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ON TOTAL VERTEX IRREGULAR LABELINGS WITH NO-HOLE WEIGHTS OF SOME CORONA GRAPHS

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Abstract. A total vertex irregular k-labeling of a graph $G = (V, E)$, $\partial : V \cup E \to \{1, 2, 3, \cdots, k\}$ is a labeling of vertices and edges of G in such a way that the weights of all vertices are distinct. A total vertex irregularity strength of graph G, denoted by $\text{tvs}(G)$ is defined as the minimum k for which a graph G has a totally irregular total k-labeling. In this paper, we investigate the no-hole total vertex irregularity strength for corona product of cyclic graphs and complete graphs.

Keywords: Total vertex irregular k-Labeling, Corona, no-hole.

AMS Subject Classification: 05C78.

1. Introduction

Let G be a finite, simple, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. In 1988, Chartrand et al. [6] defined an irregular labeling of a graph as an assignment of positive integer labels to the edges of a connected graph of order at least 3, in such a way that the sum of the weights of the edges at each vertex are distinct. The minimum of the largest label of an edge over all irregular labelings is called the irregularity strength $s(G)$ of G. Over the years many researchers investigated the calculation $s(G)$, as well as providing upper bounds for graphs with special characteristics (see [7], a dynamic survey).

Motivated by their work, and by a book of Wallis [16], Beča et al. [4] introduced the total vertex (and edge) irregularity strength of a graph. A total vertex irregular k-labeling ∂ of a graph $G = (V, E)$ is a labeling of the vertices and edges of G with labels from the set $\{1, 2, \cdots, k\}$ in such a way that for any two different vertices u and v their weights $\varphi(u)$ and $\varphi(v)$ are distinct, where the weight of a vertex v is $\varphi(v) = \partial(v) + \sum_{uv \in E(G)} \partial(uv)$, i.e. the sum of the label of the vertex v and the labels of the edges incident on v. The total vertex irregularity strength, $\text{tvs}(G)$, is minimum k for which the graph has a vertex irregular total k-labeling.

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In [4] Bača et al. determined the exact values of total vertex irregularity strength for few well known families of graphs (Paths, Cycles, Complete Graphs, Star etc.). Nurdin et al. [13] determined the total vertex irregularity strength for trees with a constraint that there is no vertex of degree 2 or 3. Anholcer and Palmer [3] determined the total vertex irregularity strength of the C_n^k, that is the circulant graphs of $\mathbb{Z}_n$ with connection set $\{1, 2, \cdots, k\}$. In [2] Ahmad et al. studied the total vertex irregularity strength of wheel related graphs (flowers, helms, generalized friendship graphs, and web graphs). Slamin, Dafik, and Winnona determined the total vertex irregularity strengths of the disjoint (isomorphic and nonisomorphic) union of sun graphs ($C_n \odot K_1$) [14]. Jeyanthi and Sudha determine the total vertex irregularity strength of $C_n \odot K_2, C_n \odot \overline{K_m}, P_n \odot K_1, P_n \odot K_2$ etc. in [11], and $C_n \odot P_m$ in [12]. In this paper, we are particularly interested in investigating the total vertex irregularity strength for the family of $C_n \odot K_m$ graphs with restriction that the weights must be consecutive or the set of weights must be a no-hole set (i.e. there is no gap between the weights).

2. Main Result

In this section, we begin with introducing a stronger version of total vertex irregularity labeling of a graph, as follows:

Definition 2.1. *The (a, d)-total vertex irregularity strength of a graph G is the minimum value of k if there exists a mapping $\partial : V \cup E \to \{1, 2, \cdots, k\}$ such that the set of all vertex weights is $\{a, a + d, a + 2d, \cdots, a + (n - 1)d\}$, where a and d are fixed integers with $d \geq 0$.*

We note that $d = 0$ and 1 generates a magic labeling, and no-hole weights respectively. Furthermore it can be easily observed that $\delta + 1 \leq a \leq \Delta + 1$. We define below the corona product of two graphs.

