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FUZZY BIATOMIC PAIRS IN FUZZY LATTICES

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Abstract. The concept of a fuzzy biatomic pair in a fuzzy lattice is introduced. Characterizations of a fuzzy biatomic pair in fuzzy lattices are obtained in terms of a fuzzy modular pair.

Keywords: Fuzzy lattices, fuzzy biatomic pair, fuzzy biatomic lattice, fuzzy weak covering property, fuzzy weak exchange property.

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1. Introduction

Bennett [2] studied the class of biatomic lattices with the nomenclature "additive lattices". In [3], she studied biatomic lattices in two possible directions in which investigations could be continued. These were (i) by defining the concept of biatomic pairs in the general lattice and (ii) by replacing an atom with a join-irreducible element.

The concept of a fuzzy set is introduced by Zadeh [16]. Many researchers have introduced fuzzy algebraic structures such as fuzzy groups by Rosenfeld [9], fuzzy lattices by Ajmal et. al. [1], Chon [4], Mezzomo et. al. [6, 8].

Wasadikar and Khubchandani [10] have introduced the concept of a fuzzy modular pair in a fuzzy lattice. In [15] Khubchandani and Khubchandani defined five distinct notions of modular pairs in a fuzzy lattice.

In this paper, we define fuzzy biatomic pairs in fuzzy lattices. Characterizations of a fuzzy biatomic pair in fuzzy lattices are obtained in terms of a fuzzy modular pair. The motivation is from the work of Khubchandani and Khubchandani [15].

2. Preliminaries

Zadeh [17] introduced the concept of a fuzzy binary relation and a fuzzy partial order relation.

Throughout this paper, X denotes a nonempty set. We recall some concepts.

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Definition 2.1. *(Chon* [4, Definition 2.1]*) A mapping $A : X \times X \to [0, 1]$ is called a fuzzy binary relation on X.*

Definition 2.2. *(Chon* [4, Definition 2.1]*) A fuzzy binary relation A on X is called:*

(i) *fuzzy reflexive: if $A(a, a) = 1$, for all $a \in X$;*
(ii) *fuzzy symmetric: if $A(a, b) = A(b, a)$, for all $a, b \in X$;*
(iii) *fuzzy transitive: if $A(a, c) \geq \sup_{b \in X} \min[A(a, b),\ A(b, c)]$;*
(iv) *fuzzy antisymmetric: if $A(a, b) > 0$ and $A(b, a) > 0$ implies $a = b$.*

Definition 2.3. *(Chon* [4, Definition 2.1]*) Let A be a fuzzy binary relation on X.*

(i) *A is called a fuzzy equivalence relation on X, if A is fuzzy reflexive, fuzzy symmetric and fuzzy transitive.*
(ii) *A is called a fuzzy partial order relation, if A is fuzzy reflexive, fuzzy antisymmetric and fuzzy transitive. The pair (X, A) is called a fuzzy partially ordered set or a fuzzy poset.*
(iii) *A is called a fuzzy total order relation, if it is a fuzzy partial order relation and $A(a, b) > 0$ or $A(b, a) > 0$, for all $a, b \in X$. In this case, the fuzzy poset (X, A) is called a fuzzy totally ordered set or a fuzzy chain.*

Several researchers have studied fuzzy lattices. Chon [4] and some others use the terms *upper bound, lower bound* and use the notations $a \vee b$ and $a \wedge b$ to denote the supremum and the infimum of two elements a, b in a fuzzy lattice X in the fuzzy sense. Since the set X is arbitrary, this gives the impression that X itself is a partially ordered set or a lattice. So we use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$, respectively.

Definition 2.4. *(Chon* [4, Definition 3.1]*) Let (X, A) be a fuzzy poset and let $Y \subseteq X$. An element $b \in X$ is said to be a fuzzy upper bound for Y iff $A(a, b) > 0$ for all $a \in Y$. A fuzzy upper bound b_0 for Y is called a least upper bound (or supremum) of Y iff $A(b_0, b) > 0$ for every fuzzy upper bound b for Y. We then write $b_0 = \sup_F Y = \vee_F Y$. If $Y = \{a, b\}$, then we write $\vee_F Y = a \vee_F b$.*

Similarly, an element $c \in X$ is said to be a fuzzy lower bound for Y iff $A(c, a) > 0$, for all $a \in Y$. A fuzzy lower bound c_0 for Y is called a fuzzy greatest lower bound (or infimum) of Y iff $A(c, c_0) > 0$ for every fuzzy lower bound c for Y. We then write $c_0 = \inf_F Y = \wedge_F Y$. If $Y = \{a, b\}$, then we write $\wedge_F Y = a \wedge_F b$.

Since A is fuzzy antisymmetric, the fuzzy least upper (fuzzy greatest lower) bound, if it exists, is unique, see Mezzomo et. al. [6, Remark 3.2].

