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FAULT-TOLERANT PARTITION RESOLVABILITY OF CHEMICAL CHAINS

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Abstract. An e partition X of the vertex set $V(H)$ of a connected graph H is the collection of e number of ordered disjoint subsets of $V(H)$, denoted as $X = \{X_1, X_2, \ldots, X_e\}$. The representation of a vertex u is a distance vector $r(u|X) = (d(u, X_1), d(u, X_2), \ldots, d(u, X_e))$, where $d(u, X_i)$ is the distance of u from X_i. Any ordered e partition X is referred as resolving partition if representations of all the vertices are distinct. The smallest integer e is referred as the partition dimension of the graph. The advancement in the concept of partition dimension is fault-tolerant partition dimension where the representations are distinct at two places for each pair of vertices. In this paper, we compute the partition dimension and fault-tolerant partition dimension of some planar graphs.

Keywords: cactus chains, starphene chains, metric dimension, partition dimension, fault-tolerant partition dimension.

AMS Subject Classification: 05C12

1. Introduction

Chemical graph theory is an important branch of computational chemistry and graph theory, having applications in pharmaceutical engineering and chemistry. The structure of chemical compounds or materials can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. A class of simple linear polymers called cactus chains, previously known as Husimi tree were emerged in the papers of Husimi and Riddell [13, 23]. In a cactus graph, no edge lies in more than one cycle so each of its blocks is either an edge or a cycle. Some aspects of cactus graphs such as bounds on the 2-domination number and k-domination on hexagonal cactus chains can be seen in [9] and [17]. Cactus graphs are used to model electronic circuits with specific properties [22] and have recently been considered for genome comparison [25].

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Starphene is a single ring of benzene which is surrounded by three identical arene substituents. Due to its physicochemical properties and characteristics inherited from arenes, starphene plays an important role in the study of organic electronics and optics. The starphene chains are used in many organic electronics to reduce the sizes of devices. The starphene molecules have also been used in the construction of NAND [29] and NOR [30] gates.

Many applications in the chemical field are based upon distance related parameters. One of these distance related parameters is metric dimension (MD) of graph that has been used in different domains of scientific research. Slater [28] and Harary and Melter [11] independently introduced the notion of minimum cardinality of resolving set within the graph, known as the metric dimension .

Suppose a network of a chain, formed by m processing devices to solve some task. Failure of any single device, will result in the replacement of another fault free unit in a chain to continue its intended operation. A fault-tolerant system empowers a system to perform its working even at a reduced level, rather failing in total. The extended concept of metric dimension as fault-tolerant metric dimension (FTMD) of graphs has been applied in different areas like resolvability of crystal structures, network analysis, chemical structures of Methylene and mathematical formalization of woven structures [1].

Chartrand et al. [8] initiated the notion of partition dimension (PD) in graphs as a generalized version of MD of graph. Its applications can be seen in areas of network discovery and verification [6], combinatorial optimization [15], image processing [18] and modelling of chemical substances [16]. The PD problem has been discussed for different classes of graphs. The PD of certain families of toeplitz graph was computed by Luo et al. [24] and Siddiqui et al. [27] computed the MD and PD of nanotubes. Chu et al. [10] and Wei et al. [32] studied the PD problem for convex polytopes and cycle related graphs respectively.

Non-deterministic polynomial time-hardness and computational complexity for parameters from the resolvability family are addressed by researches [7, 8, 15].

The progression in the area of investigation of partition dimension as fault-tolerant partition dimension (FTPD) of graph was made known by Javaid et al. [26]. Kamran et al. [4, 5, 3] computed the FTPD of mesh related networks, cycle related graphs and chemical graphs. They also gave applications of PD and FTPD in scenario of water flow in a locality [4] and sensors deployed in homes [5]. Asim et al. [19, 21] discussed FTPD of toeplitz networks and 2-partition resolvability of induced subgraphs of certain hydrocarbon nanotubes.

We protract this discipline in the paper by examining chain triangular cactus, ortho-chain square cactus, para-chain square cactus, para-chain hexagonal cactus and starphene chain and conclude that they have constant FTPD.

1.1. **Motivation.**

The computational complexity of MD and FTMD motivated researchers to compute these parameters for some specific classes. The MD and parameters related to MD of cactus graphs have been addressed by Kuziak and Sedlar et al. [14, 25], which motivated the present study to compute PD and FTPD for these important families of chains to give readers more insight of cactus and starphene chains through these newly defined distance related parameters.

