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INVESTIGATION OF A SYSTEM OF SECOND-ORDER UNDAMPED STURM–LIOUVILLE BOUNDARY VALUE PROBLEMS

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ABSTRACT. The purpose of this paper is to establish the existence of multiple positive solutions for a coupled system of second-order undamped Sturm–Liouville boundary value problems. The technique is based on the Six functionals fixed point theorem.

Keywords: Kernel, boundary value problem, system, positive solution.

AMS Subject Classification: 34B15, 34B18.

1. INTRODUCTION

Boundary value problems (BVPs) are being used widely in a variety of fields, such as pharmaceutical, electronic technologies, healthcare, and the automotive industry to simulate intricate phenomena during different stages of the design and production of high-tech objects. Positive solutions are significant in these practical settings. The BVPs have recently drawn a lot of interest because of their crucial role in application and theory [1].

There has been a considerable amount of research in recent years on the existence of one and multiple positive solutions to the BVPs for ODEqs utilizing applications of various fixed point theorems (FPTs). However, the majority of authors concentrated on the existence of positive solutions for the second-order ODEqs satisfying the Neumann boundary conditions (BCs). Here we mention few works on this line, in 2008, Chu et al. [3], studied the existence of positive solutions to the following BVP:

$$\mathtt{u}'' + \mathtt{r}^2\mathtt{u} = \mathtt{f}(\mathtt{z},\mathtt{u}) + e(\mathtt{z}), \quad \mathtt{z} \in (0,1),$$
$$\mathtt{u}'(0) = \mathtt{u}'(1) = 0,$$

where $\mathtt{r} \in \left(0, \frac{\pi}{2}\right)$, $e \in \mathtt{C}[0,1]$ by utilizing a truncation technique and the Leray–Schauder nonlinear alternative principle. On the other hand, Wang et al. [13] considered the following second-order Neumann type BVP:

$$-\mathtt{u}'' + \mathtt{r}^2\mathtt{u} = \mathtt{h}(\mathtt{z})\mathtt{f}(\mathtt{z},\mathtt{u}), \quad 0 < \mathtt{z} < 1,$$
$$\mathtt{u}'(0) = \mathtt{u}'(1) = 0,$$

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where $\mathtt{r} > 0$, by applying the FPT of cone expansion and compression of functional type, and derived the necessary conditions for the existence of positive solutions. In 2015, Henderson and Kosmatov [4] studied existence of positive solutions of the semipositone Neumann BVP:

$$-\mathtt{u}'' = \mathtt{f}\big(\mathtt{z}, \mathtt{u}(\mathtt{z})\big), \quad 0 < \mathtt{z} < 1,$$
$$\mathtt{u}'(0) = \mathtt{u}'(1) = 0,$$

by means of a Guo–Krasnosel'skii FPT. Establishing positive solutions to such problems has been the subject of extensive research, see for example [6, 7, 9, 10, 11, 12, 14] and references therein. For the modified Schrodinger system with a nonconvex diffusion term, some new findings on the existence and nonexistence of radial large positive solutions have been obtained. The successive iteration technique and the dual approach have been their key instruments, see [15, 16].

In [5], the authors established the existence and nonexistence of radial solutions to Dirichlet problem of the general k-Hessian equation by means of Leggett–Williams fixed point theorem. Recently, Prasad et. al [8] studied the existence of positive solutions to the second-order Sturm–Liouville BVP by an application of Avery–Henderson FPT. Inspired by the aforementioned studies, in this paper, we investigate the existence of multiple positive solutions for the system of second-order Sturm–Liouville BVP:

$$-\mathtt{u}_1''(\mathtt{z}) + \mathtt{r}_1^2\mathtt{u}_1(\mathtt{z}) = \mathtt{g}_1\big(\mathtt{z}, \mathtt{u}_1(\mathtt{z}), \mathtt{u}_2(\mathtt{z})\big), \quad \mathtt{z} \in (0,1), \tag{1}$$

$$-\mathtt{u}_2''(\mathtt{z}) + \mathtt{r}_2^2\mathtt{u}_2(\mathtt{z}) = \mathtt{g}_2\big(\mathtt{z}, \mathtt{u}_1(\mathtt{z}), \mathtt{u}_2(\mathtt{z})\big), \quad \mathtt{z} \in (0,1), \tag{2}$$

$$\left.\begin{aligned} \mathtt{b}_1\mathtt{u}_1(0) - \mathtt{b}_2\mathtt{u}_1'(0) = 0,\\ \mathtt{b}_3\mathtt{u}_1(1) + \mathtt{b}_4\mathtt{u}_1'(1) = 0, \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} \mathtt{d}_1\mathtt{u}_2(0) - \mathtt{d}_2\mathtt{u}_2'(0) = 0,\\ \mathtt{d}_3\mathtt{u}_2(1) + \mathtt{d}_4\mathtt{u}_2'(1) = 0, \end{aligned}\right\} \tag{4}$$

where $\mathtt{b}_\mathtt{k}, \mathtt{d}_\mathtt{k}, \mathtt{k} = \overline{1,4}$ are constants, $\mathtt{r}_1$ and $\mathtt{r}_2$ are positive constants, by means of Six Functionals Fixed Point Theorem (SFFPT) in cone and under suitable conditions. The SFFPT [2] is a generalization of the Five functionals FPT as well as the original Triple functionals FPT of Leggett–Williams. In the SFFPT, none of the functional boundaries are needed to map above or below the boundary in the functional sense.

1.1. **Hypotheses for BVP** (1)-(4).

($\mathbf{H}_1$) $\mathtt{b}_\mathtt{k} \in \mathbb{R}^+, \mathtt{k} = \overline{1,4}$ s.t. either $\mathtt{b}_1^2 + \mathtt{b}_2^2 > 0$ or $\mathtt{b}_3^2 + \mathtt{b}_4^2 > 0$,
($\mathbf{H}_2$) $\Delta_1 = \mathtt{r}_1^2(\mathtt{b}_1\mathtt{b}_4 + \mathtt{b}_2\mathtt{b}_3)\cosh(\mathtt{r}_1) + \mathtt{r}_1(\mathtt{b}_1\mathtt{b}_3 + \mathtt{b}_2\mathtt{b}_4\mathtt{r}_1^2)\sinh(\mathtt{r}_1) > 0$,
($\mathbf{H}_3$) $\mathtt{d}_\mathtt{k} \in \mathbb{R}^+, \mathtt{k} = \overline{1,4}$ s.t. either $\mathtt{d}_1^2 + \mathtt{d}_2^2 > 0$ or $\mathtt{d}_3^2 + \mathtt{d}_4^2 > 0$,
($\mathbf{H}_4$) $\Delta_2 = \mathtt{r}_2^2(\mathtt{d}_1\mathtt{d}_4 + \mathtt{d}_2\mathtt{d}_3)\cosh(\mathtt{r}_2) + \mathtt{r}_2(\mathtt{d}_1\mathtt{d}_3 + \mathtt{d}_2\mathtt{d}_4\mathtt{r}_2^2)\sinh(\mathtt{r}_2) > 0$,
($\mathbf{H}_5$) $\mathtt{g}_\mathtt{i} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^+$ are continuous, for $\mathtt{i} = \overline{1,2}$.

This article's content is organized as follows. The second Sect. includes preliminary findings. The key theorems are presented in Sect. 3, and in Sect. 4, we provide an example to show how the main result can be implemented in practice.

2. Kernel and Bounds

We convert the BVP into a kernel-based equivalent integral equation. Furthermore, we give bounds for the Kernel that are crucial to our main findings.

