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ON $\mathcal{I}_{\triangle}$-DEFERRED STATISTICALLY ROUGH CONVERGENCE WITH ORDER α

C. CHOUDHURY[1], M. KÜÇÜKASLAN[2*], §

Abstract. In this article, we introduce the concept of $\mathcal{I}_{\triangle}$-deferred statistically rough convergence of order α, $(0 < \alpha \leq 1)$ in a normed linear space. We mainly examine various properties such as closedness and convexity of the set of $\mathcal{I} - DS^{r;\alpha}_{p,q}(\triangle)-$limit for a sequence. Besides this, we prove a necessary and sufficient condition for the $\mathcal{I} - DS^{r;\alpha}_{p,q}(\triangle)-$convergence of a real valued sequence and derive some interesting implication relationships.

Keywords: Deferred statistical convergence, ideal, $\mathcal{I}-$convergence, difference of sequence, rough convergence.

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1. Introduction

The notion of the usual convergence of sequences does not always capture the fine details of the properties of a vast class of sequences that are not convergent but almost all of its terms in some sense have the properties of a convergent sequence. This leads mathematicians to generalize the idea of usual convergence to various types of convergences. In 1951, Fast [24] and Steinhaus [44] extended the concept of usual convergence to statistical convergence independently by involving the concept of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is a real number $d(A) \in [0,1]$ defined as if the limit exists

$$d(A) := \lim_{k\to\infty} \frac{|\{a \leq k : a \in A\}|}{k}$$

where the vertical bar denotes the number of elements in the set $\{a \leq k : a \in A\}$. It is clear that if A is finite then $d(A) = 0$ and $d(\mathbb{N} \setminus A) = 1 - d(A)$. A real valued sequence $x = (x_k)$ is said to be statistically convergent [26] to a number l if for every $\varepsilon > 0$, the

[1] Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India.
e-mail: chiranjibchoudhury123@gmail.com; ORCID: https://orcid.org/0000-0002-5607-9884.
[2] Department of Mathematics, Mersin University, Mersin, Turkey.
e-mail: mkucukaslan@mersin.edu.tr; ORCID: https://orcid.org/0000-0002-3183-3123.
* Corresponding author.
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natural density of the set of all k's for which the corresponding sequential term x_k lies outside the interval $(l-\varepsilon, l+\varepsilon)$ is zero. In other words for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |x_k - l| \geq \varepsilon\}\right| = 0.$$

Statistical convergence becomes one of the most active areas of research after the works of Fridy [25], Šalát [40], Tripathy [46, 47], and many others. For some most recent works on statistical convergence, [9, 11, 12, 13, 36, 42] can be addressed, where many more references can be found. In 1932, Agnew [1] generalized Cesàro mean to deferred Cesàro mean to obtain a more useful method having stronger features. Later on, Küçükaslan and Yilmaztürk [31] utilize this to introduce the notion of deferred statistical convergence mainly as a natural generalization of statistical convergence, $\lambda-$statistical convergence, lacunary statistical convergence, etc.

Let $p = (p(n))$ and $q = (q(n))$ are the sequence of non-negative integers which satisfies

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty.$$

A real valued sequence $x = (x_k)$ is said to be deferred statistical convergent to a real number l if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left|\{p(n) < k \leq q(n) : |x_k - l| \geq \varepsilon\}\right| = 0.$$

Clearly, for $p(n) = 0$ and $q(n) = n$, the above definition reduces to the definition of statistical convergence. For more details related to deferred statistical convergence and its generalizations, [2, 20, 28, 45] can be addressed where many more references can be found.

Kizmaz [27] in 1981 defined the concept of difference sequence for a real valued sequence $x = (x_k)$, by $\triangle x := (\triangle x_k) = (x_k - x_{k+1}), k \in \mathbb{N}$ and studied the inclusive relationships of the sequence spaces $c_0(\triangle) := \{x = (x_k) : \triangle x \in c_0\}$, $c(\triangle) := \{x = (x_k) : \triangle x \in c\}$ and $l_\infty(\triangle) := \{x = (x_k) : \triangle x \in l_\infty\}$, where c_0, c and l_∞ are well known spaces of all real valued null, convergent, and bounded sequences, respectively. After this study, which can be considered as a base for difference sequences, Aydın and Basar [4], Bektas et al. [8], Et [18], Et and Çolak [21], Et and Basarir [19], Et and Esi [22], Et and Nuray [23], and many others [33, 34, 35] investigated various properties of this concept.

On the other hand, in 2001, the idea of $\mathcal{I}$ and $\mathcal{I}^*-$convergence was developed by Kostyrko et al. [30] mainly as a generalization of statistical convergence. They proved that many other known notions of convergence were a particular type of $\mathcal{I}-$convergence by considering particular ideals. Consequently, this direction successively gets more attention from the researchers and became one of the most active areas of research. Generalizing the notion of deferred statistical convergence and $\mathcal{I}-$convergence, in 2019, Şengül et al. [43] introduced the notion of $\mathcal{I}-$ deferred statistical convergence of order α, $(0 < \alpha \leq 1)$. For more information on $\mathcal{I}-$convergence and its related generalizations, one may refer to Debnath and Rakshit [14], Demirci [16], Kostyrko et al. [29], Tripathy and Hazarika [48, 49, 50], Tripathy et al. [51, 53], and Tripathy and Sen [52] where many more references can be found.

