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TOWARD STABILITY INVESTIGATION OF FRACTIONAL DYNAMICAL SYSTEMS ON TIME SCALE

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Abstract. We study dynamic systems on time scales that are generalizations of classical differential or difference equations. In this paper, we present the asymptotic stability of linear fractional time-invariant systems with the Caputo $\Delta-$derivative on time scale. To ensure the asymptotic stability of this kind of system, some results about necessary and sufficient conditions are investigated, resulting in a region of asymptotic stability. Furthermore, we obtain the results of the asymptotic stability by transforming the stability region of the continuous-time case through suitable Möbious transformations.

Keywords: Time scale calculus; Linear dynamical systems; Fractional calculus; Stability.

AMS Subject Classification: 34-XX and MSC 35R07 and MSC 34A08.

1. Introduction

Time scale calculus was presented by Hilger [24, 25] to generalise and unify the study of theories of discrete and continuous differential equations, as well as to stretch these theories to other sorts of equations called dynamic equations, which have lately attracted a lot of attention. The two principal characteristics of time scale calculus are the unification and extension of discrete and continuous equations. There are numerous results concerning continuous dynamic equations that transfer over pretty readily to analogous results for discrete dynamic equations, whereas discrete dynamic equations' results may appear diametrically opposed to their continuous dual. On time scales, studying dynamic equations reveals these inconsistencies, allowing one to avoid having to repeat the proof of results twice for discrete and continuous dynamic equations. Many contributions and developments in time scale, applications of the theory, and methods have been made by many scholars in various fields [2, 4, 15, 16, 21–23]. In recent years, there has been an awful lot of interest in the study of dynamic systems on time scales because of their applications to real-world problems, consisting of electric circuits and insect populations. There are numerous application problems that may be studied more exactly with the use

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of dynamic systems on time scales. Subjects consisting of the existence and uniqueness of solutions, stability, Floquet theory, periodicity, stability, and boundedness of solutions can be studied more precisely and generally by utilizing dynamical systems on time scales.

The history of an equation's asymptotic stability on a general time scale dates back to the work of Aulbach and Hilger [9]. Results for the stability and instability of a real scalar dynamic equation are provided by Gard and Hoffacker [19]. A different approach to the asymptotic stability of linear delta dynamic equations using Lyapunov functions can be introduced by Hilger and Kloeden [26]. Fractional calculus deals with the generalization of differentiation and integration of integer order to those ones of any order. It has applications in numerous fields of science and engineering, there has been a great deal of interest in this field [28–33, 36, 44, 46, 47, 52, 54]. The stability of fractional dynamical systems has received more attention recently. Matignon [41] studied the stability of linear fractional differential systems with the Caputo derivative. Numerous researchers have conducted more studies on the stability of linear and nonlinear fractional differential systems [3, 6, 18, 37, 45, 51].

In Bastos' Ph.D. thesis [11], fractional calculus and time scale calculus were merged to introduce fractional calculus on time scales. Recently, several results have been obtained, which includes fractional time scale calculus theory [14, 48], chaotic systems [1, 55, 56], applications of fractional time scales operators to dynamic equations [12, 42], recurrent neural networks [27], optimal control [10], existence and uniqueness of solutions to dynamic equations with fractional time scales [5, 7, 8, 13, 39, 43, 50, 53]. A few authors discussed the stability of fractional dynamical equations on time scales [34, 35, 38, 49].

There is no single paper that we are aware of that investigates the asymptotic stability results for linear fractional systems with Caputo $\Delta-$derivative on time scales. This paper seeks to provide the necessary and sufficient conditions for the asymptotic stability of linear fractional time-invariant systems. We show that there is a correspondence between the stability of linear fractional time scale systems and linear fractional systems of continuous time. By using the Möbious transformations, we transform the stability region of the differential case.

2. Preliminaries

This section covers some fundamental time-scale calculus concepts.

Definition 2.1. *[16] The time scale $\mathbb{T}$ is defined as a non-empty arbitrary subset of $\mathbb{R}$ that is closed and non-empty.*

For examples, the complex numbers $\mathbb{C}$, the rational numbers $\mathbb{Q}$, $[0,1)$, $(0,1]$, $(0,1)$, and $(0,1] \cup \{2,6\}$ do not represent time scales. Whereas the integers numbers $\mathbb{Z}$, any closed interval $[a,b] \in \mathbb{R}$, the set $[0,1] \cup [4,5]$, the natural numbers $\mathbb{N}$, and the real numbers $\mathbb{R}$ represent time scales.

Definition 2.2. *[15] At $\ell \in \mathbb{T}$, the operator $\sigma : \mathbb{T} \to \mathbb{T}$ is referred to as follows:*

$$\sigma(\ell) = \inf \{r \in \mathbb{T} : r > \ell\},$$

it is called a forward jump operator. If $\sigma(\ell) = \ell$, then point ℓ is called right-dense.

Definition 2.3. *[16] At $\ell \in \mathbb{T}$, the operator $\rho : \mathbb{T} \to \mathbb{T}$ is referred to as follows:*

$$\rho(\ell) = \sup \{r \in \mathbb{T} : r < \ell\},$$

it is called a backward jump operator. If $\rho(\ell) = \ell$, and $\ell > \inf \mathbb{T}$, then point ℓ is called left-dense.

Definition 2.4. *[15] The function $\mu : \mathbb{T} \to [0, \infty)$ is known as a graininess function, and is represented by:*

$$\mu(\ell) = \sigma(\ell) - \ell, \quad \forall\, \ell \in \mathbb{T}.$$

Definition 2.5. *[15] The derived form of a time scale $\mathbb{T}$, referred to as $\mathbb{T}^\kappa$ is defined as:*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), sup\mathbb{T}], & if\ sup(\mathbb{T}) \ < \infty, \\ \mathbb{T}, & if\ sup(\mathbb{T}) \ = \infty. \end{cases}$$

Definition 2.6. *[16] The Hilger or delta derivative of $\varphi : \mathbb{T} \to \mathbb{R}$ at all $z \in \mathbb{T}^\kappa$ is denoted by $\varphi^\Delta(z)$ as follows: $\forall \varepsilon > 0$, a neighborhood exists $\mathcal{V}_\mathbb{T}$ of z, $\mathcal{V}_\mathbb{T} = (z - \delta, z + \delta) \cap \mathbb{T}$ for some $\delta > 0$, we have*

$$\left|\varphi\,(\sigma(z)) - \varphi(\ell) - \ \varphi^\Delta(z)\,(\sigma(z) \ - \ \ell)\right| \ \le \ \varepsilon\,|\sigma(z) \ - \ \ell|\,,$$

at $\ell \in \mathcal{V}_\mathbb{T}$, $\ell \neq \sigma(z)$.