2. Preliminaries

Consider a graph H of order $|V(H)|$ having vertex and edge sets as $V(H)$ and $E(H)$ respectively. The distance between two vertices $t, u \in V(H)$ is the least number of edges in $t-u$ path and is expressed as $d(t,u)$. The distance between a vertex u and $\eta \subseteq V(H)$ is defined as $\min\{d(u,y)|y \in \eta\}$ and is denoted by $d(u,\eta)$. For a vertex $u \in V(H)$, $N(u)$ denotes the open neighbourhood whereas, $N[u]$ denotes the closed neighbourhood of u in H. Consider an ordered subset $\Gamma = \{\tau_1, \tau_2, \ldots, \tau_e\}$ of vertices of H. The representation $r(u|\Gamma)$ of u in association with Γ is e-ordered distances $(d(u,\tau_1), d(u,\tau_2), \ldots, d(u,\tau_e))$. If representation of all the vertices of H in association with Γ is unique, then Γ is called a resolving set (RS) of H. The MD is defined as the least cardinality of resolving set Γ of H symbolized by $\beta(H)$.

The fault-tolerant concept of the definition of RS was introduced by Hernando et al. [12]. If for every pair of distinct vertices $\rho, u \in V(H)$, there exists at least two vertices $\alpha_1, \alpha_2 \in \Gamma$ such that $d(\rho,\alpha_m) \neq d(u,\alpha_m)$ for $m \in \{1,2\}$, then, the RS Γ of $V(H)$ is called fault-tolerant. The fault-tolerant metric dimension (FTMD) of H, symbolized by $\beta'(H)$ is the least number of members in fault-tolerant resolving set Γ.

Consider $X = \{X_1, X_2, \ldots, X_e\}$ having e partition classes of vertices of connected graph H. The representation $r(u|X)$ of vertex u associated to partition set X is e-vector $(d(u,X_1), d(u,X_2), \ldots, d(u,X_e))$. If representation of all the vertices in H are unique, then the partition X is called resolving partition (RP) of H. We define the partition dimension (PD) of graph H as, $\min\{|X| : X$ is resolving partition of $H\}$ and is denoted by $\mathcal{P}(H)$.

Let $X = \{X_1, X_2, \ldots, X_e\}$ be the ordered partition of $V(H)$ with e partition classes. The partition X is known to be fault-tolerant resolving partition (FTRP) of H if for every pair of distinct vertices $\rho, u \in V(H)$, $r(\rho|X)$ and $r(u|X)$ have at least two different coordinates. The FTPD of H is defined as the minimum number of subsets in FTRP and is denoted by $\mathcal{F}(H)$.

Some Useful Results

Chartrand et al. [8] examined successive conclusions on $\mathcal{P}(H)$.

Proposition 2.1. [8] *Let H be a connected graph with order z, then;*

(a) $\mathcal{P}(H) \leq \beta(H) + 1$.
(b) $\mathcal{P}(H) = 2$ *iff* $H = P_z$ *where* P_z *is a path.*
(c) $\mathcal{P}(H) = z$ *iff* $H = K_z$ *where* K_z *is the complete graph.*

Javaid et al. probed following conclusions on $\mathcal{F}(H)$.

Proposition 2.2. [26]

(a) *For* $z \geq 2$, $\beta'(H)$ *and* $\mathcal{F}(H)$ *are related as* $\mathcal{F}(H) \leq \beta'(H) + 1$.
(b) *For* $z \geq 3$, $3 \leq \mathcal{F}(H) \leq z$.

An important result of FTPD of graph recently provided by Asim et al. is given in the following lemma.

Lemma 2.1. [20] *Let H be a graph of order $z \geq 5$. If H has a node of degree at least 4, then $\mathcal{F}(H) \geq 4$.*

In this paper, Section 3, provides the computation of fault tolerant partition dimension of triangular and square cactus chains. Section 4, provides the computation of FTPD of para-chain hexagonal cactus graph. Section 5, is about the computation of FTPD of starphene chain. Finally, future research direction is given in the concluding section.