Definition 2.2. *The corona of two graphs G_1 and G_2 is the graph, denoted by $G_1 \odot G_2$, obtained by taking one copy of G_1, and p copies of G_2 (where $|V(G_1)| = p$), and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2.*

In this paper, we consider the family of $C_n \odot K_m$, which is corona product of a cycle of order n and the complete graph K_m, and find the no-hole total irregular edge labeling for it. Let $\{v_1, e_1, v_2, e_2, \cdots, v_n, e_n, v_1\}$ be the cycle, and $\{u_{i1}, u_{i2}, \cdots, u_{im}\}$ be the vertices adjacent to the vertex v_i, $i = 1, 2, \cdots, n$. Also e_{ij} be the edge connecting u_{ij} with v_i for $1 \leq i \leq n$ and $1 \leq j \leq m$. Note that $|V(C_n \odot K_m)| = n(m + 1)$, and $|E(C_n \odot K_m)| = n(m(m + 1)/2 + 1)$.
Since $\delta(G) = m$ and $\Delta(G) = m + 2$, from [4], we get the range for tvs(G), as

$$\left\lceil \frac{mn + m + n}{m + 3} \right\rceil \leq \text{tvs}(G) \leq mn + n - m + 3$$

We denote the total vertex irregularity strength with consecutive weights as $\gamma'(G)$, for any graph G. As it is clear that as $\gamma'(G) \geq \text{tvs}(G)$, $\gamma'(G) \geq \lceil (mn + m + n + 1)/(m + 4) \rceil$.

Theorem 2.1. $\gamma'(C_n \odot K_m) \geq \max\left\{n+1, \left\lceil \frac{m(n + 1) + 1 - s}{3} \right\rceil \right\}$, *where $s = m(m+1)/2$.*

Proof. For the graph $G = C_n \odot K_m$, the consecutive weights of the vertices form the sequence $\{m + 1, m + 2, \cdots, mn + m + n\}$. For convenience, we refer the vertices lying on C_n as the *inner* vertices and others as *outer* vertices. Since the degree of the inner vertices are more than the outer vertices, we wish to generate the highest (last) n many vertex weights, i.e., $\{mn+m+1, mn+m+2, \cdots, mn+m+n\}$ from the vertices $\{v_i : 1 \leq i \leq n\}$.

Thus the maximum of the remaining weights must occur at one the outer vertices. As the degree of any outer vertex is m, it immediately implies that $\gamma'(G) \geq \lceil (mn+m)/m \rceil = n+1$.

Without loss of generality we can assume that the weights of the outer vertices $\{u_{1j} : 1 \leq j \leq m\}$ are $\{m+1, m+2, \cdots, 2m\}$, and weight of v_1 is $mn+m+1$, minimum of the last n vertex weights. In order to generate those weights, the labels of the edges $\{e_{1j} : 1 \leq j \leq m\}$ be at most $1, 2, \cdots, m$ (by labeling 1 at the outer vertices $\{u_{1j}\}$ and all outer edges). Thus

$$\begin{aligned} mn+m+1 &= \partial(v_1) + \partial(e_1) + \partial(e_n) + \sum_{1\leq j\leq m} (e_{ij}) \\ &\leq 3\gamma'(G) + 1 + 2 + \cdots + m \\ &= 3\gamma'(G) + m(m+1)/2 \end{aligned}$$

which implies $\gamma'(G) \geq \lceil (m(n+1)+1-s)/3 \rceil$, where $s = m(m+1)/2$.

□

Next we provide the vertex irregular total labeling with consecutive weighs for some particular cases and finally for the general family of $C_n \odot K_m$ graphs, for $n > m$.