Definition 2.7. *[16] Let $\varphi : \mathbb{T} \to \mathbb{R}$ be a function, and $c, d \in \mathbb{T}$. If there exists a function $\Phi : \mathbb{T} \to \mathbb{R}$ such that $\Phi^\Delta(r) = \varphi(r)$ at all $r \in \mathbb{T}$, then Φ is called to be an antiderivative of φ. In this case the integral is given by*

$$\int_c^d \varphi(\eta)\Delta\eta = \Phi(d) - \Phi(c), \ \forall\ c, d \in \mathbb{T}. \tag{2.1}$$

Definition 2.8. *[16] Let $r \in \mathbb{T}$ and $\mu(r) > 0$,*

(1) *The definition of the Hilger complex numbers is: $\mathbb{C}_{\mu(r)} = \left\{s \in \mathbb{C} : s \neq -\frac{1}{\mu(r)}\right\}$.*
(2) *The definition of the Hilger imaginary circle is: $\mathbb{R}_{\mu(r)} = \left\{s \in \mathbb{C} : s > -\frac{1}{\mu(r)}\right\}$.*
(3) *The definition of the Hilger alternative axis is: $\mathbb{A}_{\mu(r)} = \left\{s \in \mathbb{C} : s < -\frac{1}{\mu(r)}\right\}$.*
(4) *The definition of the Hilger imaginary circle is: $\mathbb{I}_{\mu(r)} = \left\{s \in \mathbb{C} : \left|s + \frac{1}{\mu(r)}\right| = \frac{1}{\mu(r)}\right\}$.*

and for $\mu(r) = 0$, we have

$$\mathbb{C}_0 = \mathbb{C}, \quad \mathbb{R}_0 = \mathbb{R}, \quad \mathbb{A}_0 = \phi, \quad \mathbb{I}_0 = i\mathbb{R}.$$

It maps the Hilger complex numbers to the strip $\mathbb{Z}_{\mu(r)}$ defined for $h > 0$ by

$$\mathbb{Z}_{\mu(r)} = \left\{s \in \ \mathbb{C} : \ -\frac{\pi}{\mu(r)} < \operatorname{Im}(s) \le \frac{\pi}{\mu(r)}\right\}.$$

Definition 2.9. *[16] The Hilger real part of s for $\mu(r) > 0$ and $s \in \mathbb{C}\backslash\left\{-\frac{1}{\mu(r)}\right\}$ is defined by*

$$\operatorname{Re}_{\mu(r)}(s) = \frac{|s\mu(r) + 1| - 1}{\mu(r)}.$$

Definition 2.10. *[16] The Hilger imaginary part of s for $\mu(r) > 0$ and $s \in \mathbb{C}\backslash\left\{-\frac{1}{\mu(r)}\right\}$ is given by*

$$\operatorname{Im}_{\mu(r)}(s) = \frac{Arg(z\mu(r) + 1)}{\mu(r)},$$

where $Arg(s)$ denotes the principal argument of s,

$$-\pi < Arg(s) \le \pi.$$

Definition 2.11. *[17] The Hilger real part of s for $\mu(r) > 0$ and $s \in \mathbb{C} \setminus \left\{ -\frac{1}{\mu(r)} \right\}$ is given by*

$$\mathrm{Re}_{\mu(r)}(s) = \frac{|s\mu(r) + 1| - 1}{\mu(r)}.$$

Definition 2.12. *[16] For $\mu(r) > 0$, the open Hilger circle is defined as*

$$\mathcal{H}_{\mu(r)} = \left\{ s \in \mathbb{C} : \left| s + \frac{1}{\mu(r)} \right| < \frac{1}{\mu(r)} \right\}. \tag{2.2}$$

Definition 2.13. [20] *For $\mu(r) > 0$, the cylindrical transformation $\xi_{\mu(r)}(\eta) : \mathbb{C}_{\mu(r)} \to \mathbb{Z}_{\mu(r)}$, is defined as*

$$\xi_{\mu(r)}(\eta) = \frac{1}{\mu(r)} Log(1 + \eta\mu(r)),$$

where Log is the principal logarithm function. For $\mu(r) = 0$, we define $\xi_0(\eta) = \eta$ at all $\eta \in \mathbb{C}$.

Definition 2.14. [20] *If $g(t) \in \Re$, then the generalized exponential function can be defined by*

$$e_\varphi(r, \upsilon) = e^{\int_\upsilon^r \xi_{\mu(\eta)}(\varphi(\eta))\Delta\eta}, \quad for\, all\ \ \upsilon, r \in \mathbb{T}. \tag{2.3}$$

In fact, using the definition for the cylindrical transformation, we have

$$e_\varphi\,(r, \upsilon) \ = \ e^{\int_\upsilon^r \frac{1}{\mu\,(\eta)}\, Log\,(1 + \mu(\eta)\,\varphi\,(\eta))\,\Delta\,\eta} \quad for\,all \quad \upsilon, r \ \in \ \mathbb{T}. \tag{2.4}$$

Definition 2.15. [15] *The function $\varphi : \mathbb{T} \to \mathbb{R}$ is called regressive if*

$$1 + \mu(r)\,\varphi(r) \ \neq\ 0, \quad r \ \in\ \mathbb{T}^k, \tag{2.5}$$

holds. The set of all regressive and rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\Re = \Re(\mathbb{T}) = \Re(\mathbb{T}, \mathbb{R}).$$

Definition 2.16. *[16] The time scale monomials function $h_j(r, r_0) : \mathbb{T} \times \mathbb{T} \to \mathbb{R},\ j \in \mathbb{N}_0$ be defined by*

$$h_0(r, r_0) = 1 \quad \forall\ r, r_0 \in \mathbb{T},$$

and then recursively by

$$h_{j+1}(r, r_0) = \int_{r_0}^{r} h_\kappa(r, r_0)\Delta r, \ \ \forall\ r, r_0 \in \mathbb{T}.$$

As a result, the $\Delta-$derivative of h_j with respect to r satisfies for each fixed r_0.

$$h_j^\Delta(r, r_0) = h_{j-1}(r, r_0), \qquad r, r_0 \in \mathbb{T}, \ \ j \in \mathbb{N}.$$

Definition 2.17. *[15] The time scale Laplace transform of a function $\varphi : \mathbb{T} \to \mathbb{R}$ at all $r \in \mathbb{T}$, is defined by:*

$$\mathcal{L}\{\varphi(r)\}(s) = \Phi(s) := \int_0^\infty \varphi(r) e^\sigma_{\ominus z}(r, 0)\Delta r,$$

for $s \in \mathcal{D}\{\varphi\}$, such that $\mathcal{D}\{\varphi\}$ involves all complex numbers $z \in \mathbb{C}$ that have an improper integral.