3. FTPD of triangular and square cactus chains

We evaluate FTPD of triangular and square cactus chains in this section. In a graph the length of the triangular chain is represented by the number of triangles. The chain triangular cactus of length z has order $2z+1$ and is denoted by R_z. Chain triangular

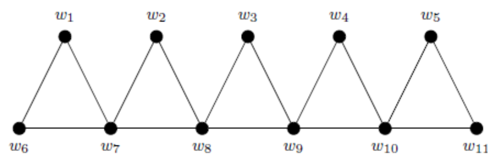

FIGURE 1. Chain triangular cactus graph R_5.

cactus R_5 is given in Figure 1. Wang et al. [31] computed the metric based parameters of chain triangular cactus graph.

Lemma 3.1. [31] *Let R_z be a chain triangular cactus graph with $z \geq 2$. Then, $\beta(R_z) = 2$.*

Lemma 3.2. [31] *Let R_z be a chain triangular cactus graph, then,*

$$\beta^{'}(R_z) = \begin{cases} 3 & \text{if } z = 2, 3; \\ 4 & \text{if } z \geq 4. \end{cases}$$

We compute $\mathcal{P}(R_z)$ in the subsequent theorem.

Theorem 3.1. *For every $z \geq 2$, the PD of chain triangular cactus R_z, is 3.*

Proof. It complies with Lemma 3.1, and Proposition 2.1(a), that $\mathcal{P}(R_z) \leq 3$. Also from Proposition 2.1(b), $\mathcal{P}(R_z) \geq 3$. Thus, $\mathcal{P}(R_z) = 3$, this completes the proof. □

Following theorem will allow us to compute the $\mathcal{F}(H)$ of chain triangular cactus.

Theorem 3.2. *For every $z \geq 2$, the FTPD of chain triangular cactus R_z, is 4.*

Proof. Consider $X = \{X_1, X_2, X_3, X_4\}$, where, $X_1 = \{w_i : 1 \leq i \leq z\}$, $X_2 = \{w_j : z+1 \leq j \leq 2z-1\}$, $X_3 = \{w_{2z}\}$ and $X_4 = \{w_{2z+1}\}$ are partition classes of $V(R_z)$. The $r(w|X)$ of R_z are shown beneath:

$$r(w_l|X) = \begin{cases} (0, 1, z-l, z-l+1) & \text{if } 1 \leq l \leq z-1; \\ (0, 2, 1, 1) & \text{if } l = z; \\ (1, 0, 2z-l, 2z-l+1) & \text{if } z+1 \leq l \leq 2z-1; \\ (1, 1, 0, 1) & \text{if } l = 2z; \\ (1, 2, 1, 0) & \text{if } l = 2z+1. \end{cases}$$

Mentioned unique identifications authenticate that X is FTRP of R_z, so, $\mathcal{F}(R_z) \leq 4$. As R_z has vertices of degree at least 4, so from Lemma 2.1, $\mathcal{F}(R_z) \geq 4$. From both inequalities, it is proved that $\mathcal{F}(R_z) = 4$. □

3.1. **FTPD of square cactus chains.** A square cacti is obtained when triangles in triangular cactus are replaced by cycles of length 4. The internal squares may connect to their neighbors in different ways. A square is an ortho-square if cut-vertices of internal squares are adjacent. A chain with ortho-squares is called ortho-chain square cactus and is denoted by O_z. A square is para-square if cut-vertices of internal squares are not adjacent. A chain with para-squares is called para-chain square cactus and is denoted by Q_s. The orders of O_z and Q_z are $6z-2$ and $3z+1$ respectively. Square cactus chains O_3 and Q_5 are shown Figure 2 and 3 respectively.

We compute the $\mathcal{P}(O_z)$, $\mathcal{F}(O_z)$, $\mathcal{P}(Q_z)$ and $\mathcal{F}(Q_z)$ in the subsequent theorems.

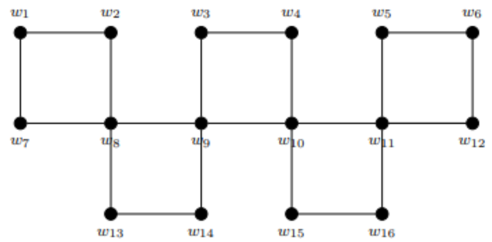

FIGURE 2. Ortho-chain square cactus O_3.

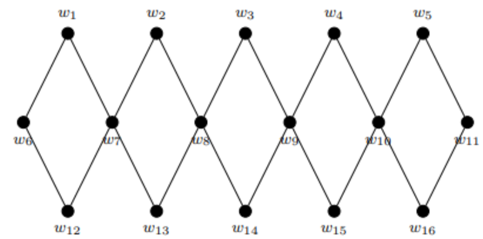

FIGURE 3. Para-chain square cactus Q_5.