Let us take $\mathtt{I}_1 = [0,1], \mathtt{I}_2 = \left[\frac{1}{8}, \frac{7}{8}\right]$ and $\mathtt{I}_3 = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Lemma 2.1. *Suppose* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *hold. If* $\mathtt{h}_1(\mathtt{z}) \in \mathtt{C}[0,1]$, *then the DEq*

$$-\mathtt{u}_1''(\mathtt{z}) + \mathtt{r}_1^2\mathtt{u}_1(\mathtt{z}) = \mathtt{h}_1(\mathtt{z}), \quad \mathtt{z} \in (0,1), \tag{5}$$

with BCs (3) *has a unique solution:*

$$\mathtt{u}_1(\mathtt{z}) = \int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})\mathtt{h}_1(\mathtt{s})d\mathtt{s},$$

where

$$\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) = \frac{1}{\Delta_1}\begin{cases}\Phi_1(\mathtt{z})\Phi_2(\mathtt{s}), & \mathtt{z} \le \mathtt{s},\\ \Phi_2(\mathtt{z})\Phi_1(\mathtt{s}), & \mathtt{s} \le \mathtt{z},\end{cases} \tag{6}$$

$$\Phi_1(\mathtt{z}) = \mathtt{b}_1\sinh(\mathtt{r}_1\mathtt{z}) + \mathtt{b}_2\mathtt{r}_1\cosh(\mathtt{r}_1\mathtt{z}),$$
$$\Phi_2(\mathtt{z}) = \mathtt{b}_3\sinh\big(\mathtt{r}_1(1-\mathtt{z})\big) + \mathtt{b}_4\mathtt{r}_1\cosh\big(\mathtt{r}_1(1-\mathtt{z})\big).$$

Proof. By algebraic calculations, we can establish the result. □

Lemma 2.2. *The Kernel* $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$ *given in* (6) *is nonnegative, for all* $\mathtt{z},\mathtt{s} \in \mathtt{I}_1$.

Proof. From the definition of the Kernel $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$, it is clear that $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) \ge 0$ for all $\mathtt{z},\mathtt{s} \in \mathtt{I}_1$. □

Lemma 2.3. *The Kernel* $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$ *given in* (6) *satisfies:*

$$\mathtt{k}_1^*\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}) \le \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) \le \mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}),\ \ for\ \mathtt{z},\mathtt{s} \in \mathtt{I}_1, \tag{7}$$

where

$$\mathtt{k}_1^* = \min\left\{\frac{\mathtt{b}_2\mathtt{r}_1}{\mathtt{b}_1\sinh(\mathtt{r}_1) + \mathtt{b}_2\mathtt{r}_1\cosh(\mathtt{r}_1)}, \frac{\mathtt{b}_4\mathtt{r}_1}{\mathtt{b}_3\sinh(\mathtt{r}_1) + \mathtt{b}_4\mathtt{r}_1\cosh(\mathtt{r}_1)}\right\}.$$

Proof. For $\mathtt{z},\mathtt{s} \in \mathtt{I}_1$,

$$\frac{\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})}{\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})} = \begin{cases}\dfrac{\Phi_1(\mathtt{z})\Phi_2(\mathtt{s})}{\Phi_1(\mathtt{s})\Phi_2(\mathtt{s})} = \dfrac{\Phi_1(\mathtt{z})}{\Phi_1(\mathtt{s})} \le 1, & \mathtt{z} \le \mathtt{s},\\ \dfrac{\Phi_2(\mathtt{z})\Phi_1(\mathtt{s})}{\Phi_2(\mathtt{s})\Phi_1(\mathtt{s})} = \dfrac{\Phi_2(\mathtt{z})}{\Phi_2(\mathtt{s})} \le 1, & \mathtt{s} \le \mathtt{z},\end{cases}$$

which proves $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) \le \mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})$. Finally for $\mathtt{z},\mathtt{s} \in \mathtt{I}_1$,

$$\frac{\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})}{\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})} = \begin{cases}\dfrac{\Phi_1(\mathtt{z})}{\Phi_1(\mathtt{s})} \ge \dfrac{\mathtt{b}_2\mathtt{r}_1}{\mathtt{b}_1\sinh(\mathtt{r}_1) + \mathtt{b}_2\mathtt{r}_1\cosh(\mathtt{r}_1)}, & \mathtt{z} \le \mathtt{s},\\ \dfrac{\Phi_2(\mathtt{z})}{\Phi_2(\mathtt{s})} \ge \dfrac{\mathtt{b}_4\mathtt{r}_1}{\mathtt{b}_3\sinh(\mathtt{r}_1) + \mathtt{b}_4\mathtt{r}_1\cosh(\mathtt{r}_1)}, & \mathtt{s} \le \mathtt{z},\end{cases}$$

that implies $\mathtt{k}_1^*\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}) \le \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$, and the lemma follows. □

Lemma 2.4. *The Kernel* $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$ *given in* (6) *satisfies:*

$$\mathtt{k}_2^*\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}) \le \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) \le \mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}),\ \ for\ \mathtt{z},\mathtt{s} \in \mathtt{I}_2, \tag{8}$$

where

$$\mathtt{k}_2^* = \min\left\{\frac{\Phi_1\left(\frac{1}{8}\right)}{\Phi_1\left(\frac{7}{8}\right)}, \frac{\Phi_2\left(\frac{7}{8}\right)}{\Phi_2\left(\frac{1}{8}\right)}\right\}.$$

Proof. For $\mathtt{z}, \mathtt{s} \in \mathtt{I}_2$,

$$\frac{\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})}{\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})} = \begin{cases} \dfrac{\Phi_1(\mathtt{z})}{\Phi_1(\mathtt{s})} \geq \dfrac{\Phi_1\left(\frac{1}{8}\right)}{\Phi_1\left(\frac{7}{8}\right)}, & \mathtt{z} \leq \mathtt{s}, \\ \dfrac{\Phi_2(\mathtt{z})}{\Phi_2(\mathtt{s})} \geq \dfrac{\Phi_2\left(\frac{7}{8}\right)}{\Phi_2\left(\frac{1}{8}\right)}, & \mathtt{s} \leq \mathtt{z}, \end{cases}$$

which implies $\mathtt{k}_2^*\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$. Finally for $\mathtt{z}, \mathtt{s} \in \mathtt{I}_2$,

$$\frac{\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})}{\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})} = \begin{cases} \dfrac{\Phi_1(\mathtt{z})}{\Phi_1(\mathtt{s})} \leq 1, & \mathtt{z} \leq \mathtt{s}, \\ \dfrac{\Phi_2(\mathtt{z})}{\Phi_2(\mathtt{s})} \leq 1, & \mathtt{s} \leq \mathtt{z}, \end{cases}$$

which proves $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})$. This completes the proof. ☐

Lemma 2.5. *The Kernel* $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$ *given in* (6) *satisfies:*

$$\mathtt{k}_3^*\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}), \ \textit{for}\ \mathtt{z}, \mathtt{s} \in \mathtt{I}_3, \tag{9}$$

where

$$\mathtt{k}_3^* = \min\left\{\frac{\Phi_1\left(\frac{1}{4}\right)}{\Phi_1\left(\frac{3}{4}\right)}, \frac{\Phi_2\left(\frac{3}{4}\right)}{\Phi_2\left(\frac{1}{4}\right)}\right\}.$$

Proof. For $\mathtt{z}, \mathtt{s} \in \mathtt{I}_3$,

$$\frac{\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})}{\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})} = \begin{cases} \dfrac{\Phi_1(\mathtt{z})}{\Phi_1(\mathtt{s})} \geq \dfrac{\Phi_1\left(\frac{1}{4}\right)}{\Phi_1\left(\frac{3}{4}\right)}, & \mathtt{z} \leq \mathtt{s}, \\ \dfrac{\Phi_2(\mathtt{z})}{\Phi_2(\mathtt{s})} \geq \dfrac{\Phi_2\left(\frac{3}{4}\right)}{\Phi_2\left(\frac{1}{4}\right)}, & \mathtt{s} \leq \mathtt{z}, \end{cases}$$

which implies $\mathtt{k}_3^*\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})$. Finally for $\mathtt{z}, \mathtt{s} \in \mathtt{I}_3$,

$$\frac{\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})}{\mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})} = \begin{cases} \dfrac{\Phi_1(\mathtt{z})}{\Phi_1(\mathtt{s})} \leq 1, & \mathtt{z} \leq \mathtt{s}, \\ \dfrac{\Phi_2(\mathtt{z})}{\Phi_2(\mathtt{s})} \leq 1, & \mathtt{s} \leq \mathtt{z}, \end{cases}$$

which proves $\mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_1}(\mathtt{s},\mathtt{s})$. This completes the proof. ☐