Theorem 2.1. *[15] Let $1 + s\mu(a) \neq 0$ for all $s \in \mathbb{C} \setminus \{0\}$, and $j \in \mathbb{N}_0$, we have*

$$\mathcal{L}(h_j(a, 0))(s) = \frac{1}{s^{j+1}}, \quad \forall\ a \in \mathbb{T}_0,$$

and

$$\lim_{a \to \infty} (h_j(a, 0) e_{\ominus s}(a, 0)) = 0.$$

Definition 2.18. *[20] On time scales $\mathbb{T}$, the generalized fractional $\Delta-$power function $h_\alpha(r, r_0)$ is*

$$h_\alpha(r, r_0) = \mathcal{L}^{-1}\left(\frac{1}{s^{\alpha+1}}\right)(r), \quad \forall\, r \geq r_0,$$

for all $s \in \mathbb{C}\backslash\{0\}$ is defined by

$$h_\alpha(r, \vartheta) = \widehat{h_\alpha(., r_0)}(r, \vartheta), \quad \vartheta, r \in \mathbb{T}, \quad r \geq \vartheta \geq r_0.$$

Definition 2.19. *[20] On time scale $\mathbb{T}$, $\alpha > 0$, at all $r \in \mathbb{T}$, and $r > r_0$. For the function $\varphi : \mathbb{T} \to \mathbb{R}$, the fractional $\Delta-$derivative of a type Riemann-Liouville for $\varphi(r)$ is given by:*

$$I^0_{\Delta,r_0}\varphi(r) = \varphi(r),$$

$$\begin{aligned}(I^\alpha_{\Delta,r_0}\varphi)(r) &= (h_{\alpha-1}(\cdot, r_0) * \varphi)(r)\\ &= \int_{r_0}^{r} \widehat{h_{\alpha-1}(\cdot, r_0)}(r, \sigma(\upsilon))\varphi(\upsilon)\Delta\upsilon\\ &= \int_{r_0}^{r} h_{\alpha-1}(r, \sigma(\upsilon))\varphi(\upsilon)\Delta\upsilon.\end{aligned}$$

Definition 2.20. *[20] On time scale $\mathbb{T}$, let $r, r_0 \in \mathbb{T}$. Then, for the function $\varphi : \mathbb{T} \to \mathbb{R}$, the fractional $\Delta-$derivative of a type Riemann-Liouville for $\varphi(r)$ is given by:*

$$D^\alpha_{\Delta,r_0}\varphi(r) = D^m_\Delta I^{m-\alpha}_{\Delta,r_0}\varphi(r), \quad \forall\, r \in \mathbb{T},$$

for α real value such that $\alpha \geq 0$ and $m = -[-\alpha]$.

Definition 2.21. *[20] On time scale $\mathbb{T}$, let $r, r_0 \in \mathbb{T}$. Then, for the function $\varphi : \mathbb{T} \to \mathbb{R}$, the fractional $\Delta-$derivative of a type Caputo for $\varphi(r)$ is given by:*

$$^C D^\alpha_{\Delta,r_0}\varphi(r) = D^\alpha_{\Delta,r_0}\left(\varphi(r) - \sum_{k=0}^{m-1} h_k(r, r_0)\varphi^{\Delta^k}(r_0)\right), \quad \forall\, r > 0, \tag{2.6}$$

for α real value such that $\alpha \geq 0$ and $m = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$, and $m = [\alpha]$ if $\alpha \in \mathbb{N}$.

In particular, when $0 < \alpha < 1$, the Eq.(2.6) takes the following forms:

$$^C D^\alpha_{\Delta,r_0}\varphi(r) = D^\alpha_{\Delta,r_0}\left(\varphi(r) - \varphi(r_0)\right), \quad \forall\, r \in \mathbb{T}, \quad r > r_0.$$

If $\varphi(r_0) = 0$, then the Caputo fractional $\Delta-$derivative coincides with the Riemann-Liouville fractional $\Delta-$derivative in the following case

$$^C D^\alpha_{\Delta,r_0}\varphi(r) = D^\alpha_{\Delta,r_0}\varphi(r), \quad \forall\, r \in \mathbb{T}, r > r_0.$$

If $\alpha = m \in \mathbb{N}$ and the delta derivative $\varphi^{\Delta^m}(r)$ of order m exists, then the Caputo fractional $\Delta-$derivative in the following case

$$^C D^m_{\Delta,r_0}\varphi(r) = \varphi^{\Delta^m}(r), \quad \forall\, r \in \mathbb{T}, \quad r > r_0.$$

The Caputo fractional $\Delta-$derivative is defined for functions $\varphi(r)$ for which the Riemann-Liouville fractional $\Delta-$derivative of the right-hand sides of Eq.(2.6) exists. Thus the following Theorems holds.

Theorem 2.2. *[20] On time scale $\mathbb{T}$, $\alpha \geq 0$ at all $r, r_0 \in \mathbb{T}$, $m = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$, and $m = [\alpha]$ if $\alpha \in \mathbb{N}$.*

(1) *If $\alpha \notin \mathbb{N}$ and $r > r_0$, then*

$$\begin{aligned} {}^{C}D^{\alpha}_{\Delta,r_0}\varphi(r) &= \left(h_{m-\alpha-1}(\cdot, r_0) * \varphi^{\Delta^m}\right)(r) \\ &= I^{m-\alpha}_{\Delta,r_0} D^m_{\Delta}\varphi(r). \end{aligned}$$

(2) *If $\alpha = m \in \mathbb{N}$ and $r > r_0$, then*

$${}^{C}D^{\alpha}_{\Delta,r_0}\varphi(r) = \varphi^{\Delta^m}(r).$$

3. Types of Stability System

Let $\mathbb{T}$ be an unbounded time scale above. Consider the following linear time-invariant system

$$y^{\Delta}(t) = Ay(t), \tag{3.1}$$

$$y(t_0) = y_0, \tag{3.2}$$

where A is an $m \times m$-constant matrix. For the system (3.1), the transfer matrix is represented by $e_A = \{(t, \ell) \in \mathbb{T} \times \mathbb{T} : t \geq \ell\} \to \mathbb{R}^{m \times m}$. Then the solution $y(t, t_0, x_0)$ of the initial system (3.1) is expressed by the formula $y\,(t, t_0, y_0) = e_A(t, t_0)\, y_0$.