Definition 2.5. *(Chon* [4, Definition 3.2]*) Let (X, A) be a fuzzy poset. Then, (X, A) is called a fuzzy lattice if and only if $a \vee_F b$ and $a \wedge_F b$ exist, for all $a, b \in X$.*

Definition 2.6. *(Mezzomo et. al.* [7, Definition 3.4]*) A fuzzy lattice (X, A) is said to be bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for every $a \in X$. In this case, $\perp$ and $\top$ are called bottom and top elements of X, respectively.*

We illustrate these concepts in the following example. In this example, the fuzzy poset (X, A) is a fuzzy lattice.

Example 2.1. *Let $X = \{\perp, a, b, c, d, e, f, g, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\perp, \perp) = A(a, a) = A(b, b) = A(c, c) = A(d, d) = A(e, e) = A(f, f) = 1$,
$A(g, g) = A(\top, \top) = 1$,
$A(\perp, a) = 0.6,\ A(\perp, b) = 0.6,\ A(\perp, c) = 0.6,\ A(\perp, d) = 0.6,\ A(\perp, e) = 0.6$,

$A(\bot, f) = 0.6$, $A(\bot, g) = 0.6$, $A(\bot, \top) = 0.06$,
$A(a, \bot) = 0$, $A(a, b) = 0$, $A(a, c) = 0$, $A(a, d) = 0$, $A(a, e) = 0.8$, $A(a, f) = 0$, $A(a, g) = 0$, $A(a, \top) = 0.06$,
$A(b, \bot) = 0$, $A(b, a) = 0$, $A(b, c) = 0$, $A(b, d) = 0$, $A(b, e) = 0.8$, $A(b, f) = 0.8$, $A(b, g) = 0$, $A(b, \top) = 0.06$,
$A(c, \bot) = 0$, $A(c, a) = 0$, $A(c, b) = 0$, $A(c, d) = 0$, $A(c, e) = 0$, $A(c, f) = 0$, $A(c, g) = 0.8$, $A(c, \top) = 0.06$,
$A(d, \bot) = 0$, $A(d, a) = 0$, $A(d, b) = 0$, $A(d, c) = 0$, $A(d, e) = 0$, $A(d, f) = 0$, $A(d, g) = 0.8$, $A(d, \top) = 0.02$,
$A(e, \bot) = 0$, $A(e, a) = 0$, $A(e, b) = 0$, $A(e, c) = 0$, $A(e, d) = 0$, $A(e, f) = 0$, $A(e, g) = 0$, $A(e, \top) = 0.06$,
$A(f, \bot) = 0$, $A(f, a) = 0$, $A(f, b) = 0$, $A(f, c) = 0$, $A(f, d) = 0$, $A(f, e) = 0$, $A(f, g) = 0$, $A(f, \top) = 0.06$,
$A(g, \bot) = 0$, $A(g, a) = 0$, $A(g, b) = 0$, $A(g, c) = 0$, $A(g, d) = 0$, $A(g, e) = 0$, $A(g, f) = 0$, $A(g, \top) = 0.06$,
$A(\top, \bot) = 0$, $A(\top, a) = 0$, $A(\top, b) = 0$, $A(\top, c) = 0$, $A(\top, d) = 0$, $A(\top, e) = 0$, $A(\top, f) = 0$, $A(\top, g) = 0$.

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	d	e	f	g	$\top$
$\bot$	1.0	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
a	0.0	1.0	0.0	0.0	0.0	0.8	0.0	0.0	0.06
b	0.0	0.0	1.0	0.0	0.0	0.8	0.8	0.0	0.06
c	0.0	0.0	0.0	1.0	0.0	0.0	0.8	0.8	0.06
d	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.8	0.06
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.06
f	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.06
g	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.06
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	d	e	f	g	$\top$
$\bot$	$\bot$	a	b	c	d	e	f	g	$\top$
a	a	a	e	$\top$	$\top$	e	$\top$	$\top$	$\top$
b	b	e	b	f	$\top$	e	f	$\top$	$\top$
c	c	$\top$	f	c	g	$\top$	f	g	$\top$
d	d	$\top$	$\top$	g	d	$\top$	$\top$	g	$\top$
e	e	e	e	$\top$	$\top$	e	$\top$	$\top$	$\top$
f	f	$\top$	f	f	$\top$	$\top$	f	$\top$	$\top$
g	g	$\top$	$\top$	g	g	$\top$	$\top$	g	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	d	e	f	g	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	$\bot$	a	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	$\bot$	$\bot$	b	b	$\bot$	b
c	$\bot$	$\bot$	$\bot$	c	$\bot$	$\bot$	c	c	c
d	$\bot$	$\bot$	$\bot$	$\bot$	d	$\bot$	$\bot$	d	d
e	$\bot$	a	b	$\bot$	$\bot$	e	b	$\bot$	e
f	$\bot$	$\bot$	b	c	$\bot$	b	f	c	f
g	$\bot$	$\bot$	$\bot$	c	d	$\bot$	c	g	g
$\top$	$\bot$	a	b	c	d	e	f	g	$\top$

We note that (X, A) is a fuzzy lattice.