Lemma 2.6. *Suppose* $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *hold. If* $\mathtt{h}_2(\mathtt{z}) \in \mathtt{C}[0,1]$, *then the DEq*

$$-\mathtt{u}_2''(\mathtt{z}) + \mathtt{r}_2^2\mathtt{u}_2(\mathtt{z}) = \mathtt{h}_2(\mathtt{z}), \quad \mathtt{z} \in (0,1), \tag{10}$$

with BCs (3) *has a unique solution:*

$$\mathtt{u}_2(\mathtt{z}) = \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})\mathtt{h}_2(\mathtt{s})d\mathtt{s},$$

where

$$\mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s}) = \frac{1}{\Delta_2}\begin{cases} \Psi_1(\mathtt{z})\Psi_2(\mathtt{s}), & \mathtt{z} \leq \mathtt{s}, \\ \Psi_2(\mathtt{z})\Psi_1(\mathtt{s}), & \mathtt{s} \leq \mathtt{z}, \end{cases} \tag{11}$$

$$\Psi_1(\mathtt{z}) = \mathtt{d}_1\sinh(\mathtt{r}_2\mathtt{z}) + \mathtt{d}_2\mathtt{r}_2\cosh(\mathtt{r}_2\mathtt{z}),$$
$$\Psi_2(\mathtt{z}) = \mathtt{d}_3\sinh\big(\mathtt{r}_2(1-\mathtt{z})\big) + \mathtt{d}_4\mathtt{r}_2\cosh\big(\mathtt{r}_2(1-\mathtt{z})\big).$$

Lemma 2.7. *The Kernel* $\mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})$ *given in* (11) *is nonnegative, for all* $\mathtt{z}, \mathtt{s} \in \mathtt{I}_1$.

Lemma 2.8. *For* $\mathtt{z}, \mathtt{s} \in \mathtt{I}_1$*, then the Kernel* $\mathtt{G}_{\mathtt{r}_2}(\mathtt{z}, \mathtt{s})$ *given in* (11) *satisfies:*

$$\mathtt{k}_1^{**}\mathtt{G}_{\mathtt{r}_2}(\mathtt{s}, \mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_2}(\mathtt{z}, \mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_2}(\mathtt{s}, \mathtt{s}), \tag{12}$$

where

$$\mathtt{k}_1^{**} = \min\left\{\frac{\mathtt{d}_2\mathtt{r}_2}{\mathtt{d}_1\sinh(\mathtt{r}_2) + \mathtt{d}_2\mathtt{r}_2\cosh(\mathtt{r}_2)}, \frac{\mathtt{d}_4\mathtt{r}_2}{\mathtt{d}_3\sinh(\mathtt{r}_2) + \mathtt{d}_4\mathtt{r}_2\cosh(\mathtt{r}_2)}\right\}.$$

Lemma 2.9. *For* $\mathtt{z}, \mathtt{s} \in \mathtt{I}_2$*, then the Kernel* $\mathtt{G}_{\mathtt{r}_2}(\mathtt{z}, \mathtt{s})$ *given in* (11) *satisfies:*

$$\mathtt{k}_2^{**}\mathtt{G}_{\mathtt{r}_2}(\mathtt{s}, \mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_2}(\mathtt{z}, \mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_2}(\mathtt{s}, \mathtt{s}), \tag{13}$$

where

$$\mathtt{k}_2^{**} = \min\left\{\frac{\Psi_1\left(\frac{1}{8}\right)}{\Psi_1\left(\frac{7}{8}\right)}, \frac{\Psi_2\left(\frac{7}{8}\right)}{\Psi_2\left(\frac{1}{8}\right)}\right\}.$$

Lemma 2.10. *For* $\mathtt{z}, \mathtt{s} \in \mathtt{I}_3$*, then the Kernel* $\mathtt{G}_{\mathtt{r}_2}(\mathtt{z}, \mathtt{s})$ *given in* (11) *satisfies:*

$$\mathtt{k}_3^{**}\mathtt{G}_{\mathtt{r}_2}(\mathtt{s}, \mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_2}(\mathtt{z}, \mathtt{s}) \leq \mathtt{G}_{\mathtt{r}_2}(\mathtt{s}, \mathtt{s}), \tag{14}$$

where

$$\mathtt{k}_3^{**} = \min\left\{\frac{\Psi_1\left(\frac{1}{4}\right)}{\Psi_1\left(\frac{3}{4}\right)}, \frac{\Psi_2\left(\frac{3}{4}\right)}{\Psi_2\left(\frac{1}{4}\right)}\right\}.$$

Consider the Banach space $\mathtt{B} = \mathtt{E} \times \mathtt{E}$, where $\mathtt{E} = \Big\{\mathtt{u}_1 : \mathtt{u}_1 \in \mathtt{C}[0, 1]\Big\}$ be endowed with the norm $\|(\mathtt{u}_1, \mathtt{u}_2)\| = \|\mathtt{u}_1\|_0 + \|\mathtt{u}_2\|_0$, for $(\mathtt{u}_1, \mathtt{u}_2) \in \mathtt{B}$ and $\|\mathtt{u}_1\|_0 = \max\limits_{\mathtt{z}\in[0,1]} |\mathtt{u}_1(\mathtt{z})|$. We can define the cone $\mathtt{K} \subset \mathtt{B}$ by

$$\mathtt{K} = \Big\{(\mathtt{u}_1, \mathtt{u}_2) \in \mathtt{B} : (\mathtt{u}_1, \mathtt{u}_2) \text{ is nonnegative, nondecreasing and concave}\Big\}.$$

Let α and β be respectively nonnegative continuous concave and convex functionals on $\mathtt{K}$. Then we define the sets for positive real numbers $\Gamma_\mathtt{r}$ and $\Theta_\mathtt{R}$:

$$\mathtt{Q}\big(\beta, \Theta_\mathtt{R}\big) = \big\{\mathtt{u} \in \mathtt{K} : \beta\big(\mathtt{u}\big) \leq \Theta_\mathtt{R}\big\},$$
$$\mathtt{Q}\big(\alpha, \beta, \Gamma_\mathtt{r}, \Theta_\mathtt{R}\big) = \big\{\mathtt{u} \in \mathtt{K} : \alpha\big(\mathtt{u}\big) \geq \Gamma_\mathtt{r} \text{ and } \beta\big(\mathtt{u}\big) \leq \Theta_\mathtt{R}\big\}.$$

Theorem 2.1. [2] *Let* $\mathtt{K}$ *be a cone in the real Banach space* $\mathtt{B}$*. Suppose* α, ψ *and* ζ *are nonnegative continuous concave functionals on* $\mathtt{K}$*, and there exist nonnegative numbers* $\Lambda_\ell, \Lambda_{\ell'}, \Gamma_\mathtt{r}, \Gamma_{\mathtt{r}'}, \Theta_\mathtt{R}$ *and* $\Theta_{\mathtt{R}'}$ *s.t.* $\mathtt{T} : \mathtt{Q}\big(\beta, \Theta_\mathtt{R}\big) \to \mathtt{K}$ *is a completely continuous operator and*