The idea of rough convergence was first introduced by H. X. Phu [38] in finite dimensional normed spaces. Phu mainly showed that the rough limit set $LIM^r x$ (where $r \geq 0$ is the degree of roughness) is bounded, closed, and convex and studied several fundamental properties of this interesting concept. Moreover in [39], Phu also investigated similar concepts for infinite dimensional normed spaces. Later on, in 2008, Aytar [5] extended it to rough statistical convergence. In 2013, Pal et al. [37] and in 2014, Dündar and Çakan

[17] independently introduced rough ideal convergence. Recently, Demir and Gümüş [15] introduced and investigated the notion of rough statistical convergence for difference sequences. For more details on rough convergence and its current generalizations, one may refer to [3, 6, 10, 32, 41].

2. Some Definitions

In this section, we present some definitions which are crucial for the results of this paper.

Definition 2.1. [30] *Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on the set X, if (i) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, and (ii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.*

Definition 2.2. [30] *Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on the set X, if (i) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and (ii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.*

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F}(\mathcal{I}) := \{X \setminus A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$.

A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

Definition 2.3. [37] *Let r be a non-negative real number. A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is said to be rough ideal convergent to $x_0 \in X$, if for every $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : \|x_k - x_0\| \geq r + \varepsilon\} \in \mathcal{I}.$$

If we choose $\mathcal{I} = \mathcal{I}_f$, the ideal consisting of all finite subsets of $\mathbb{N}$, the Definition 2.3 coincides with the definition of rough convergence given in [38] and if we choose $\mathcal{I} = \mathcal{I}_d$, the ideal consisting of all subsets of $\mathbb{N}$ having natural density zero, Definition 2.3 turns to the definition of rough statistical convergence studied in [5]. Furthermore, if we fix $r = 0$, then the choices $\mathcal{I} = \mathcal{I}_f$ and $\mathcal{I} = \mathcal{I}_d$, reduce Definition 2.3 to the definition of usual convergence and statistical convergence [26], respectively. If we choose $r = 0$, then Definition 2.3 gives us the definition of ideal convergence given in [30].

Remark 2.1. [38] *A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is bounded if and only if $LIM^r x \neq \emptyset$ for some $r \geq 0$.*

Definition 2.4. [7] *Let $x = (x_k)$ be a real valued sequence. Then, $x = (x_k)$ is said to be $\triangle-$statistically convergent to a real number l if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |\triangle x_k - l| \geq \varepsilon\}| = 0,$$

holds.

Definition 2.5. [43] *A real valued sequence $x = (x_k)$ is said to be $\mathcal{I}-$deferred statistically convergent of order $\alpha (0 < \alpha \leq 1)$ to a real number l (in short $\mathcal{I} - DS^{\alpha}_{p,q}-$convergent) if for every $\varepsilon, \delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} |\{p(n) < k \leq q(n) : |x_k - l| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

3. Main Results

In this section, we are ready to present the main results of the paper. Throughout the paper, $\mathcal{I}$ will denote a non-trivial admissible ideal in $\mathbb{N}$ and p, q will denote sequences of natural numbers satisfying $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$..

Definition 3.1. *Let $x = (x_k)$ be a sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non-negative real number. Then, x is said to be $\mathcal{I}_{\triangle}$-deferred statistically rough convergent to $x_0 \in X$ (in short $\mathcal{I} - DS^r_{p,q}(\triangle)-$convergent) if for every $\varepsilon, \delta > 0$,*

$$\left\{n \in \mathbb{N} : \tfrac{1}{q(n)-p(n)} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, x_0 is said to be $\mathcal{I}-DS^r_{p,q}(\triangle)$ limit of the sequence $x = (x_k)$ and it is denoted by $x_k \xrightarrow{\mathcal{I}-DS^r_{p,q}(\triangle)} x_0$. For $r \geq 0$, the set of all $\mathcal{I} - DS^r_{p,q}(\triangle)-$convergent sequences with roughness degree r is denoted by $\mathcal{I} - DS^r_{p,q}(\triangle)$.

Definition 3.2. *Let $x = (x_k)$ be a sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non-negative real number. Then, x is said to be $\mathcal{I}_{\triangle}-$deferred statistically rough convergent of order $\alpha (0 < \alpha \leq 1)$ to $x_0 \in X$ (in short $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)-$convergent) if for every $\varepsilon, \delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, x_0 is said to be the $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)$ limit of the sequence $x = (x_k)$ and it is denoted by $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$. For $r \geq 0$, the set of all $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)-$convergent sequences with roughness degree r is denoted by $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)$.