The following result is from Chon [4, Proposition 3.3] and Mezzomo et. al. [6, Proposition 3.5]

Proposition 2.1. *Let (X, A) be a fuzzy lattice. For every $a, b, c \in X$:*

(i) $A(a, a \vee_F b) > 0$, $A(b, a \vee_F b) > 0$, $A(a \wedge_F b, a) > 0$, $A(a \wedge_F b, b) > 0$.
(ii) $A(a, c) > 0$ *and* $A(b, c) > 0$ *implies* $A(a \vee_F b, c) > 0$.
(iii) $A(c, a) > 0$ *and* $A(c, b) > 0$ *implies* $A(c, a \wedge_F b) > 0$.
(iv) $A(a, b) > 0$ *iff* $a \vee_F b = b$.

(v) $A(a,b) > 0$ *iff* $a \wedge_F b = a$.
(vi) *If* $A(b,c) > 0$, *then* $A(a \wedge_F b, a \wedge_F c) > 0$ *and* $A(a \vee_F b, a \vee_F c) > 0$.
(vii) *If* $A(a \vee_F b, c) > 0$, *then* $A(a,c) > 0$ *and* $A(b,c) > 0$.
(viii) *If* $A(a, b \wedge_F c) > 0$, *then* $A(a,b) > 0$ *and* $A(a,c) > 0$.

Chon [4] has considered fuzzy distributivity and fuzzy modularity in fuzzy lattices.

Definition 2.7. *(Chon [4]) Let (X,A) be a fuzzy lattice. (X,A) is called a fuzzy distributive lattice, if $a \wedge_F (b \vee_F c) = (a \wedge_F b) \vee_F (a \wedge_F c)$ and $a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F (a \vee_F c)$ for all $a,b,c \in X$.*

Definition 2.8. *(Chon [4]) A fuzzy lattice (X,A) is called fuzzy modular if $A(a,c) > 0$ implies $a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F c$, for all $a,b,c \in X$.*

Corollary 2.1. *(Wasadikar and Khubchandani [14]) Let (X,A) be a fuzzy lattice and $a,b,c,d \in X$. If $A(c,a) > 0$ and $A(d,b) > 0$, then $A(c \wedge_F d, a \wedge_F b) > 0$ and $A(c \vee_F d, a \vee_F b) > 0$.*

Definition 2.9. *(Wasadikar and Khubchandani [10]) Let $\mathcal{L} = (X,A)$ be a fuzzy lattice. Let $a,b \in X$, then $b \prec_F a$ (a "fuzzy covers" b) if $0 < A(b,a) < 1$, $A(b,c) > 0$ and $A(c,a) > 0$ imply $c = b$ or $c = a$.*

Definition 2.10. *(Wasadikar and Khubchandani [10]) Let P denote the set of all $a \in X$ such that $\perp \prec_F a$. The elements of P are called fuzzy atoms.*

Dually, we the concept of fuzzy dual atom is defined.

Definition 2.11. *(Wasadikar and Khubchandani [15]) Let (X,A) be a fuzzy lattice. A pair of elements $a,b \in X$ is said to be fuzzy join-modular pair, denoted by $(a,b)_F M_j$, if the following condition holds:*
$(a,b)_F M_j : \{a \wedge_F (b \vee_F c)\} \vee_F b = (a \vee_F b) \wedge_F (b \vee_F c)$ for every $c \in X$.

Definition 2.12. *(Wasadikar and Khubchandani [11]) A fuzzy lattice $\mathcal{L} = (X,A)$ is called a FM_j-symmetric fuzzy lattice if in $\mathcal{L}$, $(a,b)_F M_j$ implies $(b,a)_F M_j$.*

Definition 2.13. *(Wasadikar and Khubchandani [12]) Let $\mathcal{L} = (X,A)$ be a fuzzy lattice with $\perp$.*
We call the following property as the fuzzy covering property:
If p is a fuzzy atom and $a \wedge_F p = \perp$, then $a \prec_F a \vee_F p$.

Definition 2.14. *(Wasadikar and Khubchandani [12]) Let $\mathcal{L} = (X,A)$ be a fuzzy lattice with $\perp$.*
The following property is called the fuzzy exchange property:
If p and q are fuzzy atoms and if $a \wedge_F p = \perp$, $A(p, a \vee_F q) > 0$ implies $A(q, a \vee_F p) > 0$ (hence implies $a \vee_F p = a \vee_F q$).