($\Re_1$) $\mathtt{Q}\big(\beta, \Theta_\mathtt{R}\big)$ *is a bounded set,*
($\Re_2$) $\mathtt{Q}\big(\eta, \Lambda_\ell\big)$ *and* $\mathtt{Q}\big(\alpha, \beta, \Gamma_\mathtt{r}, \Theta_\mathtt{R}\big)$ *are disjiont subsets of* $\mathtt{Q}\big(\beta, \Theta_\mathtt{R}\big)$,
($\Re_3$) $\big\{\mathtt{u} \in \mathtt{K} : \theta\big(\mathtt{u}\big) < \Gamma_{\mathtt{r}'}, \Gamma_\mathtt{r} < \alpha\big(\mathtt{u}\big), \Theta_{\mathtt{R}'} < \psi\big(\mathtt{u}\big), \textit{and } \beta\big(\mathtt{u}\big) < \Theta_\mathtt{R}\big\} \neq \emptyset$,
($\Re_4$) $\big\{\mathtt{u} \in \mathtt{K} : \Lambda_{\ell'} < \zeta\big(\mathtt{u}\big) \textit{ and } \eta\big(\mathtt{u}\big) < \Lambda_\ell\big\} \neq \emptyset$, *and*
($\Re_5$) $\big\{\mathtt{u} \in \mathtt{K} : \Lambda_\ell < \eta\big(\mathtt{u}\big) \textit{ and } \alpha\big(\mathtt{u}\big) < \Gamma_\mathtt{r}\big\} \neq \emptyset$.

Let the following properties be satisfied:

$(\mathbf{P}_1)$ $\alpha(\mathtt{Tu}) > \Gamma_{\mathtt{r}}$, $\forall$ $\mathtt{u} \in \mathtt{K}$ *with* $\alpha(\mathtt{u}) = \Gamma_{\mathtt{r}}, \beta(\mathtt{u}) \leq \Theta_{\mathtt{R}}$, *and* $\Gamma_{\mathtt{r}'} < \theta(\mathtt{Tu})$,

$(\mathbf{P}_2)$ $\alpha(\mathtt{Tu}) > \Gamma_{\mathtt{r}}$, $\forall$ $\mathtt{u} \in \mathtt{K}$ *with* $\alpha(\mathtt{u}) = \Gamma_{\mathtt{r}}, \beta(\mathtt{u}) \leq \Theta_{\mathtt{R}}$, *and* $\theta(\mathtt{Tu}) \leq \Gamma_{\mathtt{r}'}$,

$(\mathbf{P}_3)$ $\beta(\mathtt{Tu}) < \Theta_{\mathtt{R}}$, $\forall$ $\mathtt{u} \in \mathtt{K}$ *with* $\Gamma_{\mathtt{r}} \leq \alpha(\mathtt{u}), \beta(\mathtt{u}) = \Theta_{\mathtt{R}}$, *and* $\psi(\mathtt{Tu}) < \Theta_{\mathtt{R}'}$,

$(\mathbf{P}_4)$ $\beta(\mathtt{Tu}) < \Theta_{\mathtt{R}}$, $\forall$ $\mathtt{u} \in \mathtt{K}$ *with* $\Gamma_{\mathtt{r}} \leq \alpha(\mathtt{u}), \beta(\mathtt{u}) = \Theta_{\mathtt{R}}$, *and* $\Theta_{\mathtt{R}'} \leq \psi(\mathtt{Tu})$,

$(\mathbf{P}_5)$ $\eta(\mathtt{Tu}) < \Lambda_{\ell}$, $\forall$ $\mathtt{u} \in \mathtt{K}$ *with* $\eta(\mathtt{u}) = \Lambda_{\ell}$ *and* $\zeta(\mathtt{Tu}) < \Lambda_{\ell'}$, *and*

$(\mathbf{P}_6)$ $\eta(\mathtt{Tu}) < \Lambda_{\ell}$, $\forall$ $\mathtt{u} \in \mathtt{K}$ *with* $\eta(\mathtt{u}) = \Lambda_{\ell}$ *and* $\Lambda_{\ell'} \leq \zeta(\mathtt{Tu})$.

Then $\mathtt{T}$ *has at least three fixed points* $\mathtt{u}_1, \mathtt{u}_2$ *and* $\mathtt{u}_3$ *in* $\mathtt{Q}(\beta, \Theta_{\mathtt{R}})$ *s.t.* $\eta(\mathtt{u}_1) \leq \Lambda_{\ell}$, $\Gamma_{\mathtt{r}} \leq \alpha(\mathtt{u}_2)$ *with* $\beta(\mathtt{u}_2) \leq \Theta_{\mathtt{R}}$, *and* $\Lambda_{\ell} < \eta(\mathtt{u}_3)$ *with* $\alpha(\mathtt{u}_3) < \Gamma_{\mathtt{r}}$.

3. Main Results

We discuss the existence of multiple positive solution to the system of BVP (1)-(4) by applying SFFPT.

By a positive solution of the BVP (1)-(4), we mean $\big(\mathtt{u}_1(\mathtt{z}), \mathtt{u}_2(\mathtt{z})\big) \in \big(\mathtt{C}^2[0,1] \times \mathtt{C}^2[0,1]\big)$ satisfying (1)-(4) with $\mathtt{u}_1(\mathtt{z}) > 0, \mathtt{u}_2(\mathtt{z}) > 0, \forall \mathtt{z} \in [0,1]$.

We introduce the following notations for computational ease:

$$\mathtt{k}_1 = \min\left\{\mathtt{k}_1^*, \mathtt{k}_1^{**}\right\},\ \mathtt{k}_2 = \min\left\{\mathtt{k}_2^*, \mathtt{k}_2^{**}\right\},\ \mathtt{k}_3 = \min\left\{\mathtt{k}_3^*, \mathtt{k}_3^{**}\right\}. \tag{15}$$

$$\left.\begin{aligned} \mathtt{M}_1 &= \max_{\mathtt{z}\in \mathtt{I}_1}\left\{\int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})d\mathtt{s}, \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})d\mathtt{s}\right\},\\ \mathtt{M}_2 &= \max_{\mathtt{z}\in \mathtt{I}_2}\left\{\int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})d\mathtt{s}, \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})d\mathtt{s}\right\},\\ \mathtt{M}_3 &= \max_{\mathtt{z}\in \mathtt{I}_3}\left\{\int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})d\mathtt{s}, \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})d\mathtt{s}\right\}. \end{aligned}\right\} \tag{16}$$

The system of BVP (1)-(4) is well-known to be equivalent to an integral system:

$$\left.\begin{aligned} \mathtt{u}_1(\mathtt{z}) &= \int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})\mathtt{g}_1\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s},\\ \mathtt{u}_2(\mathtt{z}) &= \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})\mathtt{g}_2\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s}. \end{aligned}\right\} \tag{17}$$

Let $\mathtt{T}_1, \mathtt{T}_2 : \mathtt{K} \to \mathtt{E}$ be the operators defined as

$$\begin{cases} \mathtt{T}_1(\mathtt{u}_1,\mathtt{u}_2)(\mathtt{z}) = \displaystyle\int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})\mathtt{g}_1\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s},\\ \mathtt{T}_2(\mathtt{u}_1,\mathtt{u}_2)(\mathtt{z}) = \displaystyle\int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})\mathtt{g}_2\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s}. \end{cases}$$

Consider the operator $\mathtt{T} : \mathtt{K} \to \mathtt{B}$, which is defined as

$$\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2) = \Big(\mathtt{T}_1(\mathtt{u}_1,\mathtt{u}_2), \mathtt{T}_2(\mathtt{u}_1,\mathtt{u}_2)\Big),\ \text{for } (\mathtt{u}_1,\mathtt{u}_2) \in \mathtt{B}. \tag{18}$$

