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A CRANK-NICOLSON TYPE HIGHER ORDER COMPACT EXPONENTIAL SCHEME FOR SOLVING VARIABLE ORDER TIME FRACTIONAL MOBILE-IMMOBILE ADVECTION-DISPERSION MODEL

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Abstract. This paper gives the construction and analysis of a new numerical method to solve variable order time fractional mobile-immobile transport model. The proposed numerical method is based on Crank-Nicolson approach with fourth order accurate compact exponential scheme for spatial discretization. We prove that the new scheme is uniquely solvable and unconditionally stable. Furthermore, some numerical results are presented to check the effectiveness of the proposed scheme and consistency of computational results with the theoretical findings.

Keywords: Fractional order, advection-dispersion, mobile-immobile, stability, convergence.

AMS Subject Classification: 35R11, 65M12, 65M06.

1. Introduction

Recently many complex phenomena with anomalous diffusion and transport dynamics are effectively described by mathematical models using fractional derivatives than the conventional integer derivatives. Numerous physical processes have been recently discovered to display the fractional order behaviour [5, 8]. Since the models defined by the variable order fractional derivatives are extremely complex and challenging to handle analytically, it is desirable to find their solutions numerically due to the quick evolution of their applications. Classical advection-diffusion models are used to describe the transport in porous media in past, but recently this phenomena is effectively described by time fractional mobile-immobile advection-dispersion models consisting of both integer and fractional time derivatives that describles the transport in mobile and immobile regions respectively [17]. The fractional order mobile-immobile model was first developed by Schumer et al [17]. In [24], this model is used to simulate solute concentrations for an experiment along the river Dee. This model characterizes a variety of issues in physical and mathematical systems,

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including heat diffusion and ocean acoustic propagation, which behave essentially like heat dripping through a solid. The connectivity and spatial heterogeneity can be better represented by the transportation model in terms of solute movement in the total network, the anomalous transport based on connectivity is studied in [20]. Hydrologists researching the movement of water in saturated and unsaturated areas [3] have lauded mobile-immobile models in great detail. Some successful applications of mobile-immobile models to explain the anomalous movement in heterogeneous porous media can be seen in [1, 9, 21]. An alternative mobile–immobile model is employed for describing solute transport in both porous and fractured media in [2]. Gao et al [4], introduced a mobile–immobile model with an asymptotic scale-dependent dispersion function. The exploration of numerical techniques for the fractional order mobile-immobile equation has received a lot of attention in recent decades. The authors in [15] developed an unconditionally stable method for fractional mobile-immobile equation. Liu and Li [10] provide a new second-order discrete finite difference technique based on an equivalent transformative Caputo formulation for a fractal mobile-immobile transport model. Yu et al [22] introduced unconditionally stable schemes with second-order and fourth-order spatial accuracy. Kanth and Deepika [16] proposed a scheme by approximating spatial and temporal derivatives based on parametric spline and quadrature formula, and the time fractional derivative in the Caputo sense. The authors in [12] developed a fast numerical technique which is based on a fast Fourier transform for time fractional mobile-immobile advection–diffusion model. Recently in [13], a new scheme based on Crank–Nicolson difference method is constructed to solve time fractional mobile-immobile advection–diffusion model in which the fractional derivative is approximated using the Atangana–Baleanu Caputo fractional derivative. In literature, there are a few numerical schemes for mobile-immobile model with time variable fractional order. The variable order time fractional advection–dispersion equation reads as:

$$\frac{\partial y}{\partial t} + \frac{\partial^\gamma y}{\partial t^\gamma} + a\frac{\partial y}{\partial x} = d\frac{\partial^2 y}{\partial x^2} + f(x,t),\ 0 \le x \le L,\ 0 \le t \le T. \tag{1}$$

with the initial condition $u(x,0) = \phi(x), 0 \le x \le L$, and boundary conditions $u(0,t) = u(L,t) = 0,\ a \neq 0,\ d > 0,\ 0 < \gamma(x,t) < 1$. In [23], a scheme is derived based on BDF, central differences in space and an implicit Euler in time for solving (1). A Crank-Nicolson approach was utilised by Liu and Li [11] for numerically solving this model. There are other numerical methods like, Jiang and Liu [6] used the collocation method and reproducing Kernal theory to solve the time variable fractional order mobile-immobile advection-dispersion model. A numerical simulation of this model based on a finite difference scheme in time and Legendre spectral method in space is taken into account by Pourbashash [14]. In this paper, we derive a new higher order accurate numerical method based on Crank-Nicolson approach to solve (1) subject to initial and boundary conditions. The time fractional derivative is considered in the Caputo sense, the integer derivative with central difference and the spatial part is discretized with the fourth order compact exponential method. It has been demonstrated that our method is unconditionally stable, uniquely solvable and also the convergence analysis is performed. To illustrate the convergence and correctness of the new method, some test problems are presented.