Lemma 2.1. *(Wasadikar and Khubchandani [12]) If a fuzzy lattice $\mathcal{L} = (X,A)$ with $\perp$ has the fuzzy covering property, then $\mathcal{L}$ has the fuzzy exchange property.*

Definition 2.15. *(Wasadikar and Khubchandani [12]) A fuzzy poset $\mathcal{L} = (X,A)$ with the least element $\perp$ is called fuzzy atomistic if every element $x \in X$ is the least upper bound of the set of fuzzy atoms less than or equal to x.*

Lemma 2.2. *(Wasadikar and Khubchandani [12]) A fuzzy lattice $\mathcal{L} = (X,A)$ with $\perp$ is fuzzy atomistic iff $0 < A(a,b) < 1$ implies the existence of a fuzzy atom p such that $A(p,a) = 0$ and $A(p,b) > 0$.*

Definition 2.16. *(Khubchandani and Khubchandani [15]) Let (X, A) be a fuzzy lattice and $a, b \in X$ we write $(a, b)_F C_m((a, b)_F C_j)$ if the following condition is satisfied:*
$(a, b)_F C_m$: $A(a \wedge_F b, b \wedge_F c) > 0$ and $A(b, c) = 0$ together imply that $a \wedge_F b = c \wedge_F b$.
$(a, b)_F C_j$: $A(a \vee_F b, b \vee_F c) > 0$ and $A(c, b) = 0$ together imply that $a \vee_F b = c \vee_F b$.

Lemma 2.3. *(Khubchandani and Khubchandani [15]) Let (X, A) be a fuzzy lattice with $\perp$ and $a, b \in X$.*

(i) *If $A(b, a) > 0$, then $(a, b)_F C_m$.*
(ii) *If $A(b, a) = 0$, then $(a, b)_F C_m$ if and only if $a \wedge_F b \prec_F b$.*
(iii) *If p is a fuzzy atom, then $(x, p)_F C_m$ holds for every $x \in X$.*

Proposition 2.2. *(Khubchandani and Khubchandani [15]) Let (X, A) be a fuzzy lattice with $\perp$ and $a, b \in X$.*

(i) *If $(a, b)_F C_j$ implies $(a, b)_F M_j$.*
(ii) *If $(a, b)_F M_j$ and $(b, a)_F C_m$, then $(a, b)_F C_j$.*
(iii) *If p is a fuzzy atom, then $(p, b)_F C_j$ if and only if $(p, b)_F M_j$.*

3. Fuzzy biatomic pairs in fuzzy lattices

Definition 3.1. *A pair of non-zero elements a, b of a fuzzy atomistic lattice (X, A) is called fuzzy biatomic and is denoted by $(a, b)_F P$ if for every fuzzy atom $p \in X$ with $A(p, a \vee_F b) > 0$, there exists fuzzy atoms $q, r \in X$ such that $A(p, q \vee_F r) > 0$, $A(q, a) > 0$ and $A(r, b) > 0$.*

Example 3.1. *Following are the fuzzy biatomic pair in Example 2.1.*
1) As $A(b, e \vee_F f) = A(b, \top) = 0.06 > 0$, $A(a, e) = 0.8 > 0$, $A(c, f) = 0.8 > 0$ and $A(b, a \vee_F c) = A(b, \top) = 0.06 > 0$. So, $(e, f)_F P$ is a fuzzy biatomic pair in fuzzy lattice (X, A).
2) As $A(c, e \vee_F g) = A(c, \top) = 0.06 > 0$, $A(b, e) = 0.8 > 0$, $A(d, g) = 0.8 > 0$ and $A(c, b \vee_F d) = A(c, \top) = 0.06 > 0$. So, $(e, g)_F P$ is a fuzzy biatomic pair in fuzzy lattice (X, A).
3) As $A(a, f \vee_F g) = A(a, \top) = 0.06 > 0$, $A(b, f) = 0.08 > 0$, $A(c, g) = 0.08 > 0$, $A(a, f \vee_F g) = A(a, \top) = 0.06 > 0$. So $(f, g)_F P$ is fuzzy biatomic pair in fuzzy lattice (X, A).

Definition 3.2. *An fuzzy atomistic lattice (X, A) is said to be fuzzy biatomic lattice if $(a, b)_F P$ holds for all $a, b \in X$.*

It is remarked that $(a, b)_F P$ implies $(b, a)_F P$ and $A(a, b) > 0$ implies $(a, b)_F P$.

Wasadikar and Khubchandani [12] have defined fuzzy exchange property in fuzzy lattice (X, A). We define fuzzy weak exchange property.

Definition 3.3. *A fuzzy lattice (X, A) is said to have the fuzzy weak exchange property if for every fuzzy atoms $p, q, r \in X$, $A(p, q \vee_F r) > 0$ and $p \neq r$, then $A(q, p \vee_F r) > 0$.*

We establish a relation between the fuzzy biatomic pairs and the fuzzy dual modular pairs.

Lemma 3.1. *Let (X, A) be a fuzzy atomistic lattice with the fuzzy weak exchange property and let a, b be non zero elements of X. If $(a, b)_F P$, then $(a, b)_F M_j$ and converse is true if b is a fuzzy atom.*

Proof. Assume $(a, b)_F P$ holds. Let $x = (a \vee_F b) \wedge_F (b \vee_F c)$ for $c \in X$.
By Proposition 2.1(v) we have $A(x, a \vee_F b) > 0$ and $A(x, b \vee_F c) > 0$.