Define the concave functionals (α, ψ, ζ) and the convex functionals (θ, β, η) on $\mathtt{K}$ by

$$\left.\begin{aligned}\alpha(u_1,u_2) &= \min_{z\in I_1}\Big[\sum_{i=1}^{2} u_i(z)\Big], \quad \psi(u_1,u_2) = \min_{z\in I_2}\Big[\sum_{i=1}^{2} u_i(z)\Big],\\ \zeta(u_1,u_2) &= \min_{z\in I_3}\Big[\sum_{i=1}^{2} u_i(z)\Big], \quad \theta(u_1,u_2) = \max_{z\in I_1}\Big[\sum_{i=1}^{2} u_i(z)\Big],\\ \beta(u_1,u_2) &= \max_{z\in I_2}\Big[\sum_{i=1}^{2} u_i(z)\Big], \quad \eta(u_1,u_2) = \max_{z\in I_3}\Big[\sum_{i=1}^{2} u_i(z)\Big].\end{aligned}\right\} \tag{19}$$

Theorem 3.1. *Suppose* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *hold and there exist positive constants* $\Lambda_\ell, \Lambda_{\ell'}, \Gamma_r, \Gamma_{r'}, \Theta_R$ *and* $\Theta_{R'}$ *s.t.* g_i *for* $i = \overline{1,2}$ *satisfies:*

$$(\mathbf{K}_1)\ g_i(z,u_1,u_2) < \frac{\Lambda_\ell}{2M_3},\ \forall (z,u_1,u_2) \in I_3 \times \Big[0, \frac{\Gamma_r}{3}\Big]^2,$$

$$(\mathbf{K}_2)\ g_i(z,u_1,u_2) < \frac{\Theta_R}{2M_2},\ \forall (z,u_1,u_2) \in I_2 \times [0,\Theta_R]^2,$$

$$(\mathbf{K}_3)\ g_i(z,u_1,u_2) > \frac{\Gamma_r}{2k_1M_1},\ \forall (z,u_1,u_2) \in I_1 \times \Big[\Gamma_r, \frac{\Gamma_r}{k_1}\Big]^2.$$

Then the system of BVP (1)-(4) has at least three positive solutions (u_1,u_2), $(\widehat{u}_1,\widehat{u}_2)$ *and* $(\widetilde{u}_1,\widetilde{u}_2) \in Q(\beta,\Theta_R)$ *s.t.* $\eta\big((u_1,u_2)\big) \le \Lambda_\ell$, $\Gamma_r \le \alpha(\widehat{u}_1,\widehat{u}_2)$ *with* $\beta(\widehat{u}_1,\widehat{u}_2) \le \Theta_R$, *and* $\Lambda_\ell < \eta(\widetilde{u}_1,\widetilde{u}_2)$ *with* $\alpha(\widetilde{u}_1,\widetilde{u}_2) < \Gamma_r$.

Proof. Let $\Gamma'_r = \dfrac{\Gamma_r}{k_1}, \Theta_R = \dfrac{\Theta_{R'}}{k_2}, \Lambda_\ell = \dfrac{\Lambda'_\ell}{k_3}$ and $\Lambda_\ell = \dfrac{\Gamma_r}{3}$. According to the properties of g_i and $G_{ri}, i = \overline{1,2}$, we have that $T : Q(\beta,\Theta_R) \to K$ is completely continuous. The set $Q(\beta,\Theta_R)$ is bounded by a standard calculus argument. Since if $(u_1,u_2) \in Q(\beta,\Theta_R)$, then (u_1,u_2) is concave, and hence $\max_{z\in I_2}\Big[\sum_{i=1}^{2} u_i(z)\Big] < \Theta_R$. Also, it can easily be shown that

$$\begin{cases} \dfrac{\Theta_{R'} + \frac{\Gamma_r}{k_2}}{2} \in \Big\{(u_1,u_2) \in K : \theta(u_1,u_2) < \Gamma'_r, \Gamma_r < \alpha(u_1,u_2), \\ \qquad \Theta_{R'} < \psi(u_1,u_2) \text{ and } \beta(u_1,u_2) < \Theta_R\Big\}, \\ \dfrac{\Lambda_\ell}{4} \in \Big\{(u_1,u_2) \in K : \Lambda'_\ell < \zeta(u_1,u_2) \text{ and } \eta(u_1,u_2)\Big\}, \\ \dfrac{\Gamma_r + \Lambda_\ell}{2} \in \Big\{(u_1,u_2) \in K : \Lambda_\ell < \eta(u_1,u_2) \text{ and } \alpha(u_1,u_2) < \Gamma_r\Big\}. \end{cases}$$

As a result, the sets are nonempty. Consequently, if $(u_1,u_2) \in Q(\eta,\Lambda_\ell)$, then

$$\max_{z\in I_3}\Big[\sum_{i=1}^{2} u_i(z)\Big] < \Lambda_\ell,$$

and hence

$$\alpha(u_1,u_2) = \Big[\sum_{i=1}^{2} u_i(0)\Big] \le \Big[\sum_{i=1}^{2} u_i(1)\Big] \le 2\Lambda_\ell < \Gamma_r.$$

Thus, $\mathtt{Q}(\eta, \Lambda_\ell)$ and $\mathtt{Q}(\alpha, \beta, \Gamma_{\mathtt{r}}, \Theta_{\mathtt{R}})$ are disjoint subsets of $\mathtt{Q}(\beta, \Theta_{\mathtt{R}})$. As a result, Theorem 2.1's conditions $(\Re_1), (\Re_2)$, $(\Re_3), (\Re_4)$ and $(\Re_5)$ are satisfied. Now we will check the functional conditions.

Claim (i): $\alpha\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) > \Gamma_{\mathtt{r}}$, $\forall (\mathtt{u}_1, \mathtt{u}_2) \in \mathtt{Q}(\alpha, \beta, \Gamma_{\mathtt{r}}, \Theta_{\mathtt{R}})$ with $\alpha(\mathtt{u}_1, \mathtt{u}_2) = \Gamma_{\mathtt{r}}$ and $\Gamma'_{\mathtt{r}} < \theta\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big)$. Let $(\mathtt{u}_1, \mathtt{u}_2) \in \mathtt{Q}(\alpha, \beta, \Gamma_{\mathtt{r}}, \Theta_{\mathtt{R}})$ with $\alpha(\mathtt{u}_1, \mathtt{u}_2) = \Gamma_{\mathtt{r}}$, $\beta(\mathtt{u}_1, \mathtt{u}_2) \le \Theta_{\mathtt{R}}$ and $\Gamma'_{\mathtt{r}} < \theta\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big)$. Then since $\mathtt{k}_1 \mathtt{G}_{\mathtt{ri}}(\mathtt{z}, \mathtt{s}) \le \mathtt{G}_{\mathtt{ri}}(\mathtt{s}, \mathtt{s}), \mathtt{i} = \overline{1,2}$, for $\mathtt{s} \in [0,1]$, we have

$$\begin{aligned}\alpha\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) =& \min_{\mathtt{z} \in \mathtt{I}_1} \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z}, \mathtt{s}) \mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) ds \right] \\ \ge& \mathtt{k}_1 \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{s}, \mathtt{s}) \mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) ds \right] \\ =& \mathtt{k}_1 \theta\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) \\ >& \mathtt{k}_1 \Gamma'_{\mathtt{r}} = \Gamma_{\mathtt{r}}.\end{aligned}$$