Theorem 3.3. *For every $z \geq 2$, the partition dimension of ortho-chain square cactus graph O_z, is* 3.

Proof. Let $X = \{X_1, X_2, X_3\}$ be a set of $V(O_z)$ for $z \geq 2$, with 3 partition classes. The $r(w|X)$, where, $X_1 = \{w_i : 1 \leq i \leq 2z\} \cup \{w_j : 2z+2 \leq j \leq 4z-1\} \cup \{w_{4z+1}\}$, $X_2 = \{w_{2z+1}\} \cup \{w_j : 4z+2 \leq j \leq 6z-2\}$ and $X_3 = \{w_{4z}\}$ are shown below.

$$r(w_l|X) = \begin{cases} (0,1,2z) & \text{if } l = 1; \\ (0,2,2z-l+1) & \text{if } 2 \leq \mathrm{l} \leq 2\mathrm{z}-1; \\ (0,3,1) & \text{if } l = 2z; \\ (1,0,2z-1) & \text{if } l = 2z+1; \\ (0,1,4z-l) & \text{if } 2z+2 \leq l \leq 4z-1; \\ (1,2,0) & \text{if } l = 4z; \\ (0,1,2z-1) & \text{if } l = 4z+1; \\ (1,0,6z-l) & \text{if } 4z+2 \leq l \leq 6z-2. \end{cases}$$

Mentioned distinct identifications justify that X is RP of O_z, so, $\mathcal{P}(O_z) \leq 3$. Also by Proposition 2.1(b), $\mathcal{P}(O_z) \geq 3$, accomplishes the proof. □

Theorem 3.4. *For every $z \geq 2$, the FTPD of ortho-chain square cactus O_z, is* 4.

Proof. Consider $X = \{X_1, X_2, X_3, X_4\}$, where, $X_1 = \{w_i : 1 \leq i \leq 2z\} \cup \{w_j : 2z+2 \leq j \leq 4z-1\}$, $X_2 = \{w_{2z+1}\}$, $X_3 = \{w_{4z}\}$ and $X_4 = \{w_j : 4z+1 \leq j \leq 6z-2\}$ are partition classes of $V(O_z)$. The $r(w|X)$ of O_z are given below:

$$r(w_l|X) = \begin{cases} (0,1,2z,3) & \text{if } l=1; \\ (0,l,2z-l+1,2) & \text{if } 2 \le l \le 2z-1; \\ (0,2z,1,3) & \text{if } l=2z; \\ (1,0,2z-1,2) & \text{if } l=2z+1; \\ (0,l-2z-1,4z-l,1) & \text{if } 2z+2 \le l \le 4z-1; \\ (1,2z-1,0,2) & \text{if } l=4z; \\ (1,l-4z+1,6z-l,0) & \text{if } 4z+1 \le l \le 6z-2. \end{cases}$$

Mentioned distinct identifications justify that X is FTRP of O_z, so, $\mathcal{F}(O_z) \le 4$. As O_z has vertices of degree at least 4, so from Lemma 2.1, $\mathcal{F}(O_z) \ge 4$. From both inequalities $\mathcal{F}(O_z) = 4$, accomplishes the proof. □

Theorem 3.5. *For every $z \ge 2$, the PD of para-chain square cactus Q_z, is 3.*

Proof. Let $X = \{X_1, X_2, X_3\}$ be a set of $V(Q_z)$ for $z \ge 2$, with 3 partition classes. The $r(w|X)$, where, $X_1 = \{w_i : 1 \le i \le 2z\}$, $X_2 = \{w_j : 2z+1 \le j \le 3z\}$ and $X_3 = \{w_{3z+1}\}$ are shown below:

$$r(w_l|X) = \begin{cases} (0,2,2z-2l) & \text{if } 1 \le l \le z-1; \\ (0,1,2) & \text{if } l=z; \\ (0,1,4z-2l+1) & \text{if } z+1 \le l \le 2z; \\ (1,0,1) & \text{if } l=2z+1; \\ (1,0,6z-2l+2) & \text{if } 2z+2 \le l \le 3z; \\ (1,1,0) & \text{if } l=3z+1. \end{cases}$$