Definition 3.1. *[40] Let $\mathbb{T}$ be an unbounded time scale above,*

(1) *The system (3.1) is exponentially stable, if a constant $\gamma > 0$ exists such that for each $r \in \mathbb{T}$, there exists $\mathcal{K}(r) \geq 1$ and the following estimate holds:*

$$\|e_A(t, r)\| \leq \mathcal{K}(r)\, e^{-\gamma(t-r)}, \quad \forall\ t \geq r. \tag{3.3}$$

(2) *The system (3.1) is uniformly exponentially stable, if $\mathcal{K}(r)$ in the definition (1) does not depend on $r \in \mathbb{T}$.*

(3) *The system (3.1) is robustly exponentially stable, if there exists $\varepsilon > 0$ such that the exponential stability of the system*

$$|y_0| < \delta \Rightarrow |y\,(t, t_0, y_0)| < \varepsilon, \quad \forall\ t \geq t_0, \tag{3.4}$$

implies the exponential stability of the system

$$y^{\Delta}(t) = By(t), \tag{3.5}$$

for any matrix $B \in \mathbb{R}^{m \times m}$ such that $\|B - A\| \leq \varepsilon$.

(4) *The system (3.1) is uniformly exponentially stable, if a constant $\gamma > 0$ exists such that for each $r \in \mathbb{T}$, one can find $\mathcal{K}(r) \geq 1$ and the estimate*

$$\|e_A(t, \ell)\| \leq \mathcal{K}(r)\, e^{-\gamma(t-\ell)}, \tag{3.6}$$

holds at all $t \geq \ell \geq r$.

Theorem 3.1. *[40] Let $\mathbb{T}$ be an unbounded time scale above. If $\mu(t) \leq \eta$ of $\mathbb{T}$ is bounded from above, that is, there exists $\eta > 0$ such that $\mu(t) \leq \eta$ at all $t \in \mathbb{T}$, then, and only then, there exists a system (3.1) whose zero solution is uniformly exponentially stable on $\mathbb{T}$.*

Theorem 3.2. *[40] Let $\mathbb{T}$ be an unbounded time scale above, with bounded graininess. If the system (3.1) is uniformly exponentially stable, then there exists $\varepsilon > 0$ such that the system (3.5) is uniformly exponentially stable as well, as soon as $\|A - B\| \leq \varepsilon$.*

4. Stability of $\Delta-$Linear Dynamic System

In this section, we investigate the asymptotical stability of the linear time-invariant system (3.1) on a time scale $\mathbb{T}$, that is unbounded above.

First, we begin with the analysis of the scalar system on a time scale:

$$y^{\Delta}(t) = \lambda y(t), \tag{4.1}$$

$$y(0) = y_0. \tag{4.2}$$

In order to study the stability of a dynamical system on a time scale, a particular open set of the complex plane known as the Hilger circle is defined for each point $t \in \mathbb{T}$ as

$$\mathcal{H}_{\mu(t)} = \left\{ s \in \mathbb{C} : \left| s + \frac{1}{\mu(t)} \right| < \frac{1}{\mu(t)} \right\}, \tag{4.3}$$

where $\mu(t) = 0$, the Hilger circle is defined as $\mathcal{H}_0 = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\} = \mathbb{C}^-$, the open left-half complex plane. There is a link between Hilger circles and the region of asymptotic stability.

Theorem 4.1. *Let $\mathbb{T}$ be an unbounded time scale above. The scalar dynamical system (4.1) is asymptotically stable if and only if it satisfies the condition $\lambda \in \mathbb{C}$ passes through the origin and lies outside the closed disk centered at $(-\frac{1}{\mu(t)})$.*

proof: It's worth noting that

$$|1 + \mu(t)\lambda| = 1 \Leftrightarrow \left| \frac{1}{\mu(t)} + \lambda \right| = \frac{1}{\mu(t)},$$

is a circle's equation in the complex plane with a radius $(\frac{1}{\mu(t)})$ and center of $(-\frac{1}{\mu(t)})$. As a result, the condition of stability is validated.

Let $\lambda = u + iv$, we have

$$\left| \frac{1}{\mu(t)} + \lambda \right| < \frac{1}{\mu(t)} \Leftrightarrow \left| \frac{1}{\mu(t)} + u + iv \right|^2 = \left(\frac{1}{\mu(t)} + u \right)^2 + v < \frac{1}{\mu^2(t)}$$

$$\Leftrightarrow \frac{2}{\mu(t)} u + u^2 + v^2 < 0$$

$$\Leftrightarrow \mathrm{Re}(\lambda) < -\frac{\mu(t)}{2} |\lambda|^2,$$

and, as $\mu(t) \to 0$, $\mathrm{Re}(\lambda) < 0$.

Remark 4.1. *Straightforward calculations reveal that it is simple to verify if each eigenvalue of A meets the condition of Theorem 4.1 to determine whether (4.1), where A is a square matrix, is stable.*

5. Region of Stability Mapping

In this section, the Theorem 4.1 result will be further analyzed in more detail to determine the convergence region for the linear fractional system with the Caputo $\Delta-$derivative.

For the sake of simplicity, Let z represent the continuous-time system's complex variable

$$y'(t) = \lambda y(t), \tag{5.1}$$

$$y(0) = y_0, \tag{5.2}$$

and the complex variable s associated with the time scale system (4.1).

Thus, the stability regions for the system (4.1) may be deduced from the corresponding continuous-time regions for the system (5.1). These are defined by circles and lines, which can be mapped one into the other by a well-known invertible function called the Möbius transformation.

In this instance, as will be shown below, the family of Möbius transformations, which depends on the graininess function $\mu(t)$, is given as follows:

$$s = \mathcal{M}_{\mu(t)}(z) = \frac{2z}{2-\mu(t)z}, \tag{5.3}$$

where $z = x + iy$, $s = u + iv$ and the inverse given by

$$z = \mathcal{M}^{-1}_{\mu(t)}(s) = \frac{2s}{2+\mu(t)s}. \tag{5.4}$$

To begin, notice that $\mathcal{M}_{\mu(t)}$ are bijective maps of the extended complex plane $\mathbb{C} \cup \{\infty\}$, which corresponds to:

$$s \leftrightarrow z : \ 0 \leftrightarrow 0, \ -\frac{1}{\mu(t)} \leftrightarrow -\frac{2}{\mu(t)}, \ -\frac{2}{\mu(t)} \leftrightarrow \infty, \ \infty \leftrightarrow \frac{2}{\mu(t)}. \tag{5.5}$$

In fact, infinity-containing lines in the z plane are mapped to circles or lines passing through $s = -\frac{2}{\mu(t)}$.