Also let $y = \{a \wedge_F (b \vee_F c)\} \vee_F b$ for $c \in X$. By Proposition 2.1(iv) we have $A(b, y) > 0$ and $A(a \wedge_F (b \wedge_F c), y) > 0$.
To prove $(a, b)_F M_j$, it is sufficient to show that $A(x, y) > 0$.
By fuzzy atomisticity of (X, A), it is sufficient to prove that every fuzzy atom contained in x is contained in y. Let $p \in X$ be a fuzzy atom with $A(p, x) > 0$.
Case i) If $A(p, b) > 0$, then $A(p, y) > 0$ as $A(b, y) > 0$.
Case ii) If $A(p, b) = 0$, as $x = (a \vee_F b) \wedge_F (b \vee_F c)$ we have $A(x, a \vee_F b) > 0$. (I)
As $A(p, x) > 0$. (II)
From (I) and (II) by fuzzy antisymmetry of A we have $A(p, a \vee_F b) > 0$.
By $(a, b)_F P$ there exists fuzzy atoms $q, r \in X$ such that $A(p, q \vee_F r) > 0$, $A(q, a) > 0$ and $A(r, b) > 0$. Clearly $r \neq p$, as $A(p, b) = 0$ and $A(r, b) > 0$.
By the fuzzy weak exchange property $A(q, p \vee_F r) > 0$.
Moreover as $x = (a \vee_F b) \wedge_F (b \vee_F c)$ by Proposition 2.1(v) we have $A(x, b \vee_F c) > 0$.
As $A(p, x) > 0$ and $A(x, b \vee_F c) > 0$ by fuzzy transitivity of A we have
$A(p, b \vee_F c) > 0$. (III)
As $A(r, b) > 0$ and always $A(b, b \vee_F c) > 0$ by fuzzy transitivity of A we have
$A(r, b \vee_F c) > 0$. (IV)
By (III), (IV) and Corollary 2.1 we have $A(p \vee_F r, b \vee_F c) > 0$.
Taking meet a on both sides we get $A((p \vee_F r) \wedge_F a, (b \vee_F c) \wedge_F a) > 0$. (V)
As $A(q, p \vee_F r) > 0$ by Proposition 2.1(vi) we have
$A(q \wedge_F a, (p \vee_F r) \wedge_F a) > 0$. (VI)
As $A(q, a) > 0$ by Proposition 2.1(v) we have $q \wedge_F a = q$.
Therefore equation (VI) reduces to $A(q, (p \vee_F r) \wedge_F a) > 0$. (VII)
From (V) and (VII) by fuzzy transitivity of A we get $A(q, (b \vee_F c) \wedge_F a) > 0$.
As $A(q, (b \vee_F c) \wedge_F a) > 0$ and $A(r, b) > 0$, by Corollary 2.1 we get
$A(q \vee_F r, \{(b \vee_F c) \wedge_F a\} \vee_F b) > 0$. (VIII)
As $A(p, q \vee_F r) > 0$. (XI)
From (VIII) and (XI) by fuzzy transitivity of A we get $A(p, \{(b \vee_F c) \wedge_F a\} \vee_F b) > 0$.
As $y = \{a \wedge_F (b \vee_F c)\} \wedge_F b$ we get $A(p, y) > 0$. Hence by fuzzy atomisticity of (X, A) we get $A(x, y) > 0$ as required.

Conversely, assume that r is a fuzzy atom and $(a, r)_F M_j$ holds.
We shall show that $(a, r)_F P$ holds. We may assume that $A(r, a) = 0$. Let p be a fuzzy atom with $A(p, r \vee_F a) > 0$.
Case i) If $p = r$, then q may be any fuzzy atom contained in a and it will serve the purpose.
Case ii) Suppose $p \neq r$. By $(a, r)_F M_j$ we have $\{a \wedge_F (r \vee_F p)\} \vee_F r = (a \vee_F r) \wedge_F (r \vee_F p)$. Clearly $a \wedge_F (r \vee_F p) \neq \perp$. Otherwise $A(p, r) > 0$ (as $p \in (r \vee_F p) \wedge_F (a \vee_F r) = \{a \wedge_F (r \vee_F p)\} \vee_F r = r$) which is impossible. Therefore we may take any fuzzy atom q contained in $\{a \wedge_F (r \vee_F p)\}$. Using fuzzy weak exchange property for $A(q, p \vee_F r) > 0$ and $q \neq r$, we have $A(p, q \vee_F r) > 0$ and $A(q, a) > 0$. Thus $(a, r)_F P$ holds. □

Lemma 3.2. *Let a, b be non zero elements of a fuzzy atomistic lattice (X, A) with fuzzy weak exchange property. If $(b, p)_F M_j$ holds for every fuzzy atom p, then $(a, b)_F P$ iff $(a, b)_F M_j$.*