Theorem 2.2. $\gamma'(C_n \odot K_2) = \left\lceil \dfrac{2n+2}{3} \right\rceil$

Proof. Let $G = C_n \odot K_2$, and $\partial : V(G) \cup E(G) \to \{1, 2, \cdots, \lceil (2n+2)/3 \rceil\}$ be the total labeling as follows: for any integer $n \geq 3$, the labelings of the outer vertices and edges are, $\partial(e_{ij}) = \left\lceil \dfrac{2i+j}{3} \right\rceil$, $\partial(e_{i3}) = \left\lceil \dfrac{2i}{3} \right\rceil$, $\partial(u_{i1}) = \left\lfloor \dfrac{2i+1}{3} \right\rfloor$, $\partial(u_{i2}) = \left\lceil \dfrac{2i+1}{3} \right\rceil$. Finally

$$\partial(v_i) = \begin{cases} 2p-2-\lfloor i/3 \rfloor & \text{if } n = 3p+1 \text{ or } n = 3p \text{ for } p \geq 3 \\ 2p - \lfloor i/3 \rfloor & \text{if } n = 3p+2 \text{ for } p \geq 1 \end{cases}$$

$$\partial(e_i) = \begin{cases} 2p+2 & \text{if } n = 3p+1 \text{ for } p \geq 3 \\ 2p+2 & \text{if } n = 3p+2 \text{ for } p \geq 1 \\ 2p+1 & \text{if } n = 3p \text{ for } p \geq 3 \end{cases}$$

for all $i \in \{1, 2, \cdots, n\}$. Now it remains to provide the labelings for the vertices and the edges of the cycle for $n = 3, 4, 6$ and 7.

For $n = 3$: $\partial(v_i) = 1$ for $1 \leq i \leq 3$, $\partial(e_2) = \partial(e_3) = 2$, and $\partial(e_1) = 3$.

For $n = 4$: $\partial(v_i) = 1$ for $1 \leq i \leq 4$, $\partial(e_1) = \partial(e_2) = \partial(e_4) = 4$, and $\partial(e_3) = 3$.

For $n = 6$: $\partial(v_1) = \partial(v_2) = \partial(v_5) = 2$, $\partial(v_3) = \partial(v_4) = \partial(v_6) = 1$ and

$$\partial(e_i) = \begin{cases} 4 & \text{for } i = 5 \\ 5 & \text{otherwise} \end{cases}$$

For $n = 7$: $\partial(v_i) = 2p-2-\lfloor i/3 \rfloor$ for $1 \leq i \leq 7$, and

$$\partial(e_i) = \begin{cases} 5 & \text{for } i = 6 \\ 6 & \text{otherwise} \end{cases}$$

It is easy to check that the above labeling provides the upper bound on $\gamma'(C_n \odot K_2)$. On the other hand we already know that $\text{tvs}(C_n \odot K_m) = \lceil (2n+2)/3 \rceil$, which completes the proof.

□

Theorem 2.3. $\gamma'(C_n \odot K_3) = n$

Proof. Let $G = C_n \odot K_3$, and $\partial : V(G) \cup E(G) \to \{1, 2, \cdots, n\}$ be the total labeling such that for the edges of the cycle $\partial(e_n) = n$, and $\partial(e_i) = n + 1 - i$ for $1 \leq i < n$, and the vertices of the cycle get the labeling

$$\partial(v_i) = \begin{cases} n-2 & \text{for } i = 1 \\ n & \text{for } 2 \leq i \leq n-1 \\ 1 & \text{for } i = n \end{cases}$$

We define the labelings of the outer vertices as follows $\partial(u_{ij}) = j$ and $\partial(v_i u_{ij}) = \partial(u_{ij} u_{ij'}) = i$ for $2 \leq i \leq n$ and $1 \leq j, j' \leq 3$ and $\partial(u_{1j}) = 1$ for $1 \leq j \leq 3$. Finally $\partial(v_1 u_{1j}) = j$ and $\partial(u_{1j} u_{1j'}) = 1$ for $1 \leq j, j' \leq 3$ and $j \neq j'$.