Mentioned unique representations justify that X is resolving partition of Q_z, therefore, $\mathcal{P}(Q_z) \le 3$. Also by Proposition 2.1(b), $\mathcal{P}(Q_z) \ge 3$, accomplishes the proof. □

Theorem 3.6. *For every $z \ge 2$, the FTPD of para-chain square cactus Q_z, is 4.*

Proof. Consider $X = \{X_1, X_2, X_3, X_4\}$ be a set of vertices of Q_z. For $z = 2$, consider $X_1 = \{w_1, w_2, w_4\}$, $X_2 = \{w_3, w_6\}$, $X_3 = \{w_5\}$ and $X_4 = \{w_7\}$. Easily it can be verified that X is fault-tolerant partition basis. For $z \ge 3$, consider, $X_1 = \{w_i : 1 \le i \le 2z-1\}$, $X_2 = \{w_{2z}, w_{3z}, w_{3z+1}\}$, $X_3 = \{w_{2z+1}\}$ and $X_4 = \{w_j : 2z+2 \le j \le 3z-1\}$. The $r(w|X)$

of Q_z for $z \geq 3$ are given below:

$$r(w_l|X) = \begin{cases} (0, 2z-2l-2, 2z-2l+1, 2) & \text{if } 1 \leq l \leq z-2; \\ (0,1,3,2) & \text{if } l = z-1; \\ (0,1,1,4) & \text{if } l = z; \\ (0, 4z-2l-1, 4z-2l+2, 1) & \text{if } z+1 \leq l \leq 2z-1; \\ (1,0,2,3) & \text{if } l = 2z; \\ (1,1,0,5) & \text{if } l = 2z+1; \\ (1, 6z-2l, 6z-2l+3, 0) & \text{if } 2z+2 \leq l \leq 3z-1; \\ (1,0,3,2) & \text{if } l = 3z; \\ (2,0,1,4) & \text{if } l = 3z+1. \end{cases}$$

Mentioned distinct identifications authenticate that X is FTRP of Q_z, so, $\mathcal{F}(Q_z) \leq 4$. As Q_z has vertices of degree at least 4, so from Lemma 2.1, $\mathcal{F}(Q_z) \geq 4$. Thus $\mathcal{F}(Q_s) = 4$, completes the proof. □

4. FTPD of para-chain hexagonal cactus graph

We compute the FTPD of para-chain hexagonal cactus graph in this section. A hexagonal cacti is obtained when triangles in triangular cactus are replaced by cycles of length 6. The internal hexagon may connect to their neighbors in different ways. Hexagon is called an ortho-hexagon, if cut-vertices of internal hexagon are adjacent. Hexagon is called a para-hexagon, if cut-vertices of internal hexagon are not adjacent. A chain with para-hexagons is called para-chain hexagonal cactus and is denoted by L_z. The orders of L_z is $5z+1$. Para-chain hexagonal cactus L_5 is shown Figure 4.

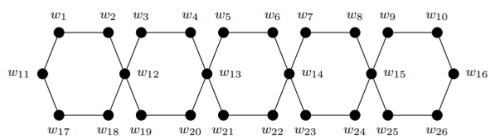

FIGURE 4. Para-chain hexagonal cactus L_5.

We compute the $\mathcal{P}(L_z)$ and $\mathcal{F}(L_z)$ in the following theorems.

Theorem 4.1. *For every $z \geq 2$, the PD of para-chain hexagonal cactus L_z, is* 3.

Proof. Let $X = \{X_1, X_2, X_3\}$ be a set of vertices of L_z for $z \geq 2$. The $r(w|X)$, where, $X_1 = \{w_i : 1 \leq i \leq 3z\} \cup \{w_{3z+2}\}$, $X_2 = \{w_{3z+1}\}$ and $X_3 = \{w_j : 3z+3 \leq j \leq 5z+1\}$

are shown as follows.

$$r(w_l|X) = \begin{cases} (0,3z,2) & \text{if } l=1; \\ (0,3z-1,3) & \text{if } l=2; \\ (0,3z-l+1,2) & \text{if } l=3p, \text{where}, 1 \leq p \leq z; \\ (0,3z-l+1,1) & \text{if } l=3p+1, \text{ where, } 1 \leq p \leq z-1; \\ (0,3z-l+1,2) & \text{if } l=3p+2, \text{where, } 1 \leq p \leq z-1; \\ (1,0,1) & \text{if } l=3z+1; \\ (0,3z-1,1) & \text{if } l=3z+2; \\ (1,\lfloor\frac{15z-3l+5}{2}\rfloor,0) & \text{if } 3z+3 \leq l \leq 5z; \\ (2,1,0) & \text{if } l=5z+1. \end{cases}$$