Also, the circle $\left|s + \frac{1}{\mu(t)}\right| = \frac{1}{\mu(t)} \Leftrightarrow |1 + \mu(t)s| = 1$ is associated with the imaginary axis on the $z = iy$ plane (the border of the stable region). Therefore, if

$$|1 + \mu(t)s| = \left|1 + \mu(t)\frac{2iy}{2 - i\mu(t)y}\right| = \left|\frac{2 + i\mu(t)y}{2 - i\mu(t)y}\right| = 1. \tag{5.6}$$

Since the numerator and denominator are conjugated, it is easy to check if $y > 0$ and $\operatorname{Im} s > 0$.

Therefore, in order to deduce that the Möbius transformations map bijectively stable regions onto stable regions, it suffices to know that, according to (5.5), the circle's center is $s = -\frac{1}{\mu(t)}$, and that all stable points are mapped to the stable region around $z = -\frac{2}{\mu(t)}$.

The regions stability of these systems are shown in Figure (2) and Figure (1), where

$$\mu_1(t) > \mu_2(t) > \mu_3(t).$$

6. Stability of $\Delta-$Fractional systems

In this section, we establish a necessary and sufficient condition for the asymptotic stability of a linear fractional system on a time scale.

$$^{C}D^{\alpha}_{\Delta,0}y(t) = Ay(t), \tag{6.1}$$

$$y^{(\ell)}(0) = y_\ell, \quad \ell = 0, \dots, m-1, \tag{6.2}$$

where $A \in \mathbb{R}^{n\times n}$, $m = [\alpha] + 1$, $\alpha \in (m-1, m)$.

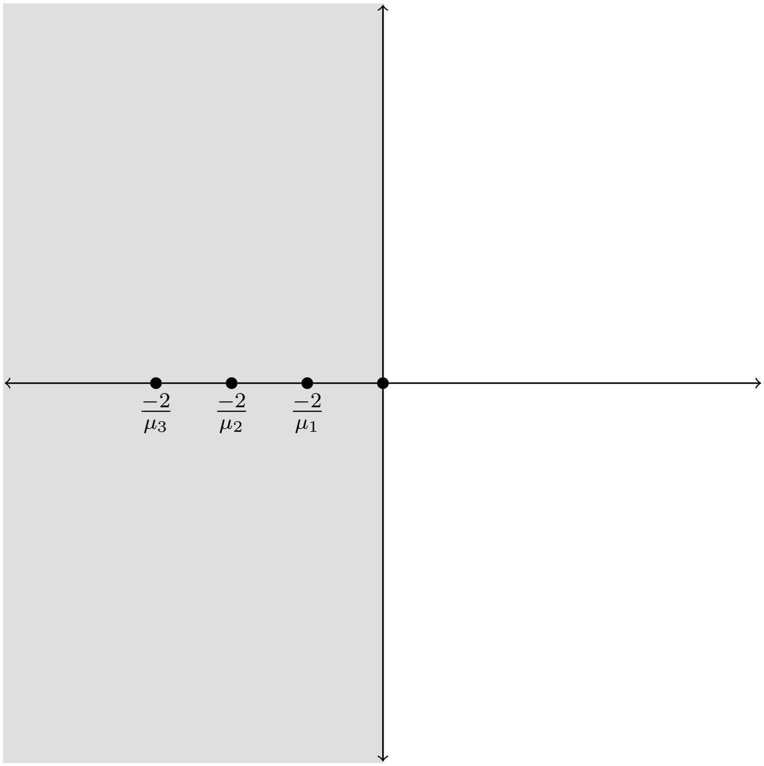

FIGURE 1. stablity region in the s plane

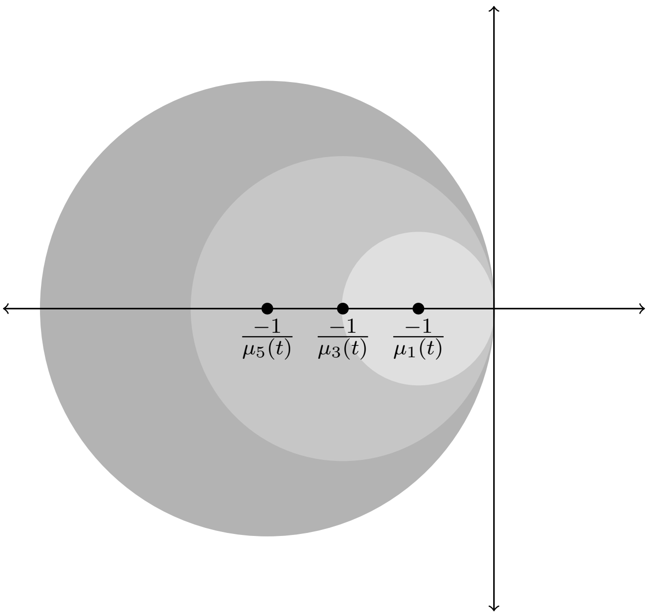

FIGURE 2. stablity region in the z plane

First, we begin with the following scalar fractional system on a time scale:

$$^{C}D^{\alpha}_{\Delta,0}y(t) = \lambda y(t), \tag{6.3}$$

$$y^{(\ell)}(0) = y_\ell, \quad \ell = 0, \dots, m-1. \tag{6.4}$$

The system's stability region (6.3) can be obtained from the corresponding continuous region $|\arg s| > \frac{\alpha\pi}{2}$ determined by [41], yielding the subsequent result.

Theorem 6.1. *Let $0 < \alpha < 1$, and $\zeta = \tan(\theta)$, $\frac{\alpha\pi}{2} < \theta < 2\pi - \frac{\alpha\pi}{2}$. The system (6.3) is thus asymptotically stable, if and only if,*

$$\operatorname{Re}(\lambda) < \frac{-\mu(t)|\lambda|^2}{2} + \zeta\,|\operatorname{Im}(\lambda)|. \tag{6.5}$$

Proof: Let $z = x + iy$ and observe that the half-lines leaving the origin with an angle of $\frac{\alpha\pi}{2}$ are represented by $\arg z = \frac{\alpha\pi}{2}$. In other words, it is identical to the equation $x = \zeta y$ where $\zeta = \cot\frac{\alpha\pi}{2}$.