This paper is organized as follows. In Section 2, we derive a new numerical technique for solving equation (1) by first applying a fourth order exponential compact scheme for the steady state problem. Then a Crank-Nicolson approach is adopted to derive the new scheme for the unsteady case. In Section 3, the uniqueness of the solution is proved. Also the convergence and unconditional stability are verified using the Fourier analysis. In

Section 4, the theoretical deductions are supported by a few numerical examples and the results are compared with some existing methods. In the end, conclusions are made.

2. Derivation of the numerical scheme

Let $\Omega = 0 \leq x \leq L$ and $h = x_{i+1} - x_i,\ i = 0, 1, 2, \ldots M-1$. Divide the time interval $[0, T]$ uniformly with step size $\tau = \frac{t_n}{n}$; t_n is the n^{th} time level.

Definition 2.1. *The fractional derivative of order $\tau^{2-\gamma}$ at $t = t_{n+\frac{1}{2}}$ is defined by*

$$\left.\frac{\partial^\gamma y}{\partial t^\gamma}\right|_{(x_i, t_{n+\frac{1}{2}})} = \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\left[g_1 y_i^n + \sum_{l=1}^{n-1}\left(g_{n-l+1} - g_{n-l}\right) y_i^l - g_n y_i^0 + \frac{y_i^{n+1} - y_i^n}{2^{1-\gamma}}\right] + O\left(\tau^{2-\gamma}\right), \tag{2}$$

where $g_j = (j+\frac{1}{2})^{1-\gamma} - (j-\frac{1}{2})^{1-\gamma}$ which satisfies $g_1 \geq g_2 \geq g_3 \geq \ldots \geq 0$, (see [7]).

The derivation of similar type of approximation of fractional derivative for solving time fractional diffusion-wave system can be found in [18].
First, consider the steady state equation of (1), given by

$$a\frac{\partial y}{\partial x} - d\frac{\partial^2 y}{\partial x^2} = F, \text{ in } \Omega. \tag{3}$$

The fourth order exponential compact scheme [19] for (3) is

$$-\omega\delta_x^2 y_i + a\delta_x y_i = F_i + \omega_1\delta_x F_i + \omega_2\delta_x^2 F_i, \tag{4}$$

where

$$\omega = \frac{ah}{2}\coth\left(\frac{ah}{2d}\right), \quad \omega_1 = \frac{d-\omega}{a}, \quad \omega_2 = \frac{d(d-\omega)}{a^2} + \frac{h^2}{6},$$

δ_x and δ_x^2, for $x \in (x_{i-1}, x_{i+1}),\ i = 1, 2, ..., M-1$, are the standard second order central difference operators for first and second order derivatives respectively. Replacing F with $-\frac{\partial y}{\partial t} - \frac{\partial^\gamma y}{\partial t^\gamma} + f$ in equation (4), a fourth order semi-discrete approximation for (1) is obtained as

$$\begin{aligned}&\left(\frac{\omega_2}{h^2} - \frac{\omega_1}{2h}\right)\left(\frac{\partial y}{\partial t} + \frac{\partial^\gamma y}{\partial t^\gamma} - f\right)_{i-1}^{n+\frac{1}{2}} + \left(1 - \frac{2\omega_2}{h^2}\right)\left(\frac{\partial y}{\partial t} + \frac{\partial^\gamma y}{\partial t^\gamma} - f\right)_{i}^{n+\frac{1}{2}} \\ &+ \left(\frac{\omega_2}{h^2} + \frac{\omega_1}{2h}\right)\left(\frac{\partial y}{\partial t} + \frac{\partial^\gamma y}{\partial t^\gamma} - f\right)_{i+1}^{n+\frac{1}{2}} = \left(\frac{\omega}{h^2} + \frac{a}{2h}\right) y_{i-1}^{n+\frac{1}{2}} - \frac{2\omega}{h^2} y_i^{n+\frac{1}{2}} + \left(\frac{\omega}{h^2} - \frac{a}{2h}\right) y_{i+1}^{n+\frac{1}{2}}.\end{aligned} \tag{5}$$