Mentioned unique identifications authenticate that X is resolving partition of L_z, thus, $\mathcal{P}(L_z) \leq 3$. Also by Proposition 2.1(b), $\mathcal{P}(L_z) \geq 3$, completes the proof. □

Theorem 4.2. *For every $z \geq 2$, the FTPD of para-chain hexagonal cactus L_z, is 4.*

Proof. Consider $X = \{X_1, X_2, X_3, X_4\}$ be a set of $V(L_z)$ with 4 partition classes, where, $X_1 = \{w_i : 1 \leq i \leq 3z-1\} \cup \{w_{5z}\}$, $X_2 = \{w_{3z}\}$, $X_3 = \{w_{3z+1}\}$ and $X_4 = \{w_j : 3z+2 \leq j \leq 5z-1\} \cup \{w_{5z+1}\}$. The $r(w|X)$ of L_z are given as follows.

$$r(w_l|X) = \begin{cases} (0,3z-l,3z-l+1,1) & \text{if } l=3p-2, \text{ where}, 1 \leq p \leq z; \\ (0,3z-l,3z-l+1,2) & \text{if } l=3p-1, \text{ where}, 1 \leq p \leq z-1; \\ (0,3z-l,3z-l+1,2) & \text{if } l=3p, \text{ where}, 1 \leq p \leq z-1; \\ (0,1,2,2) & \text{if } l=3z-1; \\ (1,0,1,2) & \text{if } l=3z; \\ (2,1,0,1) & \text{if } l=3z+1; \\ (1,\lfloor\frac{15z-3l+3}{2}\rfloor,\lfloor\frac{15z-3l+5}{2}\rfloor,0) & \text{if } 3z+2 \leq l \leq 5z-1; \\ (0,3,2,1) & \text{if } l=5z; \\ (1,2,1,0) & \text{if } l=5z+1. \end{cases}$$

Mentioned unique representations justify that X is fault-tolerant resolving partition of L_z, so, $\mathcal{F}(L_z) \leq 4$. As L_z has vertices of degree at least 4, so from Lemma 2.1, $\mathcal{F}(L_z) \geq 4$. Thus, $\mathcal{F}(L_z) = 4$, completes the proof. □

5. FTPD of Starphene Chain

Starphene chain belong to the family of polycyclic aromatic hydrocarbons (PAH), having three acene arms f, g and h connected on a centered benzene ring. Starphene chain is denoted by $St(f,g,h)$ and its order is $4(f+g+h)-2$. The vertex set of $St(f,g,h)$ is $V(St(f+g+h)) = \{a_i : 1 \leq i \leq 2(f+h)\} \cup \{b_i : 1 \leq i \leq 2(f+g)\} \cup \{c_i : 1 \leq i \leq 2(g+h-1)\}$. Graph of $St(f,g,h)$ is shown in Figure 5.

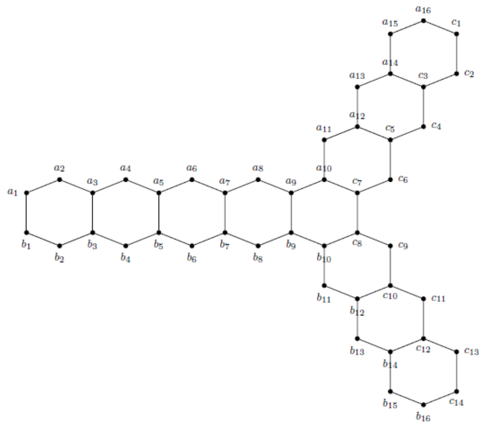

FIGURE 5. Starphene chain $St(4, 4, 4)$

Ali et al. computed the partition dimension of starphene graph $St(f, g, h)$.

Lemma 5.1. [2] *The PD of starphene graph $St(f, g, h)$ for $f, g, h \geq 2$, is 3.*

Lemma 5.2. [3] *The FTPD of z-linear benzene B_z for $z \geq 2$, is 4.*

Lemma 5.3. *Let H be a graph containing z-linear benzene B_z, then, $\mathcal{F}(H) \geq 4$.*

Proof. It is proved in [3], that $\mathcal{F}(B_z) \geq 4$, so any graph H containing z-linear benzene B_z, will have $\mathcal{F}(H) \geq 4$. □

In next theorem, we compute the FTPD of starphene graph $St(f, g, h)$.