So, based on the graph in Figure 3, where the points fulfill $x = \zeta\,|y|$, the region of stability of continuous time is highlighted by

$$\Omega = \left\{(x, y)\,\middle|\, y = \zeta x, \frac{\alpha\pi}{2} \leq \tan^{-1}(\zeta) \leq 2\pi - \frac{\alpha\pi}{2}\right\} \Leftrightarrow x < \zeta\,|y|. \tag{6.6}$$

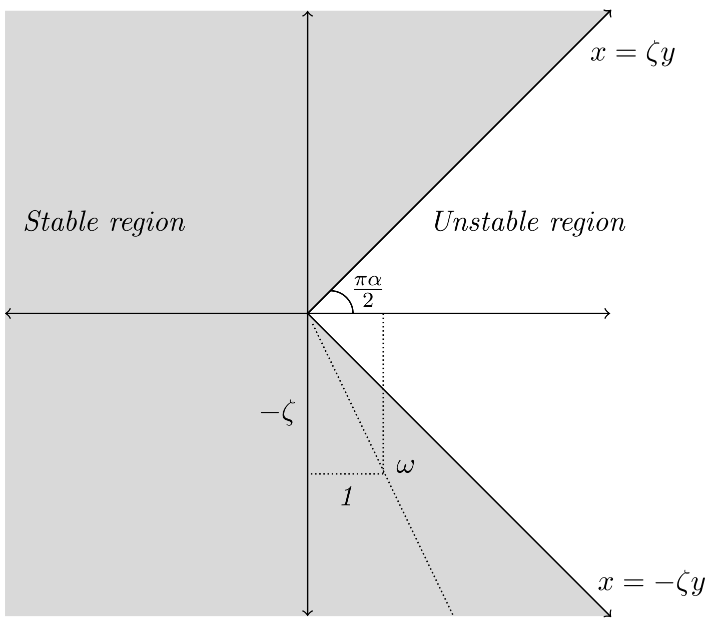

FIGURE 3. The stability region of the fractional system where $0 < \alpha < 1$.

It should be clear from Figure 3 that it is possible to write condition (6.6) without using the absolute value. In fact, consider the subsequent half planes

$$\Upsilon_+ = \{z : x < \zeta y\} \qquad \text{and} \qquad \Upsilon_- = \{z : x < -\zeta y\}.$$

when $\zeta > 0$, the stability region is $\Upsilon_+ \cup \Upsilon_-$. Despite the seeming complexity, identifying the corresponding regions is significantly simpler:

$$\Xi_+ = \mathcal{M}_{\mu(t)}(\Upsilon_+) \qquad \text{and} \qquad \Xi_- = \mathcal{M}_{\mu(t)}(\Upsilon_-).$$

of the s plane. As a consequence, the union of Ξ_+ and Ξ_-, when $\zeta > 0$, will be calculated to obtain the stability regions of system (6.3).

Therefore, the transformation of the defining condition of Υ_+*, which is*

$$x < \zeta y \Leftrightarrow x - \zeta y < 0,$$

through $\mathcal{M}_{\mu(t)}$ *in order to characterize* Ξ_+ *(*Ξ_- *is determined by symmetry).*

Let's express this condition using the complex variable z *to make it simpler to apply the transformation* $\mathcal{M}_{\mu(t)}$ *to it: First, Let*

$$\Omega_1 = \left\{ (x,y) \middle| y = \zeta x, \frac{\alpha\pi}{2} \leq \tan^{-1}(\zeta) \leq \pi \right\},$$

and define $\omega = 1 - i\zeta$ *that belongs to line* $y = -\zeta x$*, as shown in Figure 3. Then note that*

$$\begin{aligned} z\bar{\omega} + \bar{z}\omega &= 2\,\mathrm{Re}((x+iy)(1+i\zeta)) \\ &= 2(x - \zeta y) < 0. \end{aligned} \tag{6.7}$$

As a result of (5.4) and $\omega + \bar{\omega} = 2$*, we get*

$$\begin{aligned} z\bar{\omega} + \bar{z}\omega &= \frac{2s\bar{\omega}}{2+\mu(t)s} + \frac{2\bar{s}\omega}{2+\mu(t)\bar{s}} \\ &= 2\frac{s\bar{\omega}(2+\mu(t)\bar{s}) + \bar{s}\omega(2+\mu(t)s)}{|2+\mu(t)s|^2} < 0 \\ &\Leftrightarrow s\bar{\omega}(2+\mu(t)\bar{s}) + \bar{s}\omega(2+\mu(t)s) < 0 \\ &\Leftrightarrow 2s\bar{\omega} + \mu(t)s\bar{s}\bar{\omega} + 2\bar{s} + \mu(t)s\bar{s}\bar{\omega} < 0 \\ &\Leftrightarrow 2\mu(t)(s\bar{s} + s\frac{\bar{\omega}}{\mu(t)} + \bar{s}\frac{\omega}{\mu(t)}) < 0 \\ &\Leftrightarrow s\bar{s} + s\frac{\bar{\omega}}{\mu(t)} + \bar{s}\frac{\omega}{\mu(t)} < 0 \\ &\Leftrightarrow (s + \frac{\omega}{\mu(t)})(\bar{s} + \frac{\bar{\omega}}{\mu(t)}) < \frac{\omega\bar{\omega}}{\mu(t)} \\ &\Leftrightarrow \left| s + \frac{\omega}{\mu(t)} \right| < \frac{|\omega|}{\mu(t)}. \end{aligned} \tag{6.8}$$

Similarly, we can easily obtain

$$\Omega_2 = \left\{ (x,y) \middle| y = \zeta x,\ \pi \leq \tan^{-1}(\zeta) \leq 2\pi - \frac{\alpha\pi}{2} \right\},$$

with define $\omega = -1 - i\zeta$ *that belongs to line* $y = -\zeta x$*.*

Since the center is on the line containing ω *(*$\bar{\omega}$*), the circle's exterior* Ξ_+ *(*Ξ_-*) with center* $-\frac{\omega}{\mu(t)}$ *(*$-\frac{\bar{\omega}}{\mu(t)}$*), passes through the origin and is tangent to* $x = \zeta y$ *(*$x = -\zeta y$*), as illustrated in Figure 4.*

Finally, in order to obtain condition (6.5) and characterize the region analytically, observe that (6.8) is equivalent to

$$|s|^2 + \frac{2}{\mu(t)}\,\mathrm{Re}(s\bar{\omega}) < 0 \Leftrightarrow \mathrm{Re}(s\bar{\omega}) + \frac{\mu(t)}{2}|s|^2 < 0$$