Remark 3.1. *For $r = 0$, Definition 3.1 and Definition 3.2 reduces to the definition of $\mathcal{I} - DS_{p,q}(\triangle)-$convergence, and $\mathcal{I} - DS^{\alpha}_{p,q}(\triangle)-$convergence respectively, and in both the cases, the limit is unique.*

Remark 3.2. *For $r > 0$, the $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)-$limit of a sequence is not unique.*

So our main interest is to deal with the case $r > 0$. Therefore, we construct $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)-$limit set of a sequence $x = (x_k)$ denoted and defined as follows:

$$\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) := \left\{x_0 \in X : x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0\right\}.$$

For $\alpha = 1$, the above limit set reduces to

$$\mathcal{I} - LIM_x(DS^{r}_{p,q}(\triangle)) := \left\{x_0 \in X : x_k \xrightarrow{\mathcal{I}-DS^{r}_{p,q}(\triangle)} x_0\right\}.$$

It is obvious that a sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is $\mathcal{I} - DS^r_{p,q}(\triangle)-$convergent or $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)-$convergent according to as

$$\mathcal{I} - LIM_x(DS^{r}_{p,q}(\triangle)) \neq \emptyset$$

and

$$\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) \neq \emptyset$$

for some $r \geq 0$.

Theorem 3.1. *Let r be a non-negative real number. Then, $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) \subset \mathcal{I} - LIM_x(DS^{r}_{p,q}(\triangle))$ for $0 < \alpha \leq 1$.*

Proof. Let $x = (x_k)$ be a sequence in a normed linear space $(X, \|\cdot\|)$ such that $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$ holds. Then, by definition, for every $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : \|\triangle x_k - x_0\| \ge r+\varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}.$$

Since $0 < \alpha \le 1$, then the following inclusion

$$\left\{n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left|\{p(n) < k \le q(n) : \|\triangle x_k - x_0\| \ge r+\varepsilon\}\right| \ge \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : \|\triangle x_k - x_0\| \ge r+\varepsilon\}\right| \ge \delta\right\}$$

holds. Hence, the result is obtained. □

Theorem 3.2. *Let $x = (x_k)$ be a sequence in a normed linear space $(X, \|\cdot\|)$. Then,*
$$diam(\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))) = \sup\{\|y-z\| : y,z \in \mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))\} \le 2r.$$

Proof. Let us assume that $diam(\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))) > 2r$ holds. Then, there exists $y_0, z_0 \in \mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ such that $\|y_0 - z_0\| > 2r$ must be hold. Choose $\varepsilon > 0$ in such a manner that $\varepsilon < \frac{\|y_0 - z_0\|}{2} - r$. Suppose $A = \{k \in \mathbb{N} : \|\triangle x_k - y_0\| \ge r+\varepsilon\}$ and $B = \{k \in \mathbb{N} : \|\triangle x_k - z_0\| \ge r+\varepsilon\}$. Then, the following inequality

$$\frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : k \in A \cup B\}\right|$$
$$\le \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : k \in A\}\right|$$
$$+ \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : k \in B\}\right|$$

holds and so we have

$$\mathcal{I} - \lim_{n\to\infty} \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : k \in A \cup B\}\right|$$
$$\le \mathcal{I} - \lim_{n\to\infty} \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : k \in A\}\right|$$
$$+ \mathcal{I} - \lim_{n\to\infty} \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : k \in B\}\right|.$$

By assumptions, the right-hand side of the above inequality vanishes and so is the left-hand side. Thus, for any $\delta > 0$,

$$K(\delta) = \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \le q(n) : k \in A \cup B\}\right| \ge \delta\right\} \in \mathcal{I}.$$

Let $n_0 \in \mathbb{N} \setminus K(\frac{1}{2})$. Then, we have

$$\frac{1}{(q(n_0)-p(n_0))^{\alpha}} \left|\{p(n_0) < k \le q(n_0) : k \in A \cup B\}\right| < \frac{1}{2}.$$

Also,

$$\frac{1}{(q(n_0)-p(n_0))^{\alpha}} \left|\{p(n_0) < k \le q(n_0) : k \notin A \cup B\}\right| \ge 1 - \frac{1}{2} = \frac{1}{2}.$$

This implies the set $\{k \in \mathbb{N} : k \notin A \cup B\}$ is nonempty. Let $k_0 \in \mathbb{N}$ be such that $k_0 \notin A \cup B$. Then, $k_0 \in (\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B)$. Then,

$$\|\triangle x_{k_0} - y_0\| < r+\varepsilon \text{ and } \|\triangle x_{k_0} - z_0\| < r+\varepsilon.$$

Consequently,

$$\|y_0 - z_0\| \le \|\triangle x_{k_0} - y_0\| + \|\triangle x_{k_0} - z_0\| < 2(r+\varepsilon) < \|y_0 - z_0\|,$$

which is a contradiction. Hence, $diam(\mathcal{I}-LIM_x(DS^{r,\alpha}_{p,q}(\triangle)))\leq 2r$. □

Theorem 3.3. *Let $x=(x_k)$ and $y=(y_k)$ be two sequences in the normed linear space $(X,\|\cdot\|)$. Then, for a given $r\geq 0$ the following holds:*
(i) If $x_0\in\mathcal{I}-LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ then $\lambda x_0\in\mathcal{I}-LIM_{\lambda x}(DS^{r,\alpha}_{p,q}(\triangle))$, for any $\lambda\in\mathbb{R}$ with $|\lambda|\geq 1$.
(ii) If $x_0\in\mathcal{I}-LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ and $y_0\in\mathcal{I}-LIM_y(DS^{r,\alpha}_{p,q}(\triangle))$, then $x_0+y_0\in\mathcal{I}-LIM_{x+y}(DS^{r,\alpha}_{p,q}(\triangle))$.