Theorem 5.1. *The FTPD of starphene graph $St(f, g, h)$ for $f, g, h \geq 2$, is 4.*

Proof. Consider a set $X = \{X_1, X_2, X_3, X_4\}$ of vertices of $St(f, g, h)$ for $f, g, h \geq 2$. The $r(v|X)$, where $X_1 = \{a_i : 1 \leq i \leq 2(f+g)\} \cup \{b_i : 1 \leq i \leq 2(f+h)\} \cup \{c_{2g-1}, c_{2g}\}$, $X_2 = \{c_1, c_2\}$, $X_3 = \{c_i : 3 \leq i \leq 2g-2\} \cup \{c_i : 2g+1 \leq i \leq 2g+2h-3\}$ and $X_4 = \{c_{2(g+h-1)}\}$ are given as follows.

$$r(a_l|X) = \begin{cases} (0, 2f+2g-l, 2l+4-l, 2f+2h-l+2) & 1 \leq l \leq 2f+2; \\ (0, 2g-2p-1, 2, 2h+2p-1) & l = 2f+2p+1, 1 \leq p \leq g-2; \\ (0, 2g-2p-2, 1, 2h+2p) & l = 2f+2p+2, 1 \leq i \leq g-2; \\ (0, 2, 2, 2g+2h-3) & l = 2f+2g-1; \\ (0, 1, 3, 2g+2h-2) & l = 2f+2g. \end{cases}$$

$$r(b_l|X) = \begin{cases} (0, 2f+2g-l+1, 2f+4-l, 2f+2h-l+1) & 1 \leq l \leq 2f+2; \\ (0, 2g+2p-2, 2, 2h-2p) & l = 2f+2p+1, 1 \leq p \leq h-1; \\ (0, 2g+2p-l, 1, 2h-2p-1) & l = 2f+2p+2, 1 \leq i \leq h-2; \\ (0, 2g+2h-3, 2, 1) & l = 2f+2h. \end{cases}$$

$$r(c_l|X) = \begin{cases} (l, 0, 3-l, 2g+2h-l-2) & 1 \leq l \leq 2; \\ (1, 2p-1, 0, 2g+2h-2p-3) & l = 2p+1, 1 \leq p \leq g-2; \\ (2, 2p, 0, 2g+2h-2p-4) & l = 2p+2, 1 \leq p \leq g-3; \\ (1, 2g-4, 0, 2h) & l = 2g-2; \\ (0, 2g-4+p, 1, 2h-p) & l = 2g-2+p, 1 \leq p \leq 2; \\ (1, 2g-1, 0, 2h-3) & l = 2g+1; \\ (1, 2g-2+2p, 0, 2h-2p-2) & l = 2g+2p, 1 \leq p \leq h-2; \\ (2, 2g-1+2p, 0, 2h-2p-3) & l = 2g+2p+1, 1 \leq p \leq h-2; \\ (1, 2g+2h-4, 1, 0) & l = 2(g+h-1). \end{cases}$$

Above mentioned representations authenticate that X is fault-tolerant resolving partition of $St(f,g,h)$, so, $\mathcal{F}(St(f,g,h)) \leq 4$. As $St(f,g,h)$ contains z-linear Benzene, so by Lemma 5.3, $\mathcal{F}(St(f,g,h)) \geq 4$. From both inequalities, $\mathcal{F}(St(f,g,h)) = 4$, accomplishes the proof. □

6. Conclusion

We considered the cactus chains for $z \geq 2$, and computed that partition dimension of these chains is 3. We also computed fault-tolerant partition dimension of cactus chains and starphene chain. We computed that fault-tolerant partition dimension of cactus chains for $z \geq 2$, and starphene chain for $f, g, h \geq 2$, is 4. The partition dimension and fault-tolerant partition dimension of these chemical graphs are independent of number of vertices of the graph. Future research can be focused on computing the FTPD of meta chain hexagonal cactus and polyphenyl chains.

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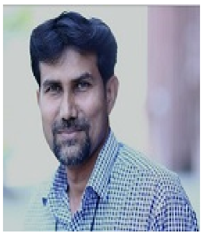

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