Claim (ii): $\alpha(\mathtt{T}\big(\mathtt{u}_1, \mathtt{u}_2)\big) > \Gamma_{\mathtt{r}}$, $\forall (\mathtt{u}_1, \mathtt{u}_2) \in \Big\{ (\mathtt{u}_1, \mathtt{u}_2) \in \mathtt{Q}(\alpha, \beta, \Gamma_{\mathtt{r}}, \Theta_{\mathtt{R}}) : \theta(\mathtt{u}_1, \mathtt{u}_2) \le \Gamma'_{\mathtt{r}} \Big\}$ with $\alpha(\mathtt{u}_1, \mathtt{u}_2) = \Gamma_{\mathtt{r}}$. Let $(\mathtt{u}_1, \mathtt{u}_2) \in \mathtt{Q}(\alpha, \beta, \Gamma_{\mathtt{r}}, \Theta_{\mathtt{R}})$ with $\alpha(\mathtt{u}_1, \mathtt{u}_2) = \Gamma_{\mathtt{r}}$, $\beta(\mathtt{u}_1, \mathtt{u}_2) \le \Theta_{\mathtt{R}}$ and $\Gamma'_{\mathtt{r}} < \theta\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big)$. Thus $\Gamma_{\mathtt{r}} \le \Big[\sum_{\mathtt{i}=1}^{2} \mathtt{u}_\mathtt{i}(\mathtt{s}) \Big] \le \dfrac{\Gamma_{\mathtt{r}}}{\mathtt{k}_1}$, for $\mathtt{s} \in [0,1]$ and hence

$$\mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) \ge \frac{\Gamma_{\mathtt{r}}}{2\mathtt{k}_1 \mathtt{M}_1}, \text{ for } \mathtt{i} = \overline{1,2}.$$

Therefore,

$$\begin{aligned}\alpha\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) =& \min_{\mathtt{z} \in \mathtt{I}_1} \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z}, \mathtt{s}) \mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) ds \right] \\ \ge& \mathtt{k}_1 \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{s}, \mathtt{s}) \mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) ds \right] \\ \ge& \mathtt{k}_1 \left(\frac{\Gamma_{\mathtt{r}}}{2\mathtt{k}_1 \mathtt{M}_1} \right) \max_{\mathtt{z} \in \mathtt{I}_1} \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z}, \mathtt{s}) ds \right] \\ =& \frac{\Gamma_{\mathtt{r}}}{2} + \frac{\Gamma_{\mathtt{r}}}{2} = \Gamma_{\mathtt{r}}.\end{aligned}$$

Claim (iii): $\beta\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) < \Theta_{\mathtt{R}}$ for all $(\mathtt{u}_1, \mathtt{u}_2) \in \mathtt{Q}(\alpha, \beta, \Gamma_{\mathtt{r}}, \Theta_{\mathtt{R}})$, $\beta(\mathtt{u}_1, \mathtt{u}_2) = \Theta_{\mathtt{R}}$, and $\psi\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) < \Theta_{\mathtt{R}'}$. Then, we have

$$\begin{aligned}\beta\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) =& \max_{\mathtt{z} \in \mathtt{I}_2} \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z}, \mathtt{s}) \mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) ds \right] \\ \le& \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{s}, \mathtt{s}) \mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) ds \right] \\ \le& \frac{1}{\mathtt{k}_2} \min_{\mathtt{z} \in \mathtt{I}_2} \sum_{\mathtt{i}=1}^{2} \left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z}, \mathtt{s}) \mathtt{g}_\mathtt{i}\big(\mathtt{s}, \mathtt{u}_1(\mathtt{s}), \mathtt{u}_2(\mathtt{s})\big) ds \right] \\ =& \frac{1}{\mathtt{k}_2} \psi\big(\mathtt{T}(\mathtt{u}_1, \mathtt{u}_2)\big) < \frac{\Theta_{\mathtt{R}'}}{\mathtt{k}_2} = \Theta_{\mathtt{R}}.\end{aligned}$$

Claim $(\boldsymbol{iv})$: $\beta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) < \Theta_{\mathtt{R}}$, $\forall(\mathtt{u}_1,\mathtt{u}_2) \in \Big\{(\mathtt{u}_1,\mathtt{u}_2) \in \mathtt{Q}(\alpha,\beta,\Gamma_{\mathtt{r}},\Theta_{\mathtt{R}}) : \Theta_{\mathtt{R}'} \leq \psi(\mathtt{u}_1,\mathtt{u}_2)\Big\}$, $\beta(\mathtt{u}_1,\mathtt{u}_2) = \Theta_{\mathtt{R}}$. Let $(\mathtt{u}_1,\mathtt{u}_2) \in \Big\{(\mathtt{u}_1,\mathtt{u}_2) \in \mathtt{Q}(\alpha,\beta,\Gamma_{\mathtt{r}},\Theta_{\mathtt{R}}) : \Theta_{\mathtt{R}'} \leq \psi(\mathtt{u}_1,\mathtt{u}_2)\Big\}$ with $\beta(\mathtt{u}_1,\mathtt{u}_2) = \Theta_{\mathtt{R}}$. Thus, $\psi \geq \Theta_{\mathtt{R}'}$, hence for $\mathtt{i} = \overline{1,2}$, $\mathtt{g}_{\mathtt{i}}\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big) < \dfrac{\Theta_{\mathtt{R}}}{2\mathtt{M}_2}$. Therefore,

$$
\begin{aligned}
\beta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) &= \max_{\mathtt{z}\in\mathtt{I}_2}\sum_{\mathtt{i}=1}^{2}\left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z},\mathtt{s})\mathtt{g}_\mathtt{i}\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s}\right]\\
&\leq \left(\frac{\Theta_{\mathtt{R}}}{2\mathtt{M}_2}\right)\max_{\mathtt{z}\in\mathtt{I}_2}\sum_{\mathtt{i}=1}^{2}\left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z},\mathtt{s})d\mathtt{s}\right]\\
&= \left(\frac{\Theta_{\mathtt{R}}}{2\mathtt{M}_2}\right)(2\mathtt{M}_2) \leq \Theta_{\mathtt{R}}.
\end{aligned}
$$

Claim $(\boldsymbol{v})$: $\eta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) < \Lambda_\ell$, $\forall(\mathtt{u}_1,\mathtt{u}_2) \in \mathtt{Q}(\eta,\Lambda_\ell)$ with $\eta(\mathtt{u}_1,\mathtt{u}_2) = \Lambda_\ell$, and $\zeta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) < \Lambda'_\ell$.

$$
\begin{aligned}
\eta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) &= \max_{\mathtt{z}\in\mathtt{I}_3}\sum_{\mathtt{i}=1}^{2}\left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z},\mathtt{s})\mathtt{g}_\mathtt{i}\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s}\right]\\
&\leq \sum_{\mathtt{i}=1}^{2}\left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{s},\mathtt{s})\mathtt{g}_\mathtt{i}\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s}\right]\\
&\leq \frac{1}{\mathtt{k}_3}\min_{\mathtt{z}\in\mathtt{I}_3}\sum_{\mathtt{i}=1}^{2}\left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z},\mathtt{s})\mathtt{g}_\mathtt{i}\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s}\right]\\
&= \frac{1}{\mathtt{k}_3}\zeta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) < \frac{\Lambda'_\ell}{\mathtt{k}_3} = \Lambda_\ell.
\end{aligned}
$$