Proof. (i) Since $x_k\xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)}x_0$, so for any given $\varepsilon,\delta>0$, following set

$$\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^\alpha}\left|\left\{p(n)<k\leq q(n):\|\triangle x_k-x_0\|\geq\frac{r+\varepsilon}{|\lambda|}\right\}\right|\geq\delta\right\} \tag{1}$$

belongs to $\mathcal{I}$. Now, we have

$$\begin{aligned}&\frac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\lambda\triangle x_k-\lambda x_0\|\geq r+\varepsilon\}\right|\\&=\frac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):|\lambda|\cdot\|\triangle x_k-x_0\|\geq r+\varepsilon\}\right|\\&\leq\frac{1}{(q(n)-p(n))^\alpha}\left|\left\{p(n)<k\leq q(n):\|\triangle x_k-x_0\|\geq\frac{r+\varepsilon}{|\lambda|}\right\}\right|.\end{aligned}$$

From the above inequation, the inclusion

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^\alpha}\left|\left\{p(n)<k\leq q(n):\|\triangle x_k-x_0\|\geq\frac{r+\varepsilon}{|\lambda|}\right\}\right|<\delta\right\}\\&\subset\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\lambda\triangle x_k-\lambda x_0\|\geq r+\varepsilon\}\right|<\delta\right\}\end{aligned}$$

holds and consequently, from (1) we have,

$$\left\{n\in\mathbb{N}:\tfrac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\lambda\triangle x_k-\lambda x_0\|\geq r+\varepsilon\}\right|\geq\delta\right\}\in\mathcal{I}.$$

Hence, $\lambda x_k\xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)}\lambda x_0$.

(ii) Since $x_k\xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)}x_0$ and $y_k\xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)}y_0$ holds, so for given $\varepsilon,\delta>0$, we have $A,B\in\mathcal{I}$, where

$$A=\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^\alpha}\left|\left\{p(n)<k\leq q(n):\|\triangle x_k-x_0\|\geq\frac{r+\varepsilon}{2}\right\}\right|\geq\frac{\delta}{2}\right\}$$

and

$$B=\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^\alpha}\left|\left\{p(n)<k\leq q(n):\|\triangle y_k-y_0\|\geq\frac{r+\varepsilon}{2}\right\}\right|\geq\frac{\delta}{2}\right\}.$$

Let $M=(\mathbb{N}\setminus A)\cap(\mathbb{N}\setminus B)$. Then, $M\in\mathcal{F}(\mathcal{I})$ and so $M\neq\emptyset$. Let us consider an element $m\in M$. Then, we have,

$$\frac{1}{(q(m)-p(m))^\alpha}\left|\left\{p(m)<k\leq q(m):\|\triangle x_k-x_0\|\geq\frac{r+\varepsilon}{2}\right\}\right|<\frac{\delta}{2}$$

and

$$\frac{1}{(q(m)-p(m))^\alpha}\left|\left\{p(m)<k\leq q(m):\|\triangle y_k-y_0\|\geq\frac{r+\varepsilon}{2}\right\}\right|<\frac{\delta}{2}.$$

Now as the inclusion

$$\{p(m) < k \le q(m) : \|\triangle(x_k + y_k) - (x_0 + y_0)\| \ge r + \varepsilon\} \subseteq \left\{p(m) < k \le q(m) : \|\triangle x_k - x_0\| \ge \frac{r+\varepsilon}{2}\right\} \cup \left\{p(m) < k \le q(m) : \|\triangle y_k - y_0\| \ge \frac{r+\varepsilon}{2}\right\}$$

holds, so we must have

$$\frac{1}{(q(m)-p(m))^\alpha} |\{p(m) < k \le q(m) : \|\triangle(x_k + y_k) - (x_0 + y_0)\| \ge r + \varepsilon\}| \le \frac{1}{(q(m)-p(m))^\alpha} \left|\left\{p(m) < k \le q(m) : \|\triangle x_k - x_0\| \ge \frac{r+\varepsilon}{2}\right\}\right| + \frac{1}{(q(m)-p(m))^\alpha} \left|\left\{p(m) < k \le q(m) : \|\triangle y_k - y_0\| \ge \frac{r+\varepsilon}{2}\right\}\right| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