Proof. From Lemma 3.1, it follows that $(a, b)_F P$ implies $(a, b)_F M_j$.
Conversely, suppose $(a, b)_F M_j$ holds. let p be a fuzzy atom with $A(p, a \vee_F b) > 0$.
Assume $A(a \wedge_F (b \vee_F p), b) > 0$. By Proposition 2.1(iv) we have $\{a \wedge_F (b \vee_F p)\} \vee_F b = b$.
By $(a, b)_F M_j$ we have $\{a \wedge_F (b \vee_F p)\} \vee_F b = (a \vee_F b) \wedge_F (b \vee_F p)$. Since $A(p, a \vee_F b) > 0$, by Corollary 2.1 we have $A(p \vee_F b, a \vee_F b) > 0$. Hence by Proposition 2.1(v) we have $(p \vee_F b) \wedge_F (a \vee_F b) = p \vee_F b$. Therefore $b = \{a \wedge_F (b \vee_F p)\} \vee_F b = (a \vee_F b) \wedge_F (b \vee_F p) = b \vee_F p$,

i.e., we get $b = b \vee_F p$. Thus by Proposition 2.1(iv) we have $A(p, b) > 0$ and hence any fuzzy atom $A(q, a) > 0$ will serve the purpose. Now suppose $A(a \wedge_F (b \vee_F p), b) = 0$. Then there exists $A(x, a \wedge_F (b \vee_F p)) > 0$ such that $A(x, b) = 0$. Since (X, A) is fuzzy atomistic, there exists a fuzzy atom $q \in X$ such that $A(q, x) > 0$ and $A(q, b) = 0$. Hence $A(q, a \wedge_F (b \vee_F p)) > 0$ as $A(q, x) > 0$. Since $(b, p)_F M_j$ holds, which is equivalent to $(b, p)_F P$ by Lemma 3.1 and $A(q, b \vee_F p) > 0$ we have a fuzzy atom $r \in X$ such that $A(q, r \vee_F p) > 0$ and $A(r, b) > 0$. Clearly $q \neq r$ and by the fuzzy weak exchange property $A(p, q \vee_F r) > 0$ as required. □

Definition 3.4. *A fuzzy lattice (X, A) is said to be fuzzy atom dual modular if $(x, p)_F M_j$ holds for all $x \in X$ and for all fuzzy atoms $p \in X$. A fuzzy lattice (X, A) is called FM_j-symmetric if $(a, b)_F M_j$ implies $(b, a)_F M_j$ for $a, b \in X$.*

Theorem 3.1. *Let $\mathcal{L} = (X, A)$ be a fuzzy atomistic lattice. Then the following statements are equivalent.*

(i) *$\mathcal{L}$ is fuzzy atom dual modular.*
(ii) *$\mathcal{L}$ has the fuzzy covering property and is FM_j-symmetric.*
(iii) *In $\mathcal{L}$, $(b, a)_F C_j \Rightarrow (a, b)_F C_m$.*
(iv) *In $\mathcal{L}$, $(a, b)_F C_m \Rightarrow (b, a)_F C_j \Rightarrow (a, b)_F M_m$.*

Proof. *(i) $\Rightarrow$ (ii)*
Clearly $\mathcal{L} = (X, A)$ has the fuzzy weak covering property. As $\mathcal{L} = (X, A)$ is fuzzy atom dual modular, i.e., $(x, p)_F M_j$ holds for all $x \in X$ and for all fuzzy atoms $p \in X$. In particular, $(q, r)_F M_j$ holds for fuzzy atoms $q, r \in X$. Hence by Lemma 2.1 $\mathcal{L}$ has fuzzy weak exchange property. Now, by Lemma 3.1 $(x, p)_F P$ holds which is also equivalent to $(p, x)_F P$ which yields $(p, x)_F M_j$ again by Lemma 3.1. Thus (X, A) has the fuzzy covering property.
To show $\mathcal{L}$ is FM_j-symmetric, suppose $(a, b)_F M_j$ holds.
Since $\mathcal{L}$ is fuzzy atom dual modular $(b, p)_F M_j$ holds for all fuzzy atoms $p \in X$. As $(a, b)_F M_j$ and $(b, p)_F M_j$ hold hence by Lemma 3.2, $(a, b)_F P$ holds and equivalently $(b, a)_F P$ holds. As noted earlier $\mathcal{L}$ has fuzzy weak exchange property and hence by Lemma 3.1 $(b, a)_F M_j$. Thus $\mathcal{L}$ is FM_j-symmetric.
(ii) $\Rightarrow$ (i)
By the fuzzy covering property $(p, x)_F M_j$ holds for all fuzzy atoms $p \in X$ and for all $x \in X$. By FM_j-symmetry we have $(x, p)_F M_j$. Thus, $\mathcal{L}$ is fuzzy atom dual modular.
(ii) $\Rightarrow$ (iii)
Suppose $(b, a)_F C_j$ holds. If $A(b, c) > 0$, then $(a, b)_F C_m$ holds by Lemma 2.3. Therefore, we assume that $A(b, a) = 0$. Suppose $(a, b)_F C_m$ does not hold. Then there exists $c \in X$ such that $0 < A(a \wedge_F b, b \wedge_F c) < 1$ and $A(b, c) = 0$. Since $A(a \wedge_F b, b \wedge_F c) > 0$, there exists $A(x, b \wedge_F c) > 0$ such that $A(x, a \wedge_F b) = 0$. As $\mathcal{L}$ is fuzzy atomistic lattice and $A(x, a) = 0$ therefore, there exists a fuzzy atom $q \in X$ such that $A(q, x) > 0$ and $A(q, a) = 0$. As $A(x, b \wedge_F c) > 0$ and $A(b \wedge_F c, b) > 0$ always holds. So, by fuzzy transitivity of A we have $A(x, b) > 0$. From $A(q, x) > 0$ and $A(x, b) > 0$ again by fuzzy transitivity of A we have $A(q, b) > 0$. As $A(q, b) > 0$ by Proposition 2.1(vi) we have $A(q \vee_F a, b \vee_F a) > 0$. This inclusion along with $A(q, a) = 0$ and $(b, a)_F C_j$ together imply $b \vee_F a = a \vee_F q$. Now by (ii), $\mathcal{L}$ has the fuzzy covering property and hence $(q, a)_F M_j$ holds. By FM_j-symmetry $(a, q)_F M_j$ holds. Hence by $(a, q)_F M_j$ and $A(q, b) > 0$ we have