The time derivatives of integer order in (5) are replaced with central differences and the fractional order is approximated by the definition 2.1. After rearranging the terms, the scheme for (1) is obtained as the following system of equations

$$\begin{aligned}&\left[\frac{1}{2}\left(\frac{\omega}{h^2} - \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} + \frac{\omega_1}{2h}\right)\right] y_{i+1}^1 + \left[-\frac{\omega}{h^2} - \mu\left(1 - \frac{2\omega_2}{h^2}\right)\right] y_i^1 \\ &+ \left[\frac{1}{2}\left(\frac{\omega}{h^2} + \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} - \frac{\omega_1}{2h}\right)\right] y_{i-1}^1 = \left[-\frac{1}{2}\left(\frac{\omega}{h^2} - \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} + \frac{\omega_1}{2h}\right)\right] y_{i+1}^0 \\ &+ \left[\frac{\omega}{h^2} - \mu\left(1 - \frac{2\omega_2}{h^2}\right)\right] y_i^0 + \left[-\frac{1}{2}\left(\frac{\omega}{h^2} + \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} - \frac{\omega_1}{2h}\right)\right] y_{i-1}^0 \\ &- \left[\left(\frac{\omega_2}{h^2} + \frac{\omega_1}{2h}\right) f_{i+1}^{\frac{1}{2}} + \left(1 - \frac{2\omega_2}{h^2}\right) f_i^{\frac{1}{2}} + \left(\frac{\omega_2}{h^2} - \frac{\omega_1}{2h}\right) f_{i-1}^{\frac{1}{2}}\right], 1 \leq i \leq M-1,\end{aligned} \tag{6}$$

and

$$\left[\frac{1}{2}\left(\frac{\omega}{h^2}-\frac{a}{2h}\right)-\mu\left(\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right)\right]y_{i+1}^{n+1}+\left[-\frac{\omega}{h^2}-\mu\left(1-\frac{2\omega_2}{h^2}\right)\right]y_i^{n+1}$$
$$+\left[\frac{1}{2}\left(\frac{\omega}{h^2}+\frac{a}{2h}\right)-\mu\left(\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right)\right]y_{i-1}^{n+1}=\left[-\frac{1}{2}\left(\frac{\omega}{h^2}-\frac{a}{2h}\right)-(\mu-g_1)\left(\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right)\right]y_{i+1}^{n}$$
$$+\left[\frac{\omega}{h^2}-(\mu-g_1)\left(1-\frac{2\omega_2}{h^2}\right)\right]y_i^n+\left[-\frac{1}{2}\left(\frac{\omega}{h^2}+\frac{a}{2h}\right)-(\mu-g_1)\left(\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right)\right]y_{i-1}^n$$
$$+\sum_{l=1}^{n-1}(g_{n-l+1}-g_{n-l})\left[\left(\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right)y_{i+1}^l+\left(1-\frac{2\omega_2}{h^2}\right)y_i^l+\left(\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right)y_{i-1}^l\right]$$
$$-g_n\left[\left(\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right)y_{i+1}^0+\left(1-\frac{2\omega_2}{h^2}\right)y_i^0+\left(\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right)y_{i-1}^0\right]$$
$$-\left[\left(\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right)f_{i+1}^{n+\frac{1}{2}}+\left(1-\frac{2\omega_2}{h^2}\right)f_i^{n+\frac{1}{2}}+\left(\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right)f_{i-1}^{n+\frac{1}{2}}\right],1\le i\le M-1,1\le n\le N, \tag{7}$$

where $\mu=\dfrac{1}{\tau}+\dfrac{\tau^{-\gamma}}{2^{1-\gamma}\Gamma(2-\gamma)}$. The solution of equations (6)-(7) for $0<\gamma(x,t)<1$, gives the solution to equation (1).

3. Analysis of the proposed scