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FORK-DECOMPOSITION OF TOTAL GRAPH OF CORONA GRAPHS

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Abstract. Let $G = (V, E)$ be a graph. Then the total graph of G is the graph $T(G)$ with vertex set $V(G) \cup E(G)$ in which two elements are adjacent if and only if they are either adjacent or incident with each other. The corona of two graphs G_1 and G_2, is the graph formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 and is denoted by $G_1 \circ G_2$. Fork is a tree obtained by subdividing any edge of a star of size three exactly once. A decomposition of G is a partition of $E(G)$ into edge disjoint subgraphs. If all the members of the partition are isomorphic to a subgraph H, then it is called a H-decomposition of G. In this paper, we investigate the existence of necessary and sufficient conditions for the fork-decomposition of Total graph of certain types of corona graphs which gives a partial solution for the conjecture of Barat and Thomassen [4] for graphs of small edge connectivity.

Keywords: Graph decomposition, Total graph, Corona graph, Fork decomposition.

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1. Introduction

We consider only simple, finite and undirected graphs. Let K_n denote the complete graph on n vertices and $\overline{K}_n$ be the null graph. Let $K_{m,n}$ denote the complete bipartite graph with parts of sizes m and n. Let P_k denote the path of length $k-1$ and S_k denote the star of size $k-1$. A vertex of degree 1 is called a *pendant vertex* and the vertex adjacent to it is called a *support*. A *tree* is a connected acyclic graph.

Definition 1.1. *The total graph of G, denoted by $T(G)$ is defined as follows: the vertex set of $T(G)$ is $V(G) \cup E(G)$; two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:*

(1) *$x, y \in V(G)$ and x is adjacent to y in G.*
(2) *$x, y \in E(G)$ and x is adjacent to y in G.*
(3) *$x \in V(G)$, $y \in E(G)$ and x is incident with y in G.*

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Remark 1.1. *The number of edges in the total graph is* $2|E(G)| + \frac{1}{2} \sum_{v \in V(G)} (d(v))^2$.

Definition 1.2. *The corona of two graphs* G_1 *and* G_2*, is the graph* $G_1 \circ G_2$ *formed from one copy of* G_1 *and* $|V(G_1)|$ *copies of* G_2 *where the* i^{th} *vertex of* G_1 *is adjacent to every vertex in the* i^{th} *copy of* G_2.

Definition 1.3. [14] *The Cartesian product of two graphs* G *and* H*, denoted by* $G \square H$*, is the graph whose vertex set is* $V(G) \times V(H)$*; two vertices* (g, h) *and* (g', h') *are adjacent in* $G \square H$ *precisely if* $g = g'$ *and* $hh' \in E(H)$*, or* $gg' \in E(G)$ *and* $h = h'$.

Terms not defined here are used in the sense of Bondy and Murty [5].

2. Literature Review

A decomposition of a graph G is a collection $\mathcal{C} = \{H_1, H_2, \ldots, H_r\}$ of subgraphs of G such that the set $\{E(H_1), E(H_2), \ldots, E(H_r)\}$ forms a partition of $E(G)$. If each H_i is isomorphic to a graph H, then $\mathcal{C}$ is called a H*-decomposition* of G. If H is a spanning subgraph of G, then the decomposition is called *a factorization*.

There are lot of applications of decomposition of graphs which include group testings, DNA library screening, scheduling problems, sharing scheme and synchronous optical networks etc. In 1995, F.K. Hwang [15] gave necessary and sufficient condition for the factorization of K_{rc} into (r, c)-cliques which is isomorphic to $K_r \square K_c$ to identify positive clones in genetic studies. This paper [15] explains how decomposition is used in DNA library screening.

Decomposition of circulant balanced graphs using algorithmic and labeling approach were studied by El. Mesady et. al. in [8, 9, 11] along with their applications. Cyclic decomposition of balanced complete bipartite graphs using novel approach was studied in [10]. Related studies were made in [12, 13].

Decomposition of arbitrary graphs into subgraphs of small size is assuming importance in the literature. There are several studies on the isomorphic decomposition of graphs into paths [19], cycles [2], trees [3], stars [20], sunlet [1] etc. The general problem of H-decompositions was proved to be NP-complete for any H of size greater than 2 by Dor and Tarsi [7].

Fork is a tree obtained by subdividing any edge of a star of size three exactly once. A tree with degree sequence $(1, 1, 1, 2, 3)$ is unique and is nothing but the fork defined above. This graph was defined by Simone and Sassano in the name of *chair graph* in 1993, when they studied the stability number of bull and chair-free graphs [6]. In 2014, Barat and Gerbner [3] studied decomposition of 191-edge connected graphs which can be decomposed into forks as a possible attempt to solve the following conjecture:

Conjecture 1. [4] For each tree T, there exists a natural number k_T such that the following holds: if G is a k_T -edge-connected simple graph such that $|E(T)|$ divides $|E(G)|$, then G has a T-decomposition.

The edge-connectivity constants in the solved cases of Conjecture 1 are seemingly far from best possible. There is very little known about lower bounds. This motivated us to concentrate on total graph of certain corona graphs which are 2-connected.

If a graph G admits a H-decomposition, then $|E(H)|$ divides $|E(G)|$. Since the size of a fork is 4, for a fork-decomposition the obvious necessary condition is

$$|E(G)| \equiv 0 \ (mod\ 4) \tag{1}$$

Decomposition of complete bipartite graphs, complete graphs and corona graphs into fork was studied in [18] by the authors. Also, they have studied the fork-decomposition of cartesian product of graph and some total graphs in [16] and [17] respectively.

The following results are used in the subsequent section.

Theorem 2.1. [18] *The complete bipartite graph $K_{m,n}$ is fork-decomposable if and only if $mn \equiv 0 \ (mod\ 4)$ except $K_{2,4i+2}$, $(i = 1, 2, ...)$.*

Theorem 2.2. [18] *The Complete graph K_n can be decomposed into forks if and only if $n = 8k$ or $n = 8k + 1$, for all $k \geq 1$.*

Theorem 2.3. [18] *$C_n \circ \overline{K}_m$ is fork-decomposable if and only if $m = 1$ and $n = 2k$ or $m = 3$.*

Theorem 2.4. [18] *For $m \geq 3$,*

(1) *$K_m \circ K_1$ is fork-decomposable if and only if $m \equiv 0,\ 7 \ (mod\ 8)$.*
(2) *$K_m \circ \overline{K}_2$ is fork-decomposable if and only if $m \equiv 0,\ 5 \ (mod\ 8)$.*

Theorem 2.5. [16] *The graph $P_2 \square C_n$ is fork-decomposable if and only if $n \equiv 0 \ (mod\ 4)$.*

In this paper, we investigate the existence of decomposition of Total graph of certain corona graphs into forks and obtain original results.

3. Total graph of corona of paths and null graphs

In this section, we investigate the necessary and sufficient conditions for the existence of fork-decomposition of total graph of corona of paths P_n and null graphs $\overline{K}_m$.

We label the vertices of $P_n \circ \overline{K}_m$ as follows:

Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{im}/1 \leq i \leq n\}$ be the set of pendant vertices adjacent to u_i and

$$\begin{aligned} E(P_n \circ \overline{K}_m) = \{e_i, f_{ij} \quad / \quad & e_i = u_i u_{i+1} \ for\ 1 \leq i \leq n-1 \ and \\ & f_{ij} = u_i v_{ij} \ for\ 1 \leq i \leq n \ and\ 1 \leq j \leq m\}. \end{aligned}$$

Then,

$$\begin{aligned} V(T(P_n \circ \overline{K}_m)) &= \{u_i/1 \leq i \leq n\} \cup \{v_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{f_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{e_i/1 \leq i \leq n-1\}. \end{aligned}$$

and

$$\begin{aligned} E(T(P_n \circ \overline{K}_m)) &= \{u_i e_i/1 \leq i \leq n-1\} \cup \{e_{i-1} u_i/1 \leq i \leq n-1\} \\ &\quad \cup \{u_i u_{i+1}/1 \leq i \leq n-1\} \cup \{e_i e_{i+1}/1 \leq i \leq n-2\} \\ &\quad \cup \{u_i f_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i v_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{f_{ij} v_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{e_i f_{ij}/1 \leq i \leq n-1, 1 \leq j \leq m\} \\ &\quad \cup \{f_{ij} f_{ik}/j \neq k, 1 \leq i \leq n, 1 \leq j, k \leq m\}. \end{aligned}$$

Remark 3.1. *The number of edges in the total graph of $P_n \circ \overline{K}_m$ is $2(mn + n - 1) + \frac{1}{2}(mn.1^2 + 2.(m+1)^2 + (n-2).(m+2)^2) = \frac{1}{2}(9mn + 8n - 4m + m^2 n - 10)$.*

The following 4 lemmas are needed for proving the necessary and sufficient conditions for the fork-decomposition of $T(P_n \circ \overline{K}_m)$.

Lemma 3.1. *$T(P_n \circ K_1)$ is fork-decomposable if and only if $n \equiv 3 \ (mod\ 4)$.*

Proof. The number of edges in $T(P_n \circ K_1)$ is $\frac{1}{2}(9(1)(n) + 8n - 4(1) + 1(n) - 10) = 9n - 7$.

If $T(P_n \circ K_1)$ is fork-decomposable, then $9n - 7 \equiv 0$ $(mod\ 4)$ which implies $8(n-1) + (n+1) \equiv 0$ $(mod\ 4)$. Since $8(n-1) \equiv 0$ $(mod\ 4)$, $n + 1 \equiv 0$ $(mod\ 4)$ which implies $n \equiv 3$ $(mod\ 4)$.

Conversely, assume that $n \equiv 3$ $(mod\ 4)$.

Then, a fork-decomposition of $T(P_n \circ K_1)$ is given by $\{u_{i-1}v_{(i-1)1}, u_{i-1}f_{(i-1)1}, u_{i-1}e_{i-1}, e_{i-1}u_i\}$, $\{e_{i-1}e_i, e_{i-1}f_{i1}, e_{i-1}f_{(i-1)1}, f_{(i-1)1}v_{(i-1)1}\}$, $\{u_iu_{i-1}, u_ie_i, u_if_{i1}, f_{i1}v_{i1}\}$, $\{u_{i+1}u_i, u_{i+1}e_i, u_{i+1}v_{(i+1)1}, u_iv_{i1}\}$, $\{f_{(i+1)1}\, v_{(i+1)1}, f_{(i+1)1}u_{i+1}, f_{(i+1)1}e_i, e_if_{i1}\}$, where $i \equiv 2$ $(mod\ 4)$ and $\{e_{i-1}e_{i-2}, e_{i-1}f_{(i-1)1}, e_{i-1}u_{i-1}, u_{i-1}u_i\}$, $\{u_ie_{i-1}, u_iv_{i1}, u_ie_i, e_if_{i1}\}$, $\{f_{i1}u_i, f_{i1}v_{i1}, f_{i1}e_{i-1}, e_{i-1}e_i\}$, $\{e_ie_{i+1}, e_if_{(i+1)1}, e_iu_{i+1}, u_{i+1}u_i\}$ where $i \equiv 0$ $(mod\ 4)$. Here the subscripts are taken modulo n. □

Lemma 3.2. *$T(P_n \circ \overline{K}_2)$ is fork-decomposable if and only if $n \equiv 3$ $(mod\ 4)$.*

Proof. The number of edges in $T(P_n \circ \overline{K}_2)$ is $\frac{1}{2}(18(n) + 8n - 8 + 4(n) - 10) = 15n - 9$.

If $T(P_n \circ \overline{K}_2)$ is fork-decomposable, then $15n - 9 \equiv 0$ $(mod\ 4)$ which implies $5n - 3 \equiv 0$ $(mod\ 4)$. This can be written as $4(n-1)+(n+1) \equiv 0$ $(mod\ 4)$. Since $4(n-1) \equiv 0$ $(mod\ 4)$, $n + 1 \equiv 0$ $(mod\ 4)$ which implies $n \equiv 3$ $(mod\ 4)$.

Conversely, assume that $n \equiv 3$ $(mod\ 4)$.

If $n \geq 3$, then, a fork-decomposition of $T(P_3 \circ \overline{K}_2)$ is given by $\{u_iv_{i2}, u_if_{i2}, u_iv_{i1}, v_{i1}f_{i1}\}$, $\{f_{i2}v_{i2}, f_{i2}e_i, f_{i2}f_{i1}, f_{i1}u_i\}$, $\{e_iu_i, e_if_{(i+1)2}, e_if_{i1}, u_iu_{i+1}\}$ where $i \equiv 1, 2$ $(mod\ 4)$ and $\{e_iu_{i+1}, e_if_{(i+1)1}\ e_ie_{i+1}, e_{i+1}f_{(i+2)1}\}$, $\{u_{i+2}e_{i+1}, u_{i+2}v_{(i+2)1}, u_{i+2}f_{(i+2)2}, f_{(i+2)2}v_{(i+2)2}\}$, $\{f_{(i+2)1}v_{(i+2)1}, f_{(i+2)1}f_{(i+2)2}, f_{(i+2)1}u_{i+2}, u_{i+2}v_{(i+2)2}\}$ where $i \equiv 1$ $(mod\ 4)$. Here the subscripts are taken modulo n.

For $n \geq 4$, the induced subgraph $\langle\{u_i, u_{i+1}, u_{i+2}, e_i, e_{i+1}, f_{i1}, f_{i2}, f_{(i+1)1}, f_{(i+1)2}, f_{(i+2)1}, f_{(i+2)2}, v_{i1}, v_{i2}, v_{(i+1)1}, v_{(i+1)2}, v_{(i+2)1}, v_{(i+2)2}\}\rangle$ where $i \equiv 0$ $(mod\ 4)$ is isomorphic to $T(P_3 \circ \overline{K}_2)$ which is fork - decomposable. Removing the above induced subgraphs, we get a subgraph which is fork - decomposable as follows: $\{e_{i-1}e_{i-2}, e_{i-1}f_{(i-1)1}, e_{i-1}u_{i-1}, u_{i-1}u_i\}$, $\{e_{i-1}e_i, e_{i-1}f_{i2}, e_{i-1}u_i, u_if_{i1}\}$, $\{f_{i1}e_i, f_{i1}f_{i2}, f_{i1}e_{i-1}, e_{i-1}f_{(i-1)2}\}$, $\{f_{i2}u_i, f_{i2}v_{i2}, f_{i2}e_i, e_if_{(i+1)1}\}$, $\{u_ie_i, u_iv_{i2}, u_iv_{i1}, v_{i1}f_{i1}\}$, $\{e_ie_{i+1}, e_if_{(i+1)2}, e_iu_{i+1}, u_{i+1}u_i\}$ where $i \equiv 0$ $(mod\ 4)$ and the subscripts are taken modulo n. □

Lemma 3.3. *$T(P_n \circ \overline{K}_5)$ is fork-decomposable if and only if $n \equiv 1$ $(mod\ 4)$.*

Proof. The number of edges in $T(P_n \circ \overline{K}_5)$ is $\frac{1}{2}(45(n) + 8n - 20 + 25(n) - 10) = 39n - 15$.

If $T(P_n \circ \overline{K}_5)$ is fork-decomposable, then $39n - 15 \equiv 0$ $(mod\ 4)$ which implies $8(5n-2) - (n-1) \equiv 0$ $(mod\ 4)$. Since $8(5n-2) \equiv 0$ $(mod\ 4)$, $n - 1 \equiv 0$ $(mod\ 4)$ which implies $n \equiv 1$ $(mod\ 4)$.

Conversely, assume that $n \equiv 1$ $(mod\ 4)$.

Then, a fork-decomposition of $T(P_n \circ \overline{K}_5)$ is given by $\{f_{ij}v_{ij}, f_{ij}f_{i(j+1)}, f_{ij}f_{i(j+2)}, v_{ij}u_i\}$ where $1 \leq i \leq n$ and $1 \leq j \leq 5$, $\{u_iu_{i-1}, u_ie_i, u_iu_{i+1}, u_{i+1}e_{i+1}\}$ where $i \equiv 2$ $(mod\ 4)$, $\{u_ie_i, u_ie_{i-1}, u_iu_{i-1}, u_{i-1}e_{i-2}\}$ where $i \equiv 0$ $(mod\ 4)$, $\{u_iu_{i-1}, u_ie_{i-1}, u_ie_i, e_iu_{i+1}\}$ where $i \equiv 5$ $(mod\ 4)$, $\{e_{i-1}f_{i1}, e_{i-1}f_{i2}, e_{i-1}f_{i3}, f_{i3}u_i\}$, $\{u_if_{i1}, u_if_{i2}, u_if_{i4}, f_{i4}e_{i-1}\}$, $\{e_if_{i1}, e_if_{i2}, e_if_{i5}, f_{i5}u_i\}$, $\{e_if_{i3}, e_if_{i4}, e_ie_{i-1}, e_{i-1}f_{i5}\}$ where $2 \leq i \leq n-1$, $\{u_1f_{12}, u_1f_{13}, u_1f_{14}, f_{14}e_1\}$, $\{e_1u_2, e_1f_{13}, e_1u_1, u_1f_{11}\}$, $\{e_1f_{11}, e_1f_{12}, e_1f_{15}, f_{15}u_1\}$, $\{u_nf_{n2}, u_nf_{n3}, u_nf_{n4}, f_{n4}e_{n-1}\}$, $\{e_{n-1}u_{n-1}, e_{n-1}f_{n3}, e_{n-1}u_5, u_5f_{n1}\}$, $\{e_{n-1}f_{n1}, e_{n-1}f_{n2}, e_{n-1}f_{n5}, f_{n5}u_5\}$. □

Lemma 3.4. *$T(P_n \circ \overline{K}_6)$ is fork-decomposable if and only if $n \equiv 1$ $(mod\ 4)$.*

Proof. The number of edges in $T(P_n \circ \overline{K}_6)$ is $\frac{1}{2}(98(n) + 8n - 24 + 36(n) - 10) = 49n - 17$.

If $T(P_n \circ \overline{K}_6)$ is fork-decomposable, then $49n - 17 \equiv 0$ $(mod\ 4)$ which implies $8(6n-2)+(n-1) \equiv 0$ $(mod\ 4)$. Since $8(6n-2) \equiv 0$ $(mod\ 4)$, $n-1 \equiv 0$ $(mod\ 4)$ which implies $n \equiv 1$ $(mod\ 4)$.

Conversely, assume that $n \equiv 1$ $(mod\ 4)$.

Then, a fork-decomposition of $T(P_n \circ \overline{K}_6)$ is given by $\{f_{ij}v_{ij},\ f_{ij}f_{i(j+1)},\ f_{ij}f_{i(j+2)},\ v_{ij}u_i\}$ where $1 \leq i \leq n$ and $1 \leq j \leq 6$, $\{u_{i-1}u_{i-2}, u_{i-1}e_{i-2}, u_{i-1}e_{i-1}, e_{i-1}f_{(i-1)3}\}$, $\{u_iu_{i-1}, u_ie_{i-1}, u_ie_i,\ e_iu_{i+1}\}$, $\{e_if_{i3},\ e_ie_{i+1},\ e_ie_{i-1},\ e_{i-1}e_{i-2}\}$, $\{e_{i+1}f_{(i+1)3},\ e_{i+1}u_{i+2},\ e_{i+1}u_{i+1},\ u_{i+1}u_i\}$ where $i \equiv 3$ $(mod\ 4)$, $\{e_if_{i3},\ e_ie_{i-1},\ e_iu_i,\ u_iu_{i-1}\}$ where $i \equiv 5$ $(mod\ 4)$, $\{f_{i1}e_{i-1},\ f_{i1}u_i,\ f_{i1}e_i,\ e_{i-1}f_{i3}\}$, $\{f_{i2}e_{i-1},\ f_{i2}u_i,\ f_{i2}e_i,\ u_if_{i3}\}$, $\{e_{i-1}f_{i4},\ e_{i-1}f_{i5},\ e_{i-1}f_{i6},\ f_{i4}f_{i1}\}$, $\{e_if_{i4},\ e_if_{i5},\ e_if_{i6},\ f_{i6}f_{i3}\}$, $\{u_if_{i4},\ u_if_{i5},\ u_if_{i6},\ f_{i5}f_{i2}\}$ where $2 \leq i \leq n-1$, $\{e_1f_{14},\ e_1f_{15},\ e_1f_{16},\ f_{16}u_1\}$, $\{f_{12}u_1,\ f_{12}e_1,\ f_{12}f_{15},\ u_1f_{11}\}$, $\{f_{13}u_1,\ f_{13}e_1,\ f_{13}f_{16},\ e_1f_{11}\}$, $\{u_1e_1,\ u_1f_{15},\ u_1f_{14},\ f_{14}f_{11}\}$, $\{e_{n-1}f_{n4},\ e_{n-1}f_{n5},\ e_{n-1}f_{n6},\ f_{n6}u_n\}$, $\{f_{n2}u_n,\ f_{n2}e_{n-1},\ f_{n2}f_{n5},\ u_nf_{n1}\}$, $\{f_{n3}u_n,\ f_{n3}e_{n-1},\ f_{n3}f_{n6},\ e_{n-1}f_{n1}\}$, $\{u_ne_{n-1},\ u_nf_{n5},\ u_nf_{n4},\ f_{n4}f_{n1}\}$. □

Theorem 3.1. *$T(P_n \circ \overline{K}_m)$ is fork-decomposable if and only if it satisfies any of the following conditions:*

(1) *$n \equiv 3$ $(mod\ 4)$ and $m \equiv 1, 2$ $(mod\ 8)$.*
(2) *$n \equiv 1$ $(mod\ 4)$ and $m \equiv 5, 6$ $(mod\ 8)$.*

Proof. Let $V(P_n \circ \overline{K}_m) = \{u_i\ /\ 1 \leq i \leq n\} \cup \{v_{ij}\ /\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$ where u_i are the support vertices and v_{ij} are the pendant vertices adjacent to corresponding u_i.

$$\begin{aligned} E(P_n \circ \overline{K}_m) &= \{e_i, f_{ij}\ /\ e_i = u_iu_{i+1}\ for\ 1 \leq i \leq n-1\ and \\ &\qquad f_{ij} = u_iv_{ij}\ for\ 1 \leq i \leq n\ and\ 1 \leq j \leq m\}. \end{aligned}$$

Then,

$$\begin{aligned} V(T(P_n \circ \overline{K}_m)) &= \{u_i/1 \leq i \leq n\} \cup \{v_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{f_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{e_i/1 \leq i \leq n-1\} \end{aligned}$$

and

$$\begin{aligned} E(T(P_n \circ \overline{K}_m)) &= \{u_ie_i/1 \leq i \leq n-1\} \cup \{e_{i-1}u_i/1 \leq i \leq n-1\} \\ &\quad \cup \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{e_ie_{i+1}/1 \leq i \leq n-2\} \\ &\quad \cup \{u_if_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_iv_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{f_{ij}v_{ij}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{e_if_{ij}/1 \leq i \leq n-1, 1 \leq j \leq m\} \\ &\quad \cup \{f_{ij}f_{ik}/j \neq k, 1 \leq i \leq n, 1 \leq j, k \leq m\}. \end{aligned}$$

The number of edges in $T(P_n \circ \overline{K}_m)$ is $\frac{1}{2}(9mn + 8n - 4m + m^2n - 10)$ which implies that $(9mn + 8n - 4m + m^2n - 10) \equiv 0$ $(mod\ 8)$. Hence, $n((m+8)(m+1)) - 4m - 10 \equiv 0$ $(mod\ 8)$. Since $8n(m+1) \equiv 0$ $(mod\ 8)$, $nm(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$.

Now assume that $T(P_n \circ \overline{K}_m)$ is fork-decomposable.

Case 1. $n \equiv 0$ $(mod\ 4)$.

Then $n = 4a$, where a is any arbitrary integer. Hence, $4am(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$. Since $m(m+1)$ is even, $4am(m+1) - 8 \equiv 0$ $(mod\ 8)$, then $-4m - 2 \equiv 0$ $(mod\ 8)$, which is not possible since m is an integer.

Case 2. $n \equiv 1$ $(mod\ 4)$.

Then $n = 4a + 1$, where a is any arbitrary integer. Here, $(4a+1)m(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$. This implies that, $m(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$, since $m(m+1)$ is even. $(m+2)(m-5) \equiv 0$ $(mod\ 8)$. Then either $m + 2 \equiv 0$ $(mod\ 8)$ or $m - 5 \equiv 0$ $(mod\ 8)$ which implies that $m \equiv 6$ $(mod\ 8)$ or $m \equiv 5$ $(mod\ 8)$.

Case 3. $n \equiv 2$ $(mod\ 4)$.

Then $n = 4a+2$, where a is any arbitrary integer. Here, $(4a+2)m(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$. This implies that, $2m(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$, since $m(m+1)$ is even. Here $m^2 + m - 2m \equiv 5$ $(mod\ 4)$ which implies $m(m-1) \equiv 5$ $(mod\ 4)$ which is a contradiction, since $m(m-1)$ is even.

Case 4. $n \equiv 3$ $(mod\ 4)$.

Then $n = 4a+3$, where a is any arbitrary integer. Here, $(4a+3)m(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$. This implies that, $3m(m+1) - 4m - 10 \equiv 0$ $(mod\ 8)$, since $m(m+1)$ is even. Here $m(3m-1) - 10 \equiv 0$ $(mod\ 8)$ which implies $m(3m-1) \equiv 2$ $(mod\ 8)$. Then either $m \equiv 2$ $(mod\ 8)$ or $3m-1 \equiv 2$ $(mod\ 8)$ which implies that $m \equiv 2$ $(mod\ 8)$ or $m \equiv 1$ $(mod\ 8)$.

Now let us prove the converse part. Let $G = P_n \circ \overline{K}_m$.

Case 1. $n \equiv 3$ $(mod\ 4)$ and $m \equiv 1$ $(mod\ 8)$.

Let $G_1 = \langle\{u_1, u_2, \ldots, u_n, e_1, e_2, \ldots, e_{n-1}, v_{1m}, v_{2m}, \ldots, v_{nm}, f_{1m}, f_{2m}, \ldots, f_{nm}\}\rangle$ be the induced subgraph of G. Then, G_1 is isomorphic to $T(P_n \circ K_1)$ which is fork-decomposable by Lemma 3.1. Let $G_2 = \langle\{(f_{1j}, v_{1j}), (f_{2j}, v_{2j}), \ldots, (f_{nj}, v_{nj}) \ /\ 1 \leq j \leq m-1\}\rangle$ be the induced subgraph of G. Then, G_2 is isomorphic to n copies of $K_{m-1} \circ K_1$. Since $m \equiv 1$ $(mod\ 8)$, G_2 is fork-decomposable by Theorem 2.4.

Let G_3 denote the collection of forks, $\{u_i v_{ij}, u_i v_{i(j+1)}, u_i f_{i(\frac{j+1}{2})}, f_{i(\frac{j+1}{2})} e_i \ /\ 1 \leq i \leq n-1$, $j \equiv 1$ $(mod\ 2)$ and $j < m\}$. Let G_4 denote the collection of forks, $\{u_n v_{nj}, u_n v_{n(j+1)}$, $u_n f_{n(\frac{j+1}{2})}, f_{n(\frac{j+1}{2})} \ /\ j \equiv 1$ $(mod\ 2)$ and $j < m\}$.

Let $H_1 = G - \bigcup_{i=1}^{4} G_i$. Let $G_5 = \langle\{u_1, e_1, f_{11}, f_{12}, \ldots, f_{1m}\}\rangle$ be the induced subgraph of H_1. The fork-decomposition of G_5 is given by $\{\langle f_{1w}u_1, f_{1w}e_1, f_{1w}f_{1m}, f_{1m}f_{1(w-\frac{m-1}{2})}\rangle \ /\ w = k, k+1, \ldots, 2k-2$ where $k = \frac{m+1}{2}\}$. Also, let $G_6 = \{u_n, e_{n-1}, f_{n1}, f_{n2}, \ldots, f_{nm}\}$ be the induced subgraph of H_1. The fork-decomposition of G_6 is given by $\{\langle f_{nw}u_n$, $f_{nw}e_{n-1}$, $f_{nw}f_{nm}$, $f_{nm}f_{n(w-\frac{m-1}{2})}\rangle \ /\ w = k, k+1, \ldots, 2k-2$ where $k = \frac{m+1}{2}\}$.

Let $H_2 = G - \bigcup_{i=1}^{6} G_i$. Let $G_7 = \langle\{u_i, e_i, f_{ik}, f_{i(k+1)}, \ldots, f_{i(2k-2)}\} \ /\ k = \frac{m+1}{2}$ and $2 \leq i \leq n-1\rangle$ be the induced subgraph of H_2. Here G_7 is isomorphic to $n-2$ copies of $K_{2,\frac{m-1}{2}}$. Let $G_8 = \langle\{e_{i-1}, f_{ij}\} \ /\ 2 \leq i \leq n-1.\ 1 \leq j \leq m\rangle$ be the induced subgraph of H_2 and it is isomorphic to $K_{2,m-1}$. Here the subgraphs G_7 and G_8 are fork-decomposable by Theorem 2.1, since $m \equiv 1$ $(mod\ 8)$.

Thus, $T(P_n \circ \overline{K}_m) = \bigcup_{i=1}^{8} G_i$ is fork-decomposable.

Case 2. $n \equiv 3$ $(mod\ 4)$ and $m \equiv 2$ $(mod\ 8)$.

Let $G_1 = \langle\{u_1, u_2, \ldots, u_n,\ e_1, e_2, \ldots, e_{n-1},\ v_{1m}, v_{2m},\ \ldots, v_{nm},\ v_{1(m-1)}, v_{2(m-1)},\ \ldots,$ $v_{n(m-1)},\ f_{1m},\ f_{2m},\ \ldots,\ f_{nm},\ f_{1(m-1)}, f_{2(m-1)},\ \ldots,\ f_{n(m-1)}\}\rangle$ be the induced subgraph of G. Then, G_1 is isomorphic to $T(P_n \circ \overline{K_2})$ which is fork-decomposable by Lemma 3.2. Let $G_2 = \langle\{f_{1j}\} \ /\ 1 \leq j \leq m-2\rangle$ and $G_3 = \langle\{f_{nj}\} \ /\ 1 \leq j \leq m-2\rangle$ be the induced subgraphs of G. Then, the graphs G_2 and G_3 are isomorphic to two copies of K_{m-2}, which are fork-decomposable by Theorem 2.2. Let $G_4 = \langle\{f_{ij}, v_{ij}\} \ /\ 2 \leq i \leq n-1, 1 \leq j \leq m-2\rangle$ be the induced subgraph of G. Then, G_4 is isomorphic to $n-2$ copies of $K_{m-2} \circ K_1$. Since $m \equiv 2$ $(mod\ 8)$, G_4 is fork-decomposable by Theorem 2.4.

Let G_5 denote the collection of forks $\{f_{1j}u_1,\ f_{1j}e_1,\ f_{1j}v_{1j},\ u_1 v_{1(m-j-1)} \ /\ 1 \leq j \leq m-2\}$, $\{f_{nj}u_n,\ f_{nj}e_{n-1},\ f_{nj}v_{nj},\ u_n v_{n(m-j-1)} \ /\ 1 \leq j \leq m-2\}$, $\{u_i v_{ij},\ u_i v_{i(j+1)},\ u_i f_{j(\frac{j+1}{2})}$, $f_{j(\frac{j+1}{2})} e_i \ /\ 2 \leq i \leq n-1, j \equiv 1$ $(mod\ 2)$ and $j < m-1\}$, $\{f_{i(\frac{m}{2}+j)}u_i,\ f_{i(\frac{m}{2}+j)}e_i,\ f_{i(\frac{m}{2}+j)}e_{i-1}$, $e_{i-1}f_{i(\frac{m-2}{2}-j)} \ /\ 2 \leq i \leq n-1, 0 \leq j \leq \frac{m-4}{2}\}$.

Let $G_6 = G - \bigcup_{i=1}^{5} G_i$. Then, $G_6 = \langle \{f_{i1},\ f_{i2},\ \ldots,\ f_{im}\}\ /\ 1 \leq i \leq n \rangle$ is isomorphic to n copies of $K_{2,m-2}$. Since $m \equiv 2\ (mod\ 8)$, G_6 is fork-decomposable by Theorem 2.1.

Thus, $T(P_n \circ \overline{K}_m) = \bigcup_{i=1}^{6} G_i$ is fork-decomposable.

The proof for the case $n \equiv 1\ (mod\ 4)$ and $m \equiv 5, 6\ (mod\ 8)$ is similar to the proof of Case 2. $\square$

4. Total graph of corona of cycles and null graphs

In this section, we investigate the existence of necessary and sufficient conditions for the fork-decomposition of total graph of corona of cycles C_n and null graphs $\overline{K}_m$.

We label the vertices of $C_n \circ \overline{K}_m$ as follows:

Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{im}\ /\ 1 \leq i \leq n\}$ be the set of pendant vertices adjacent to u_i.

Let $E(C_n \circ \overline{K}_m) = \{\{e_i, f_{ij}\}\ /\ e_i = u_i u_{i+1},\ f_{ij} = u_i v_{ij}\ where\ 1 \leq i \leq n, 1 \leq j \leq m.\}$

Then, $V(T(C_n \circ \overline{K}_m)) = \{u_i, e_i, v_{ij}, f_{ij} / 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and

$$\begin{aligned} E(T(C_n \circ \overline{K}_m)) &= \{\{u_i e_i, e_{i-1} u_i, u_i u_{i+1}, e_i e_{i+1} / 1 \leq i \leq n\} \\ &\quad \cup \{u_i f_{ij}, u_i v_{ij}, e_i f_{ij}, f_{ij} v_{ij}, e_i f_{(i+1)j} / 1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{f_{ij} f_{ik} / j \neq k, 1 \leq i \leq n, 1 \leq j, k \leq m\}\}. \end{aligned}$$

Remark 4.1. *The number of edges in Total graph of $C_n \circ \overline{K}_m$ is* $2(mn+n) + \frac{1}{2}(mn.1^2 + (n).(m+2)^2) = 2mn + 2n + \frac{1}{2}(mn + n(m^2 + 4 + 4m)) = \frac{1}{2}(m^2 n + 9mn + 8n) = \frac{1}{2}(mn(m+1) + 8n(m+1)) = \frac{1}{2}(n(m+1)(m+8))$.

The following 8 lemmas are needed for proving the necessary and sufficient conditions for the fork-decomposition of $T(C_n \circ \overline{K}_m)$.

Lemma 4.1. *$T(C_n \circ K_1)$ is fork-decomposable if and only if $n \equiv 0\ (mod\ 4)$.*

Proof. The number of edges in $T(C_n \circ K_1)$ is $\frac{1}{2}(n(1+1)(1+8)) = 9n$.

If the graph is fork-decomposable, then $9n \equiv 0\ (mod\ 4)$ which implies $n \equiv 0\ (mod\ 4)$.

Conversely, assume that $n \equiv 0\ (mod\ 4)$.

Then, a fork-decomposition of $T(C_n \circ K_1)$ is given by $\{u_i e_i, u_i f_{i1}, u_i e_{i-1}, f_{i1} v_i\}$ where $1 \leq i \leq n$, $\{u_i u_{i-1},\ u_i v_i,\ u_i u_{i+1},\ u_{i-1} v_{i-1}\}$ where $i \equiv 0\ (mod\ 2)$, $\{e_i f_{(i+1)1},\ e_i e_{i+1},\ e_i f_{i1},\ f_{i1} e_{i-1}\}$ where $i \equiv 1\ (mod\ 4)$, $\{e_i f_{i1},\ e_i f_{(i+1)1},\ e_i e_{i+1},\ e_{i+1} f_{(i+2)1}\}$ where $i \equiv 2\ (mod\ 4)$, $\{e_i e_{i+1},\ e_i f_{i1},\ e_i e_{i-1},\ e_{i-1} f_{(i-1)1}\}$ where $i \equiv 0\ (mod\ 4)$ and the subscripts are taken modulo n. Hence, the graph $T(C_n \circ K_1)$ is fork-decomposable. $\square$

Lemma 4.2. *$T(C_n \circ \overline{K}_2)$ is fork-decomposable if and only if $n \equiv 0\ (mod\ 4)$.*

Proof. The number of edges in $T(C_n \circ \overline{K}_2)$ is $\frac{1}{2}(n(2+1)(2+8)) = 15n$.

If the graph is fork-decomposable, then $15n \equiv 0\ (mod\ 4)$ which implies $n \equiv 0\ (mod\ 4)$.

Conversely, assume that $n \equiv 0\ (mod\ 4)$. Consider the collection of forks $\{u_i v_{i2},\ u_i f_{i1},\ u_i f_{i2},\ f_{i2} e_i\}$, $\{f_{i1} e_i,\ f_{i1} e_{i-1},\ f_{i1} v_{i1},\ v_{i1} u_i\}$, $\{f_{i2} f_{i1},\ f_{i2} v_{i2},\ f_{i2} e_{i-1},\ e_{i-1} u_i\}$ where $1 \leq i \leq n$ and the subscripts are taken modulo n. After removing the above collection of forks, we get the induced subgraph $\langle \{u_i, e_i / 1 \leq i \leq n\} \rangle$ isomorphic to $C_n \square P_2$, which is fork-decomposable by Theorem 2.5. Hence, the graph $T(C_n \circ \overline{K}_2)$ is fork-decomposable. $\square$

Lemma 4.3. *$T(C_n \circ \overline{K}_3)$ is fork-decomposable if and only if $n \equiv 0\ (mod\ 2)$.*

Proof. The number of edges in $T(C_n \circ \overline{K}_3)$ is $\frac{1}{2}(n(3+1)(3+8)) = 22n$.

If the graph is fork-decomposable, then $22n \equiv 0$ $(mod\ 4)$ which implies $n \equiv 0$ $(mod\ 2)$.

Conversely, assume that $n \equiv 0$ $(mod\ 2)$. Consider the collection of forks $\{f_{ij}v_{ij},\ f_{ij}u_i,\ f_{ij}f_{i(j+1)},\ f_{i(j+1)}e_{i-1}\}$ where $1 \le i \le n$, $j = 1,2,3$ and $\{e_if_{i1},\ e_if_{i2},\ e_ie_{i-1},\ e_{i-1}u_i\}$, $\{u_iv_{i1},\ u_iv_{i2},\ u_ie_i,\ e_if_{i3}\}$ where $1 \le i \le n$. After removing the above collection of forks, we get the induced subgraph $\langle\{u_i, v_{i3}\ /\ 1 \le i \le n\}\rangle$ isomorphic to $C_n \circ K_1$, which is fork-decomposable by Theorem 2.3. Here the subscripts are taken modulo n. Hence, the graph $T(C_n \circ \overline{K}_3)$ is fork-decomposable. □

Lemma 4.4. *$T(C_n \circ \overline{K}_4)$ is fork-decomposable if and only if $n \equiv 0$ $(mod\ 2)$.*

Proof. The number of edges in $T(C_n \circ \overline{K}_4)$ is $\frac{1}{2}(n(4+1)(4+8)) = 30n$.

If the graph is fork-decomposable, then $30n \equiv 0$ $(mod\ 4)$ which implies $n \equiv 0$ $(mod\ 2)$.

Conversely, assume that $n \equiv 0$ $(mod\ 2)$. Consider the collection of forks $\{u_iv_{i4},\ u_if_{i1},\ u_if_{i4},\ f_{i4}f_{i2}\}$, $\{u_iv_{i3},\ u_if_{i2},\ u_if_{i3},\ f_{i3}f_{i1}\}$, $\{u_iv_{i1},\ u_iv_{i2},\ u_ie_{i-1},\ e_{i-1}e_i\}$, $\{f_{ij}v_{ij},\ f_{ij}v_{i(j+1)},\ f_{ij}e_i,\ f_{i(j+1)}e_{i-1}\}$, where $1 \le i \le n$ and $j = 1,2,3,4$. After removing the above collection of forks, we get the induced subgraph $\langle\{u_i, e_i/1 \le i \le n\}\rangle$ isomorphic to $C_n \circ K_1$, which is fork-decomposable by Theorem 2.3. Here, the subscripts are taken modulo n. Hence, the graph $T(C_n \circ \overline{K}_4)$ is fork-decomposable. □

Lemma 4.5. *$T(C_n \circ \overline{K}_5)$ is fork-decomposable if and only if $n \equiv 0$ $(mod\ 4)$.*

Proof. The number of edges in $T(C_n \circ \overline{K}_5)$ is $\frac{1}{2}(n(5+1)(5+8)) = 39n$.

If the graph is fork-decomposable, then $39n \equiv 0$ $(mod\ 4)$ which implies $n \equiv 0$ $(mod\ 4)$.

Conversely, assume that $n \equiv 0$ $(mod\ 4)$.

Then, $T(C_n \circ \overline{K}_5)$ can be decomposed into $\{f_{ij}v_{ij},\ f_{ij}f_{i(j-1)},\ f_{ij}f_{i(j-2)},\ v_{ij}u_i\}$ where $1 \le i \le n$ and $j = 1,2,3,4,5$, $\{u_ie_{i-1},\ u_ie_i,\ u_iu_{i+1},\ u_{i+1}e_{i+1}\}$, $\{u_{i+2}e_{i+1},\ u_{i+2}u_{i+1},\ u_{i+2}u_{i+3},\ u_{i+1}e_i\}$, $\{u_{i+3}e_{i+3},\ u_{i+3}e_{i+4},\ u_{i+3}e_{i+2},\ e_{i+2}u_{i+2}\}$ where $i \equiv 1$ $(mod\ 4)$, $\{e_if_{i1},\ e_if_{i2},\ e_if_{i3},\ f_{i2}u_i\}$, $\{u_if_{i3},\ u_if_{i4},\ u_if_{i5},\ f_{i5}e_i\}$, $\{e_{i-1}f_{i3},\ e_{i-1}f_{i4},\ e_{i-1}f_{i5},\ f_{i4}e_i\}$, $\{e_{i-1}e_i,\ e_{i-1}f_{i2},\ e_{i-1}f_{i1},\ f_{i1}u_i\}$ where $1 \le i \le n$ and the subscripts are taken modulo n. Hence, the graph $T(C_n \circ \overline{K}_5)$ is fork-decomposable. □

Lemma 4.6. *$T(C_n \circ \overline{K}_6)$ is fork-decomposable if and only if $n \equiv 0$ $(mod\ 4)$.*

Proof. The number of edges in $T(C_n \circ \overline{K}_6)$ is $\frac{1}{2}(n(6+1)(6+8)) = 49n$.

If the graph is fork-decomposable, then $49n \equiv 0$ $(mod\ 4)$ which implies $n \equiv 0$ $(mod\ 4)$.

Conversely, assume that $n \equiv 0$ $(mod\ 4)$. Consider the collection of forks $\{f_{i1}e_{i-1},\ f_{i1}u_i,\ f_{i1}e_i,\ e_{i-1}f_{i5}\}$, $\{f_{i2}e_{i-1},\ f_{i2}u_i,\ f_{i2}e_i,\ u_if_{i6}\}$, $\{f_{i3}e_{i-1},\ f_{i3}u_i,\ f_{i3}e_i,\ e_{i-1}f_{i6}\}$, $\{f_{i4}\ e_{i-1},\ f_{i4}u_i,\ f_{i4}e_i,\ e_if_{i5}\}$, $\{u_ie_{i-1},\ u_if_{i5},\ u_ie_i,\ e_if_{i6}\}$, $\{f_{i5}v_{i5},\ f_{i5}f_{i6},\ f_{i5}f_{i4},\ f_{i4}v_{i4}\}$, $\{f_{i6}f_{i4},\ f_{i6}f_{i2},\ f_{i6}f_{i1},\ f_{i1}f_{i5}\}$, $\{u_iv_{i5},\ u_iv_{i6},\ u_iv_{i1},\ v_{i6}f_{i6}\}$ where $1 \le i \le n$, $\{f_{ij}v_{ij},\ f_{ij}f_{i(j+1)},\ f_{ij}f_{i(j+2)},\ f_{i(j+1)}f_{i(j+4)}\}$ where $0 \le i \le n$ and $j = 1,2,3$. Here the subscripts are taken modulo n. After removing the above collection of forks, we get the induced subgraph $\langle\{u_1, u_2, \ldots, u_n, v_{i2},\ v_{i3}, v_{i4}\}\ /\ 1 \le i \le n\rangle$ isomorphic to $C_n \circ \overline{K}_3$, which is fork-decomposable by Theorem 2.3. Hence, the graph $T(C_n \circ \overline{K}_6)$ is fork-decomposable. □

Lemma 4.7. *$T(C_n \circ \overline{K}_7)$ is fork-decomposable for all values of n.*

Proof. The number of edges in $T(C_n \circ \overline{K}_7)$ is $\frac{1}{2}(n(7+1)(7+8)) = 60n$.

If the graph is fork-decomposable, then by Equation (1), n can take all values.

Now, let us prove the converse part. The induced subgraph $\langle\{f_{i1}, f_{i2}, \ldots, f_{i7}, v_{i1}, v_{i2}, \ldots, v_{i7}\}\ /\ 1 \le i \le n\rangle$ is isomorphic to n copies of $K_7 \circ K_1$ which are fork-decomposable by Theorem 2.4. The fork-decomposition of the subgraph after removing above induced subgraph is given by $\{f_{ij}e_{i-1},\ f_{ij}u_i,\ f_{ij}e_i,\ u_iv_{ij}\}$ where $1 \le i \le n$ and $j = 1,2\ldots,5$,

$\{e_{i-1}f_{i6},\ e_{i-1}f_{i7},\ e_{i-1}u_i,\ u_iv_{i6}\}$, $\{u_if_{i6},\ u_if_{i7},\ u_iv_{i7},\ f_{i7}e_i\}$, $\{e_if_{i6},\ e_iu_i,\ e_ie_{i-1},\ u_iu_{i+1}\}$ where $1 \leq i \leq n$ and the subscripts are taken modulo n. Hence, the graph $T(C_n \circ \overline{K}_7)$ is fork-decomposable. □

Lemma 4.8. *$T(C_n \circ \overline{K}_8)$ is fork-decomposable for all values of n.*

Proof. The number of edges in $T(C_n \circ \overline{K}_8)$ is $\frac{1}{2}(n(8+1)(8+8)) = 72n$.

If the graph is fork-decomposable, then by Equation (1), n can take all values.

Now, let us prove the converse part. The induced subgraph $\langle\{f_{i1}, f_{i2},\ \ldots,\ f_{i8}, v_{i1}, v_{i2}, \ldots, v_{i8}\ /\ 1 \leq i \leq n\}\rangle$ is isomorphic to n copies of $K_8 \circ K_1$ which are fork-decomposable by Theorem 2.4. The fork-decomposition of the subgraph after removing the above induced subgraph is given by $\{e_iu_{i+1},\ e_ie_{i-1},\ e_iu_i,\ u_iu_{i-1}\}$ where $1 \leq i \leq n$, $\{f_{ij}e_{i-1},\ f_{ij}e_i,\ f_{ij}u_i, u_iv_{ij}\}$ where $1 \leq i \leq n$ and $j = 1, 2 \ldots, 8$ and the subscripts are taken modulo n. Hence, the graph $T(C_n \circ \overline{K}_8)$ is fork-decomposable. □

Lemma 4.9. *$T(C_n \circ \overline{K}_9)$ is fork-decomposable if and only if $n \equiv 0\ (mod\ 4)$.*

Proof. The number of edges in $T(C_n \circ \overline{K}_9)$ is $\frac{1}{2}(n(9+1)(9+8)) = 85n$.

If the graph is fork-decomposable, then $8n \equiv 0\ (mod\ 4)$ which implies $n \equiv 0\ (mod\ 4)$.

Now, let us prove the converse part. The induced subgraph $\langle\{u_i, e_i, v_{i9}, f_{i9}\}\ /\ 1 \leq i \leq n\rangle$ is isomorphic to $T(C_n \circ K_1)$ which is fork-decomposable by Lemma 4.1. The induced subgraph $\langle\{f_{i1}, f_{i2}, \ldots, f_{i8}, v_{i1}, v_{i2}, \ldots, v_{i8}\}\ /\ 1 \leq i \leq n\rangle$ is isomorphic to n copies of $K_8 \circ K_1$ which are fork-decomposable by Theorem 2.2. Consider the subgraph obtained after removing the above $T(C_n \circ K_1)$ and n copies of $K_8 \circ K_1$. The induced subgraph $\langle\{f_{i5}, f_{i6}, \ldots, f_{i9}, u_i, e_i, e_{i-1}\}\ /\ 1 \leq i \leq n\rangle$ is isomorphic to n copies of $K_{4,4}$ which are fork-decomposable by Theorem 2.1. The induced subgraph $\langle\{f_{i1}, f_{i2}, f_{i3}, f_{i4},\ f_{i9}, e_{i-1}\}\ /\ 1 \leq i \leq n\rangle$ is isomorphic to $K_{2,4}$ which is fork-decomposable by Theorem 2.1. The fork-decomposition of the induced subgraph obtained after removing the above subgraphs is given by $\{u_iv_{ik}, u_iv_{i(k+4)}, u_if_{ik},\ f_{ik}e_i\}$ where $1 \leq i \leq n$ and $k = 1, 2, 3, 4$. Hence, the graph $T(C_n \circ \overline{K}_9)$ is fork-decomposable. □

Theorem 4.1. *$T(C_n \circ \overline{K}_m)$ is fork-decomposable if and only if it satisfies any one of the following conditions:*

(1) $m \equiv 0, 7\ (mod\ 8)$.
(2) $n \equiv 0\ (mod\ 2)$ *and* $m \equiv 0, 3\ (mod\ 4)$.
(3) $n \equiv 0\ (mod\ 4)$.

Proof. Let $V(C_n \circ \overline{K}_m) = \{u_i\ /\ 1 \leq i \leq n\} \cup \{v_{ij}\ /\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where u_i are the support vertices and v_{ij} are the pendant vertices adjacent to corresponding u_i.

Let $E(C_n \circ \overline{K}_m) = \{e_i, f_{ij}\ /\ e_i = u_iu_{i+1},\ f_{ij} = u_iv_{ij}\ for\ 1 \leq i \leq n, 1 \leq j \leq m.\}$

Then, $V(T(C_n \circ \overline{K}_m)) = \{u_i, e_i, v_{ij}, f_{ij}/1 \leq i \leq n,\ 1 \leq j \leq m\}$ and

$$\begin{aligned} E(T(C_n \circ \overline{K}_m)) &= \{u_ie_i, e_{i-1}u_i, u_iu_{i+1}, e_ie_{i+1}/1 \leq i \leq n\} \\ &\quad \cup\{u_if_{ij}, u_iv_{ij}, e_if_{ij}, f_{ij}v_{ij}, e_if_{(i+1)j}/1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup\{f_{ij}f_{ik}/j \neq k, 1 \leq i \leq n, 1 \leq j, k \leq m\}. \end{aligned}$$

The number of edges in $T(C_n \circ \overline{K}_m)$ is $\frac{1}{2}(n(m+1)(m+8))$. If the graph is fork-decomposable, then $n(m+1)(m+8) \equiv 0\ (mod\ 8)$ which implies $nm(m+1) \equiv 0\ (mod\ 8)$.

If n is odd, then $m(m+1) \equiv 0\ (mod\ 8)$ which implies $m \equiv 0\ (mod\ 8)$ or $m+1 \equiv 0\ (mod\ 8)$. Hence, $m \equiv 0\ (mod\ 8)$ or $m \equiv 7\ (mod\ 8)$ for all values of n which is condition 1. If n is even, then $m(m+1) \equiv 0\ (mod\ 4)$. This implies that, $m \equiv 0\ (mod\ 4)$ or $m+1 \equiv 0\ (mod\ 4)$. Hence, $m \equiv 0\ (mod\ 4)$ or $m \equiv 3\ (mod\ 4)$ which is condition 2. Also, since $m(m+1)$ is even, $n \equiv 0\ (mod\ 4)$ which is condition 3.

Now, let us prove the converse part.

Case 1. $m \equiv 0, 7$ $(mod\ 8)$.

Subcase (a) $m \equiv 0$ $(mod\ 8)$.