It is easy to check that the above labeling provides the upper bound on $\gamma'(C_n \odot K_3)$. Recall, we already know that $\gamma'(C_n \odot K_3) \geq \lceil (3n-2)/3 \rceil = n$, which completes the proof. □

Theorem 2.4. *$\gamma'(C_3 \odot K_4) = 4$, and $\gamma'(C_n \odot K_4) = \left\lceil \frac{4n-5}{3} \right\rceil$ for $n \geq 4$.*

Proof. Let $G = C_n \odot K_4$, we first prove the total irregularity strength with no-hole weights for $n \geq 4$, and consider $n = 3$ as a special case later. Theorem 2.1 leads us to the inequality $\gamma'(C_n \odot K_4) \geq \lceil (4n-5)/3 \rceil$ for $n \geq 3$. To prove the tightness first we assume

$$i^* = \begin{cases} \left\lfloor \frac{4p+5}{3} \right\rfloor & \text{for } n = 3p \\ \left\lfloor \frac{4p+8}{3} \right\rfloor & \text{for } n = 3p+1 \\ \left\lfloor \frac{4p+9}{3} \right\rfloor & \text{for } n = 3p+2 \end{cases}$$

and define a total labeling $\partial : V(G) \cup E(G) \to \{1, 2, \cdots, \lceil (4n-5)/3 \rceil\}$, that assign the following labeling to the outer edge and vertices: for $1 \leq j \leq 4$,

$$\partial(u_{ij}) = \begin{cases} 1 & \text{for } i = 1 \\ \left\lceil \frac{j}{2} \right\rceil & \text{for } i = 2 \\ j & \text{for } 3 \leq i \leq i^* \\ j + i - i^* & \text{for } i^* < i \leq n \end{cases}$$

$$\partial(e_{ij}) = \begin{cases} j & \text{for } i = 1 \\ \left\lceil \frac{j+3}{2} \right\rceil & \text{for } i = 2 \\ i & \text{for } 3 \leq i \leq i^* \\ i^* & \text{for } i^* < i \leq n \end{cases}$$

Now let $n = 3p + q$ for p, q being a non-negative integers where $p \geq 2$ and $q \in \{0, 1, 2\}$. It can easily be observed that $\lceil (4n-5)/3 \rceil = 4p + q - 1$. Now the vertices and edges of the cycle get labeled as follows:

$$\partial(e_i) = 4p + q - 1 \text{ for } 1 \leq i \leq n$$

$$\partial(v_i) = \begin{cases} 4p + 2q - 2 - i & \text{for } i = 1, 2 \\ 4p + 2q + 6 - 3i & \text{for } 3 \leq i \leq i^* \\ 4p + 2q + 6 + i - 4i^* & \text{for } i^* < i \leq n \end{cases}$$

We see that all the vertex and edge labels are at most $\lceil (4n-5)/3 \rceil$. Also as $\varphi(v_i) = 4n+4+i$ and $\varphi(u_{ij}) = 4i+j$ for $1 \leq i \leq n$ and $1 \leq j \leq 4$, the weights of the vertices formed under the labeling ∂ are $\{5, 6, \cdots, 4n{+}4, 4n{+}5, 4n{+}6, \cdots, 5n{+}4\}$. Hence $\gamma'(C_n \odot K_4) = \lceil (4n-5)/3 \rceil$. It is easy to check that the above labeling provides the upper bound on $\gamma'(C_n \odot K_m)$. Whereas we already have $\text{tvs}(C_n \odot K_m) = \lceil (4n-5)/3 \rceil$, which completes the proof.