From the above inequality, we get

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^\alpha} |\{p(n) < k \le q(n) : \|\triangle(x_k + y_k) - (x_0 + y_0)\| \ge r + \varepsilon\}| \ge \delta\right\}$$

is the subset of $\mathbb{N} \setminus M$. Since $A, B \in \mathcal{I}$, so $\mathbb{N} \setminus M = A \cup B \in \mathcal{I}$ and consequently,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^\alpha} |\{p(n) < k \le q(n) : \|\triangle(x_k + y_k) - (x_0 + y_0)\| \ge r + \varepsilon\}| \ge \delta\right\} \in \mathcal{I}$$

and the proof ends. □

Theorem 3.4. *Let $0 < \alpha \le \beta \le 1$, then $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) \subset \mathcal{I} - LIM_x(DS^{r,\beta}_{p,q}(\triangle))$ holds for a fixed $r \ge 0$.*

Proof. Let $x = (x_k)$ be a sequence in a normed linear space $(X, \|\cdot\|)$ such that $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$. Then, the result follows from the following inclusion

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^\beta} |\{p(n) < k \le q(n) : \|\triangle x_k - x_0\| \ge r + \varepsilon\}| \ge \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^\alpha} |\{p(n) < k \le q(n) : \|\triangle x_k - x_0\| \ge r + \varepsilon\}| \ge \delta\right\}.$$

□

Definition 3.3. *A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is said to be $\mathcal{I}_\triangle$-deferred statistically bounded of order α $(0 < \alpha \le 1)$ (in short $\mathcal{I} - DS^\alpha_{p,q}(\triangle)-$bounded) if there exists a real number $M > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^\alpha} |\{p(n) < k \le q(n) : \|\triangle x_k\| \ge M\}| \ge \delta\right\} \in \mathcal{I}.$$

If we consider the case $\alpha = 1$ in Definition 3.7, then the sequence $x = (x_k)$ is called $\mathcal{I}_\triangle$- deferred statistically bounded (in short $\mathcal{I} - DS_{p,q}(\triangle)-$bounded). In view of the definitions, we formulate the following result without proof.

Theorem 3.5. *Every $\mathcal{I} - DS^\alpha_{p,q}(\triangle)-$bounded sequence is also $\mathcal{I} - DS_{p,q}(\triangle)-$bounded for $0 < \alpha \le 1$.*

Theorem 3.6. *A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is $\mathcal{I}_{\triangle}$-deferred statistically bounded of order α $(0 < \alpha \leq 1)$ if and only if there exists some $r \geq 0$ such that $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) \neq \emptyset$.*

Proof. Firstly, let us assume that $x = (x_k)$ be a $\mathcal{I}(\triangle)-$deferred statistically bounded sequence of order $\alpha(0 < \alpha \leq 1)$ in X. Then,

$$A := \left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k\| \geq M\}\right| \geq \delta\right\} \in \mathcal{I}$$

for some $M > 0$. Suppose $r' = \sup\{\|x_k\| : p(m) < k \leq q(m), m \in \mathbb{N} \setminus A\}$. Then, the set $\mathcal{I} - LIM_x(DS^{r',\alpha}_{p,q}(\triangle))$ contains the origin of X so that $\mathcal{I} - LIM_x(DS^{r',\alpha}_{p,q}(\triangle)) \neq \emptyset$.

Conversely, assume that there is a $r \geq 0$ such that $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) \neq \emptyset$. Let $x_0 \in \mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$. Then, for every $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Fix $\varepsilon = \|x_0\|$ and let $M = r + \|x_0\|$. Then, for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq M\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Hence, $x = (x_k)$ is $\mathcal{I}(\triangle)-$deferred statistically bounded of order $\alpha(0 < \alpha \leq 1)$. □

Theorem 3.7. *Let $x = (x_k)$ be a sequence in a normed linear space $(X, \|\cdot\|)$. Then, the set $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ is convex for some $r \geq 0$.*

Proof. Let $y_0, y_1 \in \mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ and $\varepsilon > 0$ be given. Suppose $A_0 = \{k \in \mathbb{N} : \|\triangle x_k - y_0\| \geq r + \varepsilon\}$ and $A_1 = \{k \in \mathbb{N} : \|\triangle x_k - y_1\| \geq r + \varepsilon\}$. Then, by Theorem 3.2, for any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : k \in A_0 \cup A_1\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Let

$$A := \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : k \in A_0 \cup A_1\}\right| \geq 1 - \delta_1\right\}$$

Then, $A \in \mathcal{I}$ and for every $n \notin A$ we have

$$\frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : k \in A_0 \cup A_1\}\right| < 1 - \delta_1$$
$$\Rightarrow \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : k \notin A_0 \cup A_1\}\right| \geq 1 - (1 - \delta_1) = \delta_1.$$

Therefore, the set $\{k \in \mathbb{N} : k \notin A_0 \cup A_1\}$ is nonempty. Suppose $k_0 \in (\mathbb{N} \setminus A_0) \cap (\mathbb{N} \setminus A_1)$. Then, for any $0 \leq \lambda \leq 1$,

$$\begin{aligned}\|\triangle x_{k_0} - (1 - \lambda)y_0 - \lambda y_1\| &\leq (1 - \lambda)\|\triangle x_{k_0} - y_0\| + \lambda\|\triangle x_{k_0} - y_1\| \\ &< (1 - \lambda)(r + \varepsilon) + \lambda(r + \varepsilon) = r + \varepsilon.\end{aligned}$$