If $m = 8$, the result is proved for all values of n by Lemma 4.8. For $m > 8$, consider the induced subgraph $\langle \{u_i, e_i, v_{im}, v_{i(m-1)}, \ldots, v_{i(m-7)}, f_{im}, f_{i(m-1)}, \ldots, f_{i(m-7)}\} \ / \ 1 \leq i \leq n\rangle$ is isomorphic to $T(C_n \circ \overline{K}_8)$ which is fork-decomposable by Lemma 4.8.

The induced subgraph obtained by removing the above $T(C_n \circ \overline{K}_8)$ from $T(C_n \circ \overline{K}_m)$ is isomorphic to a graph which can be decomposed into n copies of $K_{8,m-8}$ and n copies of H_1 which is given in figure 1. The graph $K_{8,m-8}$ is fork-decomposable by Theorem 2.1. The vertices inside the dotted ellipse in figure 1 is adjacent to each other and hence it forms a complete graph K_{m-8}.

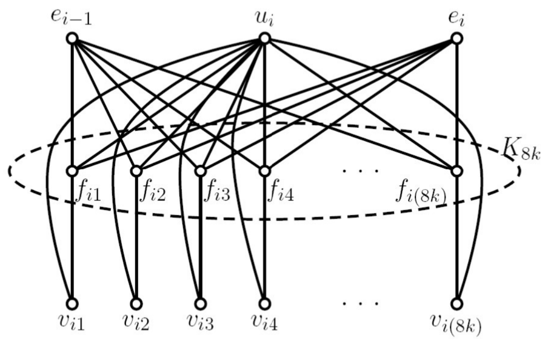

FIGURE 1. H_1

The induced subgraph $\langle\{f_{i1}, f_{i2}, \ldots, f_{i(m-8)}, v_{i1}, v_{i2}, \ldots, v_{i(m-8)}\} \ / \ 1 \leq i \leq n\rangle$ is isomorphic to n copies of $K_{m-8} \circ K_1$ which is fork-decomposable by Theorem 2.4. The fork-decomposition of the remaining subgraph after removing $K_{8,m-8}$ and $K_{m-8} \circ K_1$ from H_1 is given by $\{f_{ij}e_{i-1}, \ f_{ij}e_i, \ f_{ij}u_i, \ u_iv_{ij}\}$ where $1 \leq i \leq n, 1 \leq j \leq m-8$.

Subcase (b) $m \equiv 7$ $(mod\ 8)$.

If $m = 7$, the result is proved for all values of n by Lemma 4.7. For $m > 7$, the induced subgraph $\langle\{u_i, e_i, \ v_{im}, \ v_{i(m-1)}, \ \ldots, \ v_{i(m-6)}, \ f_{im}, \ f_{i(m-1)}, \ \ldots, \ f_{i(m-6)}\} \ / \ 1 \leq i \leq n\rangle$ is isomorphic to $T(C_n \circ \overline{K}_7)$ which is fork-decomposable by Lemma 4.7. The subgraph obtained after removing the above $T(C_n \circ \overline{K}_7)$ from $T(C_n \circ \overline{K}_m)$ is isomorphic to a graph which can be decomposed into n copies of $K_{7,m-7}$ which are fork-decomposable by Theorem 2.1 and n copies of H_1 (Figure 1) which are also fork-decomposable.

Case 2. $n \equiv 0$ $(mod\ 2)$ and $m \equiv 0, 3$ $(mod\ 4)$.

Subcase (a) $n \equiv 0$ $(mod\ 2)$ and $m \equiv 0$ $(mod\ 4)$.

If $m = 4$, the result is proved by Lemma 4.4. For $m > 4$, the induced subgraph $\langle\{u_i, e_i, \ v_{im}, \ v_{i(m-1)}, \ \ldots, \ v_{i(m-3)}, \ f_{im}, \ f_{i(m-1)}, \ \ldots, \ f_{i(m-3)}\} \ / \ 1 \leq i \leq n\rangle$ is isomorphic to $T(C_n \circ \overline{K}_4)$ which is fork-decomposable by Lemma 4.4. The subgraph obtained after removing the above $T(C_n \circ \overline{K}_4)$ from $T(C_n \circ \overline{K}_m)$ is isomorphic to a graph which can be decomposed into n copies of $K_{4,m-4}$ which are fork-decomposable by Theorem 2.1 and n copies of H_1 which are also fork-decomposable.

Subcase (b) $m \equiv 3$ $(mod\ 4)$ and $n \equiv 0$ $(mod\ 2)$.

If $m = 3$, the result is proved by Lemma 4.3. For $m > 3$. The induced subgraph obtained by $\langle\{u_i, e_i, v_{im}, v_{i(m-1)}, v_{i(m-2)}, f_{im}, f_{i(m-1)}, f_{i(m-2)}\} \ / \ 1 \leq i \leq n\rangle$ is isomorphic to $T(C_n \circ \overline{K}_3)$ which is fork-decomposable by Lemma 4.3. The subgraph obtained after removing the above $T(C_n \circ \overline{K}_3)$ from $T(C_n \circ \overline{K}_m)$ is isomorphic to a graph which can be decomposed into n copies of $K_{3,m-3}$ which are fork-decomposable by Theorem 2.1 and n copies of H_1 which are also fork-decomposable.

Case 3. $n \equiv 0 \ (mod\ 4)$.

It is enough to prove the result for $m \equiv 1, 2 \ (mod\ 4)$. Let $m \equiv 1 \ (mod\ 4)$. For $m = 1, 5, 9$, the result is proved by Lemmas 4.1, 4.5 and 4.9. For $m > 9$, the induced subgraph $\langle \{u_i, e_i, v_{im}, v_{i(m-1)}, \ldots, v_{i(m-8)}, f_{im}, f_{i(m-1)}, \ldots, f_{i(m-8)}\} \ / \ 1 \leq i \leq n \rangle$ is isomorphic to $T(C_n \circ \overline{K}_9)$ which is fork-decomposable by Lemma 4.9. The subgraph obtained after removing the above $T(C_n \circ \overline{K}_9)$ from $T(C_n \circ \overline{K}_m)$ is isomorphic to a graph which can be decomposed into n copies of $K_{9,m-9}$ which are fork-decomposable by Theorem 2.1 and n copies of H_1 which are also fork-decomposable.

Now, let $m \equiv 2 \ (mod\ 4)$. For $m = 2$, the result is proved by Lemma 4.2. For $m > 2$, the induced subgraph $\langle \{u_i, e_i, v_{im}, v_{i(m-1)}, f_{im}, f_{i(m-1)}\} \ / \ 1 \leq i \leq n \rangle$ is isomorphic to $T(C_n \circ \overline{K}_2)$ which is fork-decomposable by Lemma 4.2. The subgraph obtained after removing the above $T(C_n \circ \overline{K}_2)$ from $T(C_n \circ \overline{K}_m)$ is isomorphic to a graph which can be decomposed into n copies of $K_{2,m-2}$ which are fork-decomposable by Theorem 2.1 and n copies of H_1 which are also fork-decomposable. □

5. Conclusion

In this paper, we have reviewed the literature on decomposition of graphs and its applications with special reference to the subgraph fork. Fork-decomposition of 191-edge connected graphs has already been studied in the literature. But this constant is far from best possible. Very little is known about lower bounds. In this paper, we have investigated and characterized some class of 2-edge connected graphs for fork-decomposition. In Section 3, we have characterized the fork-decomposition of total graph of paths and null graphs. In Section 4, we have characterized the fork-decomposition of total graph of cycles and null graphs. A similar characterization for fork-decomposition of $T(G \circ \overline{K}_m)$ where $G \in \{K_n, K_{m,n}, W_n\}$ seems to be an interesting open problem for further research.

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