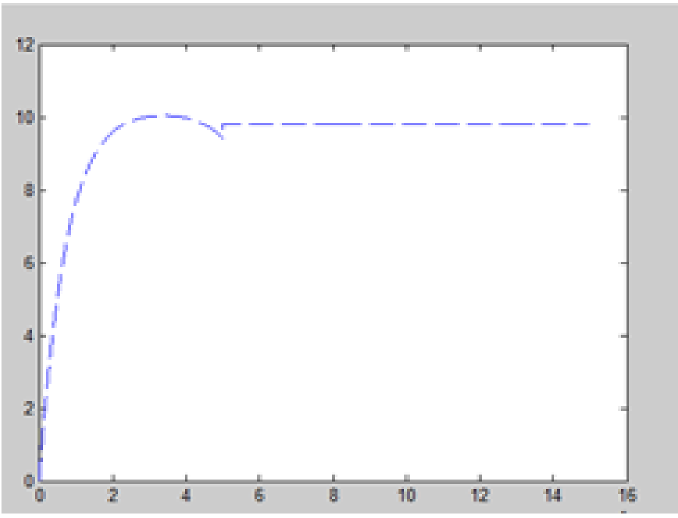

Figure 3. Changing $u_1(i)$ on the interval $0 \leq i \leq N + \Delta$.

6. Conclusions

In the current paper, a discrete linear quadratic optimization problem was investigated, where in a known part of the time interval some coordinates of the control actions in the form of a penalty function with a certain weight are added to the quadratic functional. Constructing the discrete Euler-Lagrange equations and obtaining the control and trajectories using a discrete fundamental matrix.

References

[1] Larin V.B., Tunik A.A. Ilnytska S.I., (2020), Some Algorithms for Unmanned Aerial Vehicles Navigation, Outskirts, 204 p.

[2] Aliev F.A., Larin V.B., (1998), Optimization of Linear Control System, London: Gordon and Breach, 270 p.

[3] Aliev F.A., (1989), Methods for solving applied problems of optimization of dynamic systems, Baku: Elm, 320 p. (in Russian)

[4] Bryson A., Ho Yu Shi, (1972), Applied theory of optimal control, M: Mir. (in Russian)

[5] Aliev F.A., (1986), Optimal control problem for a linear system with nonseparated two-point boundary conditions, Differential equations, (2), pp.345-347. (in Russian)
[6] Aliev F.A., (1985), Optimization problem with two-point boundary conditions, News of the USSR Academy of Sciences, Ser.Techn. Cybernetics, (6), pp.138-146. (in Russian)
[7] Mutallimov M.M., Hajiyeva N.S., (2023), Construction of optimal program trajectories and controls for vertical movement of a quadcopter, Proceedings of IAM, 12(2), pp.166-180. (in Russian)
[8] Aliev F.A., Mutallimov M.M., Velieva N.I., Safarova N.A., Jafarov A.G., (2023), Algorithms for solving the stabilization problems of the quadcopter motion by GPS data, Applied and Computational Mathematics, 22(4), pp.520-527.
[9] Aliev F.A., Javadov G.N., Mutallimov M.M. Problems of stabilizing the movement of a quadcopter according to GPS data, ANAS, Reports, 79(1-2), pp.10-16. (in Russian)
[10] Aliev F.A., Sushchenko O.A., Mutallimov M.M., Javadov N.G., Mammadov F.F., Maharramov R.R., (2023), Algorithm for quadcopter motion stabilization taking into account data of inertial navigation system, TWMS J. Pure Appl. Math., 14(2), pp.278-289.
[11] Kvakernaak H., Sivan R., (1977), Linear Optimal Control Systems, Moscow: Mir, 656 p. (in Russian)
[12] Aliev F.A., Larin V.B., Naumenko K.I., Suntsev V.N., (1978), Optimization of Linear Time-Invariant Control Systems, Kyiv: Naukova Dumka, 327 p. (in Russian)
[13] Andreev Y.N., (1976), Control of Finite-Dimensional Linear Objects, Moscow, Science, 424 p. (in Russian)
[14] Aliev F.A., Mutallimov M.M., Maharramov I.A., Huseynova N.S., Amirova L.I., (2020), About One Sweep Algorithm for Solving Linear-Quadratic Optimization Problem with Unseparated Two-Point Boundary Conditions, Sahand Communications in Mathematical Analysis Search, 17(1), pp.99-10.
[15] Polak E., (1971), Computational Methods in Optimization, Unified Approach, Academic Press.
[16] Abramov A.A., (1961), On the transfer of boundary conditions for systems of linear ordinary differential equations (a variant of the sweep method), Journal of Computational Mathematics and Mathematical Physics, 1, pp.542-545. (in Russian).
[17] Moszynski K., (1964), A method of solving the boundary value problem for a system of linear ordinary differensial equations, Algorytmy, 11, pp.25-43.
[18] Ismaylov N.A., Mukhtarova N.S., Askerov I.M., (2013), Algorithm for solving an optimization problem with boundary control, Report of National Academy of Sciences of Azerbaijan.
[19] Ismaylov N.A., Mukhtarova N.S., (2013), Method for solution of the problem of discrete optimization with boundary control, Proceedings of IAM, 2(1), pp.20-27.
[20] Aliev F.A., Safarova N.A., (2000), Inverse problem for discrete optimal periodic control systems, Journal of Automation and Information Sciences, 32(6), pp.43-49.
[21] Aliev F.A., (1980), Optimization of Discrete Systems, Dokl. AN of Az. SSR, 36(1).
[22] Hajiyeva N.S., Namazov A.A., Veliyeva N.I., Mutallimov M.M., Salehov A., Aliev F.A., Algorithm for solution of linear quadratic optimization problem with constraint in the form of equalities on control, Applied and Computational Mathematics (in Press).
[23] Mohiuddine S.A., Arun Kajla, Sheetal Desshwall, Abdullah Alotabi, (2023), Approximation by Szasz -Durrmeyer type operators, TWMS J. Pure Appl. Math., V.14, N.2, pp.214-231.
[24] Kalantarova J.V., Aliyeva G.N.,(2022), Decay of solutions of damped Kirchhoff and beam equations, TWMS J. Pure Appl. Math., V.13, N.1, pp.119-128.
[25] Temirbekov N.M., Temirbekova L.N., Nurmangaliyeva, (2022), Numerical solution of the first kind Fredholm ıntegral equations by projection methods with wavelets as basis functions. TWMS J. Pure Appl. Math., V.13, N.1, pp.105-118
[26] Aliyev F.A., Mutallimov M.M.,Tunik A.A., Velieva N.I., Rasulova U.Z., Mirsaabov S.M., (2022), Constructing an optimal controller maneuver of quadrotor in 3-D space. TWMS J. Pure Appl. Math., V.13, N.2, pp.211-221.
[27] Tunk A.A., Sushchenko O.A., Ilnytska S.I., Kondratuk V. M., Determination of the quadrotor mathematical models for control systems synthesis and simulation, (2023), Appl. Comput. Math., V.22, N.1, pp.116-132

Aliev Fikret for the photo and short autobiography, see TWMS J. of Appl. and Engin. Math., Vol.13, No.4.

Hajiyeva Nazile for the photo and short autobiography, see TWMS J. of Appl. and Engin. Math., Vol.13, No.4.