Now we complete our proof by providing the labeling for the particular case $C_3 \odot K_4$. $\partial(v_1) = \partial(v_3) = \partial(u_{ij}) = 1$ for $1 \leq j \leq 4$ and $\partial(v_2) = 4$. $\partial(u_{ij}) = j$ and $\partial(e_{ij}) = \partial(u_{ij}u_{ij'}) = i$ for $2 \leq i \leq 3$ and $1 \leq j \leq 4$. $\partial(e_i) = 3$ for $1 \leq i \leq 3$. Finally $\partial(e_{1j}) = j$ and $\partial(u_{1j}u_{1j'}) = 1$ for $1 \leq j \leq 4$ □

Now after we provide the total vertex irregular labeling with no-hole weights for the cases $m = 2, 3$, and 4, instead of proving more cases based on m, we look into the problem asymptotically. We begin with providing a lower bound for $\gamma'(C_n \odot K_m)$ for some n, and then propose a conjecture for the for almost all n.

Theorem 2.5. *$\gamma'(C_n \odot K_m) > \max\left\{\left\lceil \frac{m(n+1)}{m+1} \right\rceil, \left\lceil \frac{m(n+1)+1-s}{3} \right\rceil\right\}$, where $s = m(m+1)/2$ if $\left\lceil \frac{3+\sqrt{8m+25}}{2} \right\rceil \leq n \leq \left\lfloor \frac{m^3+3m-2}{2m^2-4m} \right\rfloor$ when $m \geq 13$.*

Proof. First we can easily verify that if $m \leq 12$, then $\left\lceil \frac{3+\sqrt{8m+25}}{2} \right\rceil \geq \left\lfloor \frac{m^3+3m-2}{2m^2-4m} \right\rfloor$. Hence now onward we will only consider $m \geq 13$. As $\deg(u_{ij}) = m$ and $\deg(v_i) = m+2$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, it is evident that in order to minimize k, the weights $\{mn+m+1, mn+m+1, \cdots, mn+m+n\}$ must be generated on the vertices of the cycle C_n. Without loss of generality let us assume that the weight of the vertex v_1 must be $mn+m+1$.

According to our assumption, as $n \leq \left\lfloor \frac{m^3+3m-2}{2m^2-4m} \right\rfloor$, we get that $m^3+3m-2 > n(2m^2-4m)$, which simplifies to $4mn+3m > 2m^2n - m^3 + 2$. From there we can easily derive $2mn+3m > (m+1)(mn+m+1-m(m+1)/2)$, which concludes $\left\lceil \frac{m(n+1)}{m+1} \right\rceil > \left\lceil \frac{m(n+1)+1-s}{3} \right\rceil$.

On the other hand, as $m > \left\lfloor \frac{m^3+3m-2}{2m^2-4m} \right\rfloor \geq n$, $\left\lceil \frac{m(n+1)}{m+1} \right\rceil = n+1$. Clearly in this case $\left\lceil \frac{m(n+1)+1-s}{3} \right\rceil \leq n$ if

$$\gamma'(C_n \odot K_m) = \max\left\{\left\lceil \frac{m(n+1)}{m+1} \right\rceil, \left\lceil \frac{m(n+1)+1-s}{3} \right\rceil\right\}$$

Now in order to generate the $\varphi(v_1) = mn+m+1$, we can make this observation.

$$\begin{aligned} mn+m+1 &= \sum_{j=1}^{m} e_{1j} + e_1 + e_n + v_1 \\ &\leq 1 + 2 + \cdots + (n+1) + (m-n-1)(n+1) + 3(n+1) \\ &= (n+1)((n+2/2) + m - n + 2) \end{aligned}$$

The above inequality implies to $n^2 - 3n - (2m+4) \leq 0$. But if $n > \frac{3+\sqrt{8m+25}}{2}$, then $n^2 - 3n - (2m+4) > 0$. Thus the possible maximum labels for and around v_1, can not generate $mn+m+1$.

□

Conjecture 2.1. $\gamma'(C_n \odot K_m) = \max\left\{\left\lceil\frac{m(n+1)}{m+1}\right\rceil, \left\lceil\frac{m(n+1)+1-s}{3}\right\rceil\right\}$, *where* $n > \left\lfloor\frac{m^3+3m-2}{2m^2-4m}\right\rfloor$ *and* $s = m(m+1)/2$.

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