Let $B = \{k \in \mathbb{N} : \|\triangle x_k - ((1 - \lambda)y_0 + \lambda y_1)\| \geq r + \varepsilon\}$. Then, clearly

$$(\mathbb{N} \setminus A_0) \cap (\mathbb{N} \setminus A_1) \subseteq \mathbb{N} \setminus B.$$

So, for $n \notin A$,

$$\delta_1 \leq \frac{1}{(q(n)-p(n))^\alpha} \left|\{p(n) < k \leq q(n) : k \notin A_0 \cup A_1\}\right| \leq \frac{1}{(q(n)-p(n))^\alpha} \left|\{p(n) < k \leq q(n) : k \notin B\}\right|.$$

In other words,

$$\frac{1}{(q(n)-p(n))^\alpha} \left|\{p(n) < k \leq q(n) : k \in B\}\right| < 1-\delta_1 < \delta.$$

This implies that

$$\mathbb{N} \setminus A \subseteq \Big\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^\alpha} \left|\{p(n) < k \leq q(n) : k \in B\}\right| < \delta\Big\}.$$

Now by taking complements on both sides of the above inclusion and using the fact that $A \in \mathcal{I}$, we obtain

$$\Big\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^\alpha} \left|\{p(n) < k \leq q(n) : k \in B\}\right| \geq \delta\Big\} \in \mathcal{I}.$$

This completes the proof. □

Theorem 3.8. *Let $x = (x_k)$ be a sequence in a normed linear space $(X, \|\cdot\|)$. Then, the set $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ is closed for some $r \geq 0$.*

Proof. If $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) = \emptyset$, then there is nothing to prove. So let us assume that

$$\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle)) \neq \emptyset.$$

Consider a sequence (y_k) in $\mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ such that

$$\lim_{k\to\infty} y_k = y.$$

Choose $\varepsilon > 0$ and $\delta > 0$. Then, there exists $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\|y_k - y\| < \tfrac{\varepsilon}{2} \text{ for all } k > i_{\frac{\varepsilon}{2}}.$$

Let $k_0 > i_{\frac{\varepsilon}{2}}$. Then, $y_{k_0} \in \mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ and consequently

$$A = \Big\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^\alpha} \left|\left\{p(n) < k \leq q(n) : \|\triangle x_k - y_{k_0}\| \geq r + \tfrac{\varepsilon}{2}\right\}\right| \geq \delta\Big\} \in \mathcal{I}.$$

Let M denotes the set $\mathbb{N} \setminus A$. Then, $M \in \mathcal{F}(\mathcal{I})$ and so is non-empty. Take $n \in M$. Then, we have,

$$\frac{1}{(q(n)-p(n))^\alpha} \left|\left\{p(n) < k \leq q(n) : \|\triangle x_k - y_{k_0}\| \geq r + \frac{\varepsilon}{2}\right\}\right| < \delta$$

$$\Rightarrow \frac{1}{(q(n)-p(n))^\alpha} \left|\left\{p(n) < k \leq q(n) : \|\triangle x_k - y_{k_0}\| < r + \frac{\varepsilon}{2}\right\}\right| \geq 1-\delta.$$

Let $B_n = \left\{p(n) < k \leq q(n) : \|\triangle x_k - y_{k_0}\| < r + \frac{\varepsilon}{2}\right\}$. Then, for $k \in B_n$,

$$\|\triangle x_k - y\| \leq \|\triangle x_k - y_{k_0}\| + \|y_{k_0} - y\| < r + \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = r + \varepsilon.$$

Hence, $B_n \subseteq \{p(n) < k \leq q(n) : \|\triangle x_k - y\| < r + \varepsilon\}$ which implies that

$$1-\delta \leq \tfrac{|B_n|}{(q(n)-p(n))^\alpha} \leq \tfrac{1}{(q(n)-p(n))^\alpha} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - y\| < r + \varepsilon\}\right|.$$

Therefore,

$$\tfrac{1}{(q(n)-p(n))^\alpha} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - y\| \geq r + \varepsilon\}\right| < 1 - (1-\delta) = \delta.$$

This implies,

$$\left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - y\| \geq r + \varepsilon\}\right| \geq \delta\right\} \subseteq A \in \mathcal{I}.$$

Hence, $y \in \mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ and this completes the proof. □

Theorem 3.9. *A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)-$convergent to $x_0 \in X$ with roughness degree $r \geq 0$ if and only if there exists a sequence $y = (y_k)$ in X which is $\mathcal{I} - DS^{\alpha}_{p,q}(\triangle)-$convergent to x_0 and $\|\triangle x_k - \triangle y_k\| \leq r$ for all $k \in \mathbb{N}$.*

Proof. Let $y = (y_k)$ be a sequence in X such that $\|\triangle x_k - \triangle y_k\| \leq r$ for all $k \in \mathbb{N}$ and $y_k \xrightarrow{\mathcal{I}-DS^{\alpha}_{p,q}(\triangle)} x_0$.