Claim $(\boldsymbol{vi})$: $\eta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) < \Lambda_\ell$, $\forall(\mathtt{u}_1,\mathtt{u}_2) \in \big\{(\mathtt{u}_1,\mathtt{u}_2) \in \mathtt{Q}(\eta,\Lambda_\ell) : \Lambda'_\ell \leq \zeta(\mathtt{u}_1,\mathtt{u}_2)\big\}$, with $\eta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) < \Lambda_\ell$. Let $(\mathtt{u}_1,\mathtt{u}_2) \in \big\{(\mathtt{u}_1,\mathtt{u}_2) \in \mathtt{Q}(\eta,\Lambda_\ell) : \Lambda'_\ell \leq \zeta(\mathtt{u}_1,\mathtt{u}_2)\big\}$, with $\eta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) < \Lambda_\ell$. Thus, $\zeta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) \geq \Lambda'_\ell$, and hence $\mathtt{g}_\mathtt{i}\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big) < \dfrac{\Lambda_\ell}{2\mathtt{M}_3}$, for $\mathtt{i} = \overline{1,2}$. Therefore,

$$
\begin{aligned}
\eta\big(\mathtt{T}(\mathtt{u}_1,\mathtt{u}_2)\big) &= \max_{\mathtt{z}\in\mathtt{I}_3}\sum_{\mathtt{i}=1}^{2}\left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z},\mathtt{s})\mathtt{g}_\mathtt{i}\big(\mathtt{s},\mathtt{u}_1(\mathtt{s}),\mathtt{u}_2(\mathtt{s})\big)d\mathtt{s}\right]\\
&\leq \left(\frac{\Lambda_\ell}{2\mathtt{M}_3}\right)\max_{\mathtt{z}\in\mathtt{I}_3}\sum_{\mathtt{i}=1}^{2}\left[\int_0^1 \mathtt{G}_{\mathtt{r}_\mathtt{i}}(\mathtt{z},\mathtt{s})d\mathtt{s}\right]\\
&= \left(\frac{\Lambda_\ell}{2\mathtt{M}_3}\right)(2\mathtt{M}_3) \leq \Lambda_\ell.
\end{aligned}
$$

Therefore the conditions $(\mathbf{P}_1)$-$(\mathbf{P}_6)$ of Theorem 2.1 have been satisfied. Thus $\mathtt{T}$ has at least three positive solutions $(\mathtt{u}_1,\mathtt{u}_2)$, $(\widehat{\mathtt{u}}_1,\widehat{\mathtt{u}}_2)$ and $(\widetilde{\mathtt{u}}_1,\widetilde{\mathtt{u}}_2) \in \mathtt{Q}(\beta,\Theta_{\mathtt{R}})$ of the system of BVP (1)-(4) s.t. $\eta\big((\mathtt{u}_1,\mathtt{u}_2)\big) \leq \Lambda_\ell$, $\Gamma_{\mathtt{r}} \leq \alpha\big(\widehat{\mathtt{u}}_1,\widehat{\mathtt{u}}_2\big)$ with $\beta\big(\widehat{\mathtt{u}}_1,\widehat{\mathtt{u}}_2\big) \leq \Theta_{\mathtt{R}}$, and $\Lambda_\ell < \eta\big(\widetilde{\mathtt{u}}_1,\widetilde{\mathtt{u}}_2\big)$ with $\alpha\big(\widetilde{\mathtt{u}}_1,\widetilde{\mathtt{u}}_2\big) < \Gamma_{\mathtt{r}}$. □

Corollary 3.1. *Suppose* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *hold and there exist positive constants* $\Gamma_{\mathtt{rj}}$, $\Gamma'_{\mathtt{rj}}$,$\Theta_{\mathtt{Rj}}$, $\Theta'_{\mathtt{Rj}}$, $\Lambda_{\ell\mathtt{j}}$ *and* $\Lambda'_{\ell\mathtt{j}}$ $(1 \le \mathtt{j} \le \mathtt{n})$ *s.t.* $\mathtt{g_i}$, *for* $\mathtt{i} = \overline{1,2}$ *satisfies:*

$$(\mathbf{K}_4)\ \mathtt{g_i}\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) < \frac{\Lambda_{\ell\mathtt{j}}}{2\mathtt{M}_3},\ \forall\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) \in \mathtt{I}_3 \times \left[0, \frac{\Gamma_{\mathtt{rj}}}{3}\right]^2, 1 \le \mathtt{j} \le \mathtt{n},$$

$$(\mathbf{K}_5)\ \mathtt{g_i}\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) < \frac{\Theta_{\mathtt{Rq}}}{2\mathtt{M}_2},\ \forall\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) \in \mathtt{I}_2 \times [0,\Theta_{\mathtt{Rq}}]^2, 1 \le \mathtt{q} \le \mathtt{n},$$

$$(\mathbf{K}_6)\ \mathtt{g_i}\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) > \frac{\Gamma_{\mathtt{rp}}}{2\mathtt{k}_1\mathtt{M}_1}, \forall\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) \in \mathtt{I}_1 \times \left[\Gamma_{\mathtt{rp}}, \frac{\Gamma_{\mathtt{rp}}}{\mathtt{k}_1}\right]^2, 1 \le \mathtt{p} \le \mathtt{n}.$$

Then the BVP (1)-(4) *has at least* $2\mathtt{n}+1$ *positive solutions in* $\mathtt{Q}(\beta,\Theta_{\mathtt{Rn}})$.

Proof. On $\mathtt{n}$, we employ induction. Initially, for $\mathtt{n} = 1$, we know that the system of BVP (1)-(4) has at least three positive solutions in $\mathtt{Q}(\beta,\Theta_{\mathtt{R1}})$ according to Theorem 3.1. Then we assume that this result holds true for $\mathtt{n} = \mathtt{k}$. To clearly show that this conclusion holds true for $\mathtt{n} = \mathtt{k}+1$. We assume that there are positive constants $\Gamma_{\mathtt{rj}}, \Gamma'_{\mathtt{rj}}, \Theta_{\mathtt{Rj}}, \Theta'_{\mathtt{Rj}}, \Lambda_{\ell\mathtt{j}}$ and $\Lambda'_{\ell\mathtt{j}}$ $(1 \le \mathtt{j} \le \mathtt{k}+1)$ s.t. $\mathtt{g_i}$, for $\mathtt{i} = \overline{1,2}$ satisfies:

$$\left.\begin{aligned} \mathtt{g_i}\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) < \frac{\Lambda_{\ell\mathtt{j}}}{2\mathtt{M}_3},\ \forall\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) \in \mathtt{I}_3 \times \left[0, \frac{\Gamma_{\mathtt{rj}}}{3}\right]^2, \\ 1 \le \mathtt{j} \le \mathtt{k}+1, \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} \mathtt{g_i}\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) < \frac{\Theta_{\mathtt{Rq}}}{2\mathtt{M}_2},\ \forall\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) \in \mathtt{I}_2 \times [0,\Theta_{\mathtt{Rq}}]^2, \\ 1 \le \mathtt{q} \le \mathtt{k}+1, \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} \mathtt{g_i}\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) > \frac{\Gamma_{\mathtt{rp}}}{2\mathtt{k}_1\mathtt{M}_1},\ \forall\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big) \in \mathtt{I}_1 \times \left[\Gamma_{\mathtt{rp}}, \frac{\Gamma_{\mathtt{rp}}}{\mathtt{k}_1}\right]^2, \\ 1 \le \mathtt{p} \le \mathtt{k}+1. \end{aligned}\right\} \tag{22}$$

According to a supposition, the BVP (1)-(4) has at least $2\mathtt{k}+1$ positive solutions $(\mathtt{u}_j, \mathtt{u}_j^*), j = 1,2,3,\cdots,2\mathtt{k}+1$ in $\mathtt{Q}(\beta,\Theta_{\mathtt{Rk}})$. Moreover, it follows from Theorem 3.1, (20), (21) and (22) that the system of BVP (1)-(4) has at least three positive solutions $(\mathtt{u}_1,\mathtt{u}_2)$, $(\widehat{\mathtt{u}}_1,\widehat{\mathtt{u}}_2)$ and $(\widetilde{\mathtt{u}}_1,\widetilde{\mathtt{u}}_2)$ in $\mathtt{Q}(\beta,\Theta_{\mathtt{Rk}})$ s.t. $\eta(\mathtt{u}_1,\mathtt{u}_2) \le \Lambda_{\ell\mathtt{k}+1}, \Gamma_{\mathtt{rk}+1} \le \alpha(\widehat{\mathtt{u}}_1,\widehat{\mathtt{u}}_2)$ with $\beta(\widehat{\mathtt{u}}_1,\widehat{\mathtt{u}}_2) \le \Theta_{\mathtt{Rk}+1}$ and $\Lambda_{\ell\mathtt{k}+1} < \eta(\widetilde{\mathtt{u}}_1,\widetilde{\mathtt{u}}_2)$ with $\alpha(\widetilde{\mathtt{u}}_1,\widetilde{\mathtt{u}}_2) < \Gamma_{\mathtt{rk}+1}$. Obviously, $(\widehat{\mathtt{u}}_1,\widehat{\mathtt{u}}_2)$ and $(\widetilde{\mathtt{u}}_1,\widetilde{\mathtt{u}}_2)$ are distinct from $(\mathtt{u}_j,\mathtt{u}_j^*), j = 1,2,\cdots,2\mathtt{k}+1$. Hence this result also holds true for $\mathtt{n} = \mathtt{k}+1$ because the system of BVP (1)-(4) has at least $2\mathtt{k}+3$ positive solutions in $\mathtt{Q}(\beta,\Theta_{\mathtt{Rk}+1})$. □