$$\begin{aligned} b \wedge_F (a \vee_F q) &= (a \vee_F q) \wedge_F (q \vee_F b), \\ &= \{a \wedge_F (q \vee_F b)\} \vee_F q, \quad \text{since } (a, q)_F M_j \\ &= (a \wedge_F b) \vee_F q, \quad \text{since } A(q, b) > 0 \text{ so we have } q \vee_F b = b \end{aligned}$$

i.e., we get $b \wedge_F (a \vee_F q) = (a \wedge_F b) \vee_F q$. (I)
Thus

$$\begin{aligned} b &= b \wedge_F (a \vee_F b), && \text{by absorption} \\ &= b \wedge_F (a \vee_F q), && \text{as } b \vee_F a = a \vee_F q \\ &= (a \wedge_F b) \vee_F q, && \text{from (I)} \end{aligned}$$

i.e., we have $b = (a \wedge_F b) \vee_F q$.
Clearly $A((a \wedge_F b) \vee_F q, b \wedge_F c) > 0$,
i.e., $A(b, b \wedge_F c) > 0$ (II)
and $A(b \wedge_F c, b) > 0$ always holds. (III)
From (II) and (III) by fuzzy antisymmetry of A we get $b \wedge_F c = b$.
By Proposition 2.1(v) we have $A(b, c) > 0$, a contradiction to $A(b, c) = 0$.

Conversly, suppose that $(a, b)_F C_m$ holds.
To show $(b, a)_F C_j$ holds.
Let $A(b \vee_F a, a \vee_F c) > 0$ and $A(c, a) = 0$.
Case i) If $A(b, a) > 0$, then $(b, a)_F C_j$ holds by Lemma 2.3.
Case ii) Assume that $A(b, a) = 0$. Since $\mathcal{L}$ is fuzzy atomistic, $A(b, a) = 0$ and $A(c, a) = 0$, there exists fuzzy atoms $p, q \in X$ such that $A(p, b) > 0$, $A(p, a) = 0$, $A(q, c) > 0$ and $A(q, a) = 0$. Hence $A(b \vee_F a, a \vee_F p) > 0$. We claim that $b \vee_F a = a \vee_F p$. If not, then there exists $y \in X$ such that $A(a \vee_F p, y) > 0$ with $A(b \vee_F a, y) = 0$. As $A(a, a \vee_F p) > 0$ and $A(a \vee_F p, y) > 0$ by fuzzy transitivity we have $A(a, y) > 0$. As $A(a, y) > 0$ by Proposition 2.1(vi) we have $A(q \vee_F b, a \vee_F b) > 0$.
Note that $A(b, y) = 0$, therefore by using $(a, b)_F C_m$ we have $a \wedge_F b = b \wedge_F y$. (IV)
Clearly $A(p, b \wedge_F y) > 0$ from (IV) we get $A(p, a \wedge_F b) > 0$, a contradiction to $A(p, a) = 0$.
Hence $b \vee_F a = a \vee_F p$. (V)
Since $A(q, c) > 0$ and $A(c, a \vee_F b) > 0$ (as $A(b \vee_F a, a \vee_F c) > 0$) by fuzzy transitivity of A we have $A(q, a \vee_F b) > 0$. As $A(q, a) > 0$ by Proposition 2.1(vi) we have $A(q \vee_F b, a \vee_F b) > 0$. As $A(q, q \vee_F b) > 0$ and $A(q \vee_F b, a \vee_F b) > 0$ by fuzzy transitivity of A we get $A(q, a \vee_F b) > 0$. By (V) we have $A(q, a \vee_F p) > 0$. Since $\mathcal{L}$ has the fuzzy covering property, therefore by Lemma 2.1, $\mathcal{L}$ has the fuzzy exchange property. Now, by the fuzzy exchange property we have $A(p, q \vee_F a) > 0$. As we have $A(p, q \vee_F a) > 0$, moreover $A(q \vee_F a, a \vee_F c) > 0$, hence by fuzzy transitivity of A we get $A(p, a \vee_F c) > 0$. This yields $A(a \vee_F c, p \vee_F a) > 0$.
But $p \vee_F a = b \vee_F a$, which yields $A(a \vee_F c, b \vee_F a) > 0$. (VII)
Hence $b \vee_F a = a \vee_F c$. Thus $(b, a)_F C_j$ holds.
$(iii) \Rightarrow (iv)$
Follows from Proposition 2.2(i).
$(iv) \Rightarrow (iii)$
Suppose that $(b, a)_F C_j$ holds. Therefore by (iv) we get $(a, b)_F$ holds. By Lemma 2.2, we have $(a, b)_F C_m$.
Conversely, suppose that $(a, b)_F C_m$ holds. By (iv) we have $(b, a)_F C_j$ holds.
$(iii) \Rightarrow (i)$
In view of Lemma 2.3, it is clear that $(x, p)_F C_m$ holds for all fuzzy atoms p and for all $x \in X$. Hence by assumption, $(p, x)_F C_j$ holds. By Proposition 2.2 $(p, x)_F M_j$ holds for fuzzy atoms p and for all $x \in X$. Therefore $\mathcal{L}$ has the fuzzy covering property and hence, the fuzzy weak exchange property by Lemma 2.1. In view of Lemma 3.1, to prove $\mathcal{L}$ is fuzzy atom dual modular, it is sufficient to show that $(x, q)_F P$ holds for every non zero elements $x \in X$ and for every fuzzy atom $q \in X$. Let $p, q \in X$ be fuzzy atoms and x be a non zero elements of X such that $A(p, x \vee_F q) > 0$.
To show $(x, p)_F P$, we need a fuzzy atom $r \in X$ with $A(r, x) > 0$ such that $A(p, r \vee_F q) > 0$.