Then, by definition for every $\varepsilon > 0$ and $\delta > 0$,

$$A = \left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle y_k - x_0\| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Let $n \notin A$. Then, we have

$$\tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle y_k - x_0\| \geq \varepsilon\}\right| < \delta$$

which further gives

$$\frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle y_k - x_0\| < \varepsilon\}\right| \geq 1 - \delta. \tag{2}$$

Let for $n \in \mathbb{N}$, A_n denotes the set $\{p(n) < k \leq q(n) : \|\triangle y_k - x_0\| < \varepsilon\}$.

Then, for $k \in A_n$,

$$\|\triangle x_k - x_0\| \leq \|\triangle x_k - \triangle y_k\| + \|\triangle y_k - x_0\| < r + \varepsilon.$$

This proves that the inclusion

$$A_n \subseteq \{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| < r + \varepsilon\}$$

holds and eventually from (2) we obtain

$$\frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| < r + \varepsilon\}\right| \geq \frac{|A_n|}{(q(n)-p(n))^{\alpha}} \geq 1 - \delta$$

That is;

$$\frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}\right| < 1 - (1 - \delta) = \delta.$$

Thus,

$$\left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}\right| \geq \delta\right\} \subseteq A \in \mathcal{I}.$$

Hence, $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$.

For the converse part, let us assume that $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$. Then, by definition for every $\varepsilon, \delta > 0$,

$$B = \left\{n \in \mathbb{N} : \tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Let $n \notin B$. Then, we have

$$\tfrac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}\right| < \delta$$

which further gives

$$\frac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\triangle x_k-x_0\|<r+\varepsilon\}\right|\geq 1-\delta. \tag{3}$$

Suppose for $n\in\mathbb{N}$, B_n denotes the set $\{p(n)<k\leq q(n):\|\triangle x_k-x_0\|<r+\varepsilon\}$.

Define a sequence $y=(y_k)$ by

$$\triangle y_k=\begin{cases}x_0, & if\ \|\triangle x_k-x_0\|\leq r\\ \triangle x_k+r\frac{x_0-\triangle x_k}{\|\triangle x_k-x_0\|}, & otherwise.\end{cases}$$

Then, it is easy to observe that $\|\triangle x_k-\triangle y_k\|\leq r$ for all $k\in\mathbb{N}$. Moreover, $\|\triangle y_k-x_0\|=0$, whenever $\|\triangle x_k-x_0\|\leq r$ and $\|\triangle y_k-x_0\|<\varepsilon$, whenever $r<\|\triangle x_k-x_0\|<r+\varepsilon$.

The above fact gives the following inclusion

$$B_n\subseteq\{p(n)<k\leq q(n):\|\triangle y_k-x_0\|<\varepsilon\}$$

holds and subsequently from (3), we have

$$\frac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\triangle y_k-x_0\|<\varepsilon\}\right|\geq\frac{|B_n|}{(q(n)-p(n))^\alpha}\geq 1-\delta$$

i.e.,

$$\frac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\triangle y_k-x_0\|\geq\varepsilon\}\right|<\delta.$$

Thus,

$$\left\{n\in\mathbb{N}:\tfrac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\triangle y_k-x_0\|\geq\varepsilon\}\right|\geq\delta\right\}\subseteq B\in\mathcal{I}.$$

Hence, $y_k\xrightarrow{\mathcal{I}-DS^\alpha_{p,q}(\triangle)}x_0$. This completes the proof. □

Definition 3.4. *A point γ is said to be $\mathcal{I}(\triangle)-$deferred statistically cluster point of order α, if*

$$\left\{n\in\mathbb{N}:\tfrac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\triangle x_k-\gamma\|\geq\varepsilon\}\right|<\delta\right\}\notin\mathcal{I}.$$

For any sequence $x=(x_k)$, the set of all $\mathcal{I}(\triangle)-$deferred statistically cluster point of order α is denoted by $\Gamma_x\left(\mathcal{I}-DS^\alpha_{p,q}(\triangle)\right)$.

Theorem 3.10. *Let $x=(x_k)$ be a sequence in a normed linear space $(X,\|\cdot\|)$. Then, for an arbitrary $\gamma\in\Gamma_x\left(\mathcal{I}-DS^\alpha_{p,q}(\triangle)\right)$ and $r>0$, $\|x_0-\gamma\|<r$ for all $x_0\in\mathcal{I}-LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$.*

Proof. If possible let us assume that for $\gamma\in\Gamma_x\left(\mathcal{I}-DS^\alpha_{p,q}(\triangle)\right)$ there exists a $x_0\in\mathcal{I}-LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$ satisfying $\|x_0-\gamma\|\geq r$. Fix $\varepsilon=\frac{\|x_0-\gamma\|-r}{2}$. Then, we have for any $\delta>0$,

$$A=\left\{n\in\mathbb{N}:\tfrac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\triangle x_k-\gamma\|\geq\varepsilon\}\right|<\delta\right\}\notin\mathcal{I}.$$

Suppose

$$B=\left\{n\in\mathbb{N}:\tfrac{1}{(q(n)-p(n))^\alpha}\left|\{p(n)<k\leq q(n):\|\triangle x_k-x_0\|\geq r+\varepsilon\}\right|\geq\delta\right\}.$$