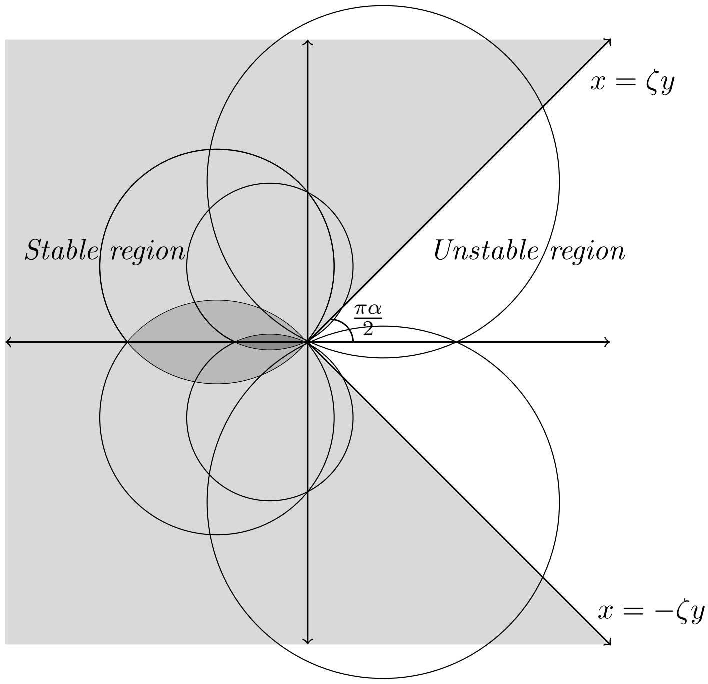

FIGURE 4. The stability region of the fractional system where $0 < \alpha < 1$.

Consequently, by $s = u + i\upsilon$, *the condition* Ξ_+ *becomes*

$$\mathrm{Re}((u + i\upsilon)(1 + i\zeta)) + \frac{\mu(t)}{2}|s|^2 = u - \upsilon\zeta + \frac{\mu(t)}{2}|s|^2 < 0$$

$$\Leftrightarrow u + \frac{\mu(t)}{2}|s|^2 < \upsilon\zeta.$$

Analogously, $u + \frac{\mu(t)}{2}|s|^2 < -\upsilon\xi$ *defines* Ξ_-.

As a result, the conditions that characterize Ξ_+ *and* Ξ_- *are*

$$s = u + i\upsilon \in \Xi_\pm \Leftrightarrow u + \frac{\mu(t)}{2}|s|^2 < \pm\upsilon\zeta.$$

Note that, the union is satisfied if and only if at least one of the two conditions is met

$$u + \frac{\mu(t)}{2}|s|^2 < |\upsilon\zeta|\,.$$

However, $\Xi_+ \cup \Xi_-$ *is only the stability region when* $\zeta > 0$, *in which case the condition becomes*

$$u + \frac{\mu(t)}{2}|s|^2 < |\upsilon|\,\zeta.$$

Therefore, $\mathrm{Re}(s) < -\frac{\mu(t)}{2}|s|^2 + \zeta\,|\mathrm{Im}(s)|$.

Theorem 6.2. *Let* $1 < \alpha < 2$, *and* $\zeta = \tan(\theta)\,,\ \frac{\alpha\pi}{2} < \theta < \pi - \frac{\alpha\pi}{2}$. *The system (6.3) is thus asymptotically stable, if and only if,*

$$\mathrm{Re}(\lambda) < \frac{-\mu(t)|\lambda|^2}{2} + \zeta\,|\mathrm{Im}(\lambda)|\,. \tag{6.9}$$

Proof: Let $z = x + iy$ *and observe that the half-lines leaving the origin with an angle of* $\frac{\alpha\pi}{2}$ *are represented by* $\arg z = \frac{\alpha\pi}{2}$. *In other words, it is identical to the equation* $x = \zeta y$

where $\zeta = \cot \frac{\alpha\pi}{2}$.

So, based on the graph in Figure 5, where the points fulfill $x = \zeta\,|y|$, *the region of stability of continuous time is highlighted by*

$$\Omega = \left\{ (x,y) \,\middle|\, y = \zeta x, \frac{\alpha\pi}{2} \le \tan^{-1}(\zeta) \le \pi - \frac{\alpha\pi}{2} \right\} \Leftrightarrow x < \zeta\,|y|\,. \tag{6.10}$$

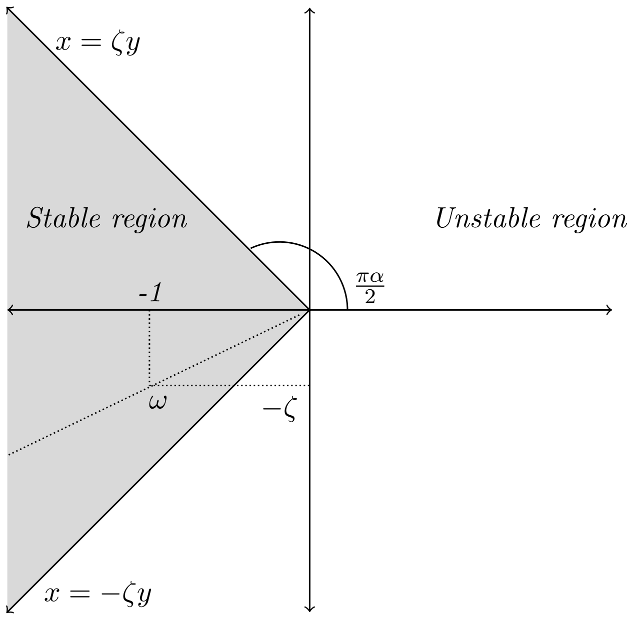

FIGURE 5. The stability region of the fractional system where $1 < \alpha < 2$.