When $p = q$ any fuzzy atom $A(r, x) > 0$ may be used and when $A(q, x) > 0$, $r = p$ may be used. Hence we may assume that $p \neq q$ and $q \wedge_F x = \perp$, by Lemma 2.3, $(x, q)_F C_m$ trivially holds for a fuzzy atom q and hence by (iii), $(q, x)_F C_j$ holds. As $A(p, q \vee_F x) > 0$, by Proposition 2.1(iv) we get $x \vee_F p \vee_F q = q \vee_F x$.
Now, we show $(p \vee_F q, x)_F C_j$ holds.
It is clear that if $A(p \vee_F q, x) > 0$, then by Lemma 2.3 $(p \vee_F q, x)_F C_j$ holds trivially.
Therefore suppose that $(p \vee_F q) \wedge_F x = \perp$ holds.
To prove that $(p \vee_F q, x)_F C_j$ holds.
Let $A(p \vee_F q \vee_F x, x \vee_F y) > 0$ (VI)
and $y \wedge_F x = \perp$. Since $q \vee_F x = x \vee_F p \vee_F q$ from (VI) we have $A(q \vee_F x, x \vee_F y) > 0$ and by $(q, x)_F C_j$ we get $q \vee_F x = x \vee_F y$. Hence $p \vee_F q \vee_F x = x \vee_F y$ as required. Thus $(p \vee_F q, x)_F C_j$ holds. Again by (iii) $(x, p \vee_F q)_F C_m$ holds. We assert that $x \wedge_F (p \vee_F q) \neq \perp$. If $x \wedge_F (p \vee_F q) = \perp$, then clearly $A(x \wedge_F (p \vee_F q), (p \vee_F q) \wedge_F p) > 0$ and moreover $(p \vee_F q) \wedge_F p \neq \perp$ and $p \neq q$. Hence by $(x, p \vee_F q)_F C_m$ we have $\perp = x \wedge_F (p \vee_F q) = (p \vee_F q) \wedge_F p = p$, a contradiction to the fact that p is a fuzzy atom. Hence $x \wedge_F (p \vee_F q) \neq \perp$ and must contain a fuzzy atom say $r \in X$. As $q \wedge_F x = \perp$ and $A(r, x) > 0$ and $q \neq r$. Since $A(r, p \vee_F q) > 0$ and as noted earlier $(q, x)_F C_j$ holds for all fuzzy atoms q and for all $x \in X$. Therefore by Proposition 2.2, $\mathcal{L}$ has the fuzzy covering property. Hence by Lemma 2.1, $\mathcal{L}$ has the exchange property. Now using the fuzzy exchange property we have $A(p, r \vee_F q) > 0$ and $A(r, x) > 0$ as required. □

4. Conclusion

In this paper, we have introduced the notion of fuzzy biatomic pair in fuzzy lattice. Also, we have obtained some properties.

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