Let $m\in A$ be such that for $p(m)<k\leq q(m)$, $\|\triangle x_k-\gamma\|<\varepsilon$. Then, we have,

$$\|\triangle x_k-x_0\|\geq\|x_0-\gamma\|-\|\triangle x_k-\gamma\|>r+\varepsilon.$$

This implies that $A \subseteq B$ and consequently, $B \notin \mathcal{I}$, which contradicts to the fact that $x_0 \in \mathcal{I} - LIM_x(DS^{r,\alpha}_{p,q}(\triangle))$. This completes the proof. □

Theorem 3.11. *Let $t = (t(n))$ and $s = (s(n))$ be two sequences of positive integers such that $p(n) \leq t(n) < s(n) \leq q(n)$ holds for all $n \in \mathbb{N}$. Then, for a fixed $r \geq 0$,*

(i) $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{t,s}(\triangle)} x_0$ implies $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$, provided that both the sets $\{k \in \mathbb{N} : p(n) < k \leq t(n)\}$ and $\{k \in \mathbb{N} : s(n) < k \leq q(n)\}$ are finite for all $n \in \mathbb{N}$.

(ii) $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$ implies $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{t,s}(\triangle)} x_0$, provided that

$$\mathcal{I} - \liminf_{n\to\infty} \left(\frac{s(n)-t(n)}{q(n)-p(n)}\right)^{\alpha} > 0.$$

Proof. The proof is parallel to that of Theorem 2.3.1 and Theorem 2.3.2 in [31], so omitted. □

Theorem 3.12. *Let $t = (t(n))$ and $s = (s(n))$ be two sequences of positive integers such that $p(n) \leq t(n) < s(n) < q(n)$ and $\{k \in \mathbb{N} : t(n) < k \leq s(n)\}$ is a finite set for all $n \in \mathbb{N}$. If $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,t}(\triangle)} x_0$ and $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{s,q}(\triangle)} x_0$ holds for a fixed $r \geq 0$, simultaneously, then $x_k \xrightarrow{\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)} x_0$.*

Proof. For any $\varepsilon > 0$ and a fixed $r \geq 0$, the following inclusion

$$\begin{aligned}
&\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\} \\
&\qquad = \{p(n) < k \leq t(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\} \\
&\qquad\qquad \cup \{t(n) < k \leq s(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\} \\
&\qquad\qquad\qquad \cup \{s(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}
\end{aligned}$$

and related inequality

$$\begin{aligned}
&\frac{|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}|}{(q(n) - p(n))^{\alpha}} \\
&\qquad \leq \frac{|\{p(n) < k \leq t(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}|}{(t(n) - p(n))^{\alpha}} \\
&\qquad\qquad + \frac{|\{t(n) < k \leq s(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}|}{(s(n) - t(n))^{\alpha}} \\
&\qquad\qquad\qquad + \frac{|\{s(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}|}{(q(n) - s(n))^{\alpha}}
\end{aligned}$$

holds. From the assumption of the theorem, the $\mathcal{I}-$limit of the right-hand side of the above inequality is zero. Therefore,

$$\mathcal{I} - \lim_{n\to\infty} \frac{|\{p(n) < k \leq q(n) : \|\triangle x_k - x_0\| \geq r + \varepsilon\}|}{(q(n) - p(n))^{\alpha}} = 0.$$

Therefore, the proof of the theorem is ended. □

4. Conclusions

In this study, the concept of $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)-$convergence is introduced in a normed linear space setting. Several interesting properties and implication relationships have been presented and proved whenever one varies the parameter α $(0 < \alpha \leq 1)$, the sequences p, q, and the roughness degree r. But the main emphasis was given to the $\mathcal{I} - DS^{r,\alpha}_{p,q}(\triangle)$-limit set which is proved to be a closed and convex subset of the normed linear space considered. Furthermore, Theorem 3.6 gives a condition under which $\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)-$limit set of a sequence is non-empty and Theorem 3.9 gives an equivalent condition for $\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)-$convergence of a sequence in a normed linear space $(X,\|\cdot\|)$.

In the future, this work can be extended over higher order difference sequences and several analytical properties such as solidity, monotonicity, symmetry, etc. of the corresponding sequence spaces can be investigated.

Also, based on Theorem 3.7 and Theorem 3.8, the following question can be asked: Let A be an arbitrary convex (or closed) set in normed vector space. Is there a sequence (x_k) whose $\mathcal{I}-DS^{r,\alpha}_{p,q}(\triangle)-$limit set will be A?

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Chiranjib Choudhury for the photo and short autobiography, see TWMS J. of Appl. and Engin. Math., Vol.14, No.1.

Mehmet Küçükaslan completed his undergraduate education in Mathematics at Karadeniz Technical University in 1994, and his graduate education at Mersin University in 1997, and his doctoral education at Çukurova University in 2003. Since 2003, he has worked in different positions at Mersin University, Faculty of Science, Department of Mathematics. He still continues to work as a professor in the same faculty. His areas of study are Orthogonal Polynomials, Fourier Series, Analysis in Metric Spaces, and Different Types of Convergence in Summability Theory.