It should be clear from Figure 5 that it is possible to write condition (6.10) without using the absolute value. In fact, consider the subsequent half planes

$$\Upsilon_+ = \{z : x < \zeta y\} \qquad \text{and} \qquad \Upsilon_- = \{z : x < -\zeta y\}\,.$$

Despite the seeming complexity, identifying the corresponding regions is significantly simpler:

$$\Xi_+ = \mathcal{M}_{\mu(t)}\left(\Upsilon_+\right) \qquad \text{and} \qquad \Xi_- = \mathcal{M}_{\mu(t)}\left(\Upsilon_-\right).$$

of the s plane. As a consequence, the union of Ξ_+ *and* Ξ_-, *when* $\zeta < 0$, *will be calculated to obtain the stability zone of system (6.3).*

Therefore, the transformation of the defining condition of Υ_-, *which is*

$$x < -\zeta y \Leftrightarrow x + \zeta y < 0,$$

through $\mathcal{M}_{\mu(t)}$ *in order to characterize* Ξ_+ *(*Ξ_- *is determined by symmetry).*

Let's express this condition using the complex variable z *to make it simpler to apply the transformation* $\mathcal{M}_{\mu(t)}$ *to it: First, let* $\omega = -1 - i\zeta$ *that belongs to line* $y = \zeta x$, *as illustrated in Figure 5. Then observe that*

$$\begin{aligned}
z\bar{\omega}+\bar{z}\omega &= 2\operatorname{Re}((x+iy)(-1+i\zeta)) \\
&= 2(-x-\zeta y)<0 \\
&= -2(x+\zeta y)<0 \\
&= 2(x+\zeta y)>0.
\end{aligned}$$

As a result of (5.4) and $\omega+\bar{\omega}=-2$, we get

$$\begin{aligned}
z\bar{\omega}+\bar{z}\omega &= \frac{2s\bar{\omega}}{2+\mu(t)s}+\frac{2\bar{s}\omega}{2+\mu(t)\bar{s}} \\
&= 2\frac{s\bar{\omega}(2+\mu(t)\bar{s})+\bar{s}\omega(2+\mu(t)s)}{|2+\mu(t)s|^2}>0 \\
&\Leftrightarrow s\bar{\omega}(2+\mu(t)\bar{s})+\bar{s}\omega(2+\mu(t)s)>0 \\
&\Leftrightarrow 2s\bar{\omega}+\mu(t)s\bar{s}\bar{\omega}+2\bar{s}+\mu(t)s\bar{s}\bar{\omega}>0 \\
&\Leftrightarrow -2\mu(t)(s\bar{s}+s\frac{\bar{\omega}}{\mu(t)}+\bar{s}\frac{\omega}{\mu(t)})>0 \\
&\Leftrightarrow s\bar{s}-s\frac{\bar{\omega}}{\mu(t)}-\bar{s}\frac{\omega}{\mu(t)}<0 \qquad (6.11) \\
&\Leftrightarrow (s-\frac{\omega}{\mu(t)})(\bar{s}-\frac{\bar{\omega}}{\mu(t)})<\frac{\omega\bar{\omega}}{\mu(t)} \\
&\Leftrightarrow \left|s-\frac{\omega}{\mu(t)}\right|<\frac{|\omega|}{\mu(t)}.
\end{aligned}$$

Since the center is on the line containing ω ($\bar{\omega}$), the circle's exterior Ξ_+ (Ξ_-) with center $\frac{\omega}{\mu(t)}$ ($\frac{\bar{\omega}}{\mu(t)}$), passes through the origin and is tangent to $x=\zeta y$ ($x=-\zeta y$), as illustrated in Figure 6.

Finally, in order to obtain condition (6.9) and characterize the region analytically, observe that (6.11) is equivalent to

$$|s|^2-\frac{2}{\mu(t)}\operatorname{Re}(s\bar{\omega})<0 \Leftrightarrow \operatorname{Re}(s\bar{\omega})-\frac{\mu(t)}{2}|s|^2>0.$$

Consequently, by $s=u+iv$, the condition Ξ_+ becomes

$$\begin{aligned}
\operatorname{Re}((u+iv)(-1+i\zeta))-\frac{\mu(t)}{2}|s|^2 &= -u-\upsilon\zeta-\frac{\mu(t)}{2}|s|^2>0 \\
&\Leftrightarrow u+\frac{\mu(t)}{2}|s|^2<-\upsilon\zeta.
\end{aligned}$$

Analogously, $u+\frac{\mu(t)}{2}|s|^2<\upsilon\zeta$ defines Ξ_+.

As a result, the conditions that characterize Ξ_+ and Ξ_- are

$$s=u+iv\in\Xi_\pm \Leftrightarrow u+\frac{\mu(t)}{2}|s|^2<\pm\upsilon\zeta.$$

Note that, the intersection is satisfied if and only if at least one of the two conditions is met

$$u+\frac{\mu(t)}{2}|s|^2<-|\upsilon\zeta|.$$

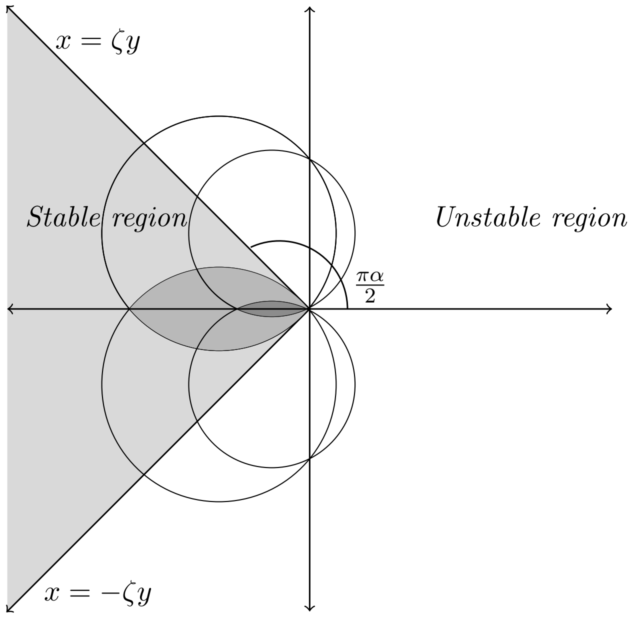

FIGURE 6. The stability region of the fractional system where $1 < \alpha < 2$.

When $\zeta < 0$ *and the stability region is equal to* $\Xi_+ \cap \Xi_-$, *the equivalent condition is*

$$u + \frac{\mu(t)}{2}|s|^2 < -|v|(-\zeta) = |v|\zeta.$$

As a result, $\mathrm{Re}(s) < -\frac{\mu(t)}{2}|s|^2 + \zeta\,|\mathrm{Im}\, s|$ *in both cases.*

7. Conclusions

Studying dynamic equations on a time scale allows one to avoid having to repeat the proof of results twice for discrete and continuous dynamic equations. The main reason against why study the time scale calculus is that it allows one to study a given dynamic system on any time scale set $\mathbb{T}$, and then this set will be selected later based on the type of dynamic system. This feature enables us to reflect the results between $\mathbb{Z}$ and $\mathbb{R}$. In this paper, we established a necessary and sufficient condition for the asymptotic stability of linear fractional invariant-time systems. Although the system becomes taken into consideration in the scalar case, the vector case is a straightforward generalization. Instead of analyzing the convergence of the system's trajectories, as was done before for an analogous continuous-time system, the result was obtained by transforming the stability region of the continuous-time case using appropriate Möbius transformations.

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