4. Example

Consider the system of BVP:

$$-\mathtt{u}_1'' + \frac{1}{4}\mathtt{u}_1 = \mathtt{g}_1\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big), \quad \mathtt{z} \in (0,1), \tag{23}$$

$$-\mathtt{u}_2'' + \frac{1}{9}\mathtt{u}_2 = \mathtt{g}_2\big(\mathtt{z},\mathtt{u}_1,\mathtt{u}_2\big), \quad \mathtt{z} \in (0,1), \tag{24}$$

$$\left.\begin{aligned} 3\mathtt{u}_1(0) - \mathtt{u}_1'(0) = 0, \\ \mathtt{u}_1(1) + 2\mathtt{u}_1'(1) = 0, \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} 4\mathtt{u}_2(0) - \mathtt{u}_2'(0) = 0, \\ 2\mathtt{u}_2(1) + 3\mathtt{u}_2'(1) = 0, \end{aligned}\right\} \tag{26}$$

where we take $\mathtt{b}_1 = 3, \mathtt{b}_2 = 1, \mathtt{b}_3 = 1, \mathtt{b}_4 = 2, \mathtt{d}_1 = 4, \mathtt{d}_2 = 1, \mathtt{d}_3 = 2, \mathtt{d}_4 = 3,\ \mathtt{r}_1 = \frac{1}{2}$, $\mathtt{r}_2 = \frac{1}{3}$, $\mathtt{I}_1 = [0,1], \mathtt{I}_2 = \left[\frac{1}{8}, \frac{7}{8}\right]$, $\mathtt{I}_3 = \left[\frac{1}{4}, \frac{3}{4}\right]$,

$$\mathtt{g}_1(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2) = \frac{79}{40}\sin^2(\mathtt{z}) + \frac{\mathtt{u}_1^{\frac{3}{2}}\mathtt{u}_2^2}{154} + \frac{\pi}{159},$$

$$\mathtt{g}_2(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2) = \frac{19\pi}{20}\cos^2(\mathtt{z}) + \frac{\mathtt{u}_1^2\mathtt{u}_2^{\frac{3}{2}}}{145} + \frac{\pi}{90}.$$

By simple calculation, we obtain

$$\Delta_1 = \frac{7\sqrt{e}}{4} \approx 2.8852622237, \Delta_2 = \frac{39e^{\frac{2}{3}} - 11}{18\sqrt[3]{e}} \approx 2.5859466756, \text{ and}$$

$$\begin{cases} \Phi_1(\mathtt{z}) = 3\sinh\left(\frac{\mathtt{z}}{2}\right) + \frac{1}{2}\cosh\left(\frac{\mathtt{z}}{2}\right), \\ \Phi_2(\mathtt{z}) = \sinh\left(\frac{1-\mathtt{z}}{2}\right) + \cosh\left(\frac{1-\mathtt{z}}{2}\right), \\ \Psi_1(\mathtt{z}) = 4\sinh\left(\frac{\mathtt{z}}{3}\right) + \frac{1}{3}\cosh\left(\frac{\mathtt{z}}{3}\right), \\ \Psi_2(\mathtt{z}) = 2\sinh\left(\frac{1-\mathtt{z}}{3}\right) + \cosh\left(\frac{1-\mathtt{z}}{3}\right), \end{cases}$$

$\mathtt{k}_1^* \approx 0.2350619429, \mathtt{k}_2^* \approx 0.3617745806, \mathtt{k}_3^* \approx 0.5215319025, \mathtt{k}_1^{**} \approx 0.1949105524,\ \mathtt{k}_2^{**} \approx 0.3268277639, \mathtt{k}_3^{**} \approx 0.4934161989$.

It follows from that $\mathtt{k}_1 = \min\left\{\mathtt{k}_1^*, \mathtt{k}_1^{**}\right\} \approx 0.1949105524, \mathtt{k}_2 = \min\left\{\mathtt{k}_2^*, \mathtt{k}_2^{**}\right\}$ $\approx 0.3268277639, \mathtt{k}_3 = \min\left\{\mathtt{k}_3^*, \mathtt{k}_3^{**}\right\} \approx 0.4934161989$, and

$$\begin{cases} \mathtt{M}_1 &= \max\limits_{\mathtt{z}\in\mathtt{I}_1}\left\{\int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})d\mathtt{s}, \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})d\mathtt{s}\right\} \\ &= \max\left\{0.95651222919, 0.8961486851\right\} \approx 0.95651222919, \\ \mathtt{M}_2 &= \max\limits_{\mathtt{z}\in\mathtt{I}_2}\left\{\int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})d\mathtt{s}, \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})d\mathtt{s}\right\} \\ &= \max\left\{0.8559160434, 0.8021967470\right\} \approx 0.8559160434, \\ \mathtt{M}_3 &= \max\limits_{\mathtt{z}\in\mathtt{I}_3}\left\{\int_0^1 \mathtt{G}_{\mathtt{r}_1}(\mathtt{z},\mathtt{s})d\mathtt{s}, \int_0^1 \mathtt{G}_{\mathtt{r}_2}(\mathtt{z},\mathtt{s})d\mathtt{s}\right\} \\ &= \max\left\{0.7586643681, 0.7096377130\right\} \approx 0.7586643681. \end{cases}$$

Choosing $\Lambda_\ell = 5, \Gamma_{\mathtt{r}} = \frac{20\pi}{9}, \Theta_{\mathtt{R}} = \frac{73\pi}{50}$ s.t. $\mathtt{g}_\mathtt{i}$ for $\mathtt{i} = \overline{1,2}$ satisfies:

- $\mathtt{g}_\mathtt{i}\left(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2\right) < \frac{\Lambda_\ell}{2\mathtt{M}_3} \approx 3.2952648168,\ \forall\left(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2\right) \in \mathtt{I}_3 \times \left[0, \frac{20\pi}{27}\right]^2$,
- $\mathtt{g}_\mathtt{i}\left(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2\right) < \frac{\Theta_\mathtt{R}}{2\mathtt{M}_2} \approx 2.6794247576,\ \forall\left(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2\right) \in \mathtt{I}_2 \times \left[0, \frac{73\pi}{50}\right]^2$,
- $\mathtt{g}_\mathtt{i}\left(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2\right) > \frac{\Gamma_\mathtt{r}}{2\mathtt{k}_1\mathtt{M}_1} \approx 18.7232606824,\ \forall\left(\mathtt{z}, \mathtt{u}_1, \mathtt{u}_2\right) \in \mathtt{I}_1 \times \left[\frac{20\pi}{9}, \frac{57\pi}{5}\right]^2$.

So, all conditions of Theorem 2.1 are satisfied. Hence the system of BVP (23)-(26) has at least three positive solutions that belong to $\mathtt{Q}\left(\beta, \frac{73\pi}{50}\right)$.

5. Conclusion

We have established the existence of multiple positive solutions for a coupled system of second-order undamped Sturm–Liouville boundary value problems using the Six functionals fixed point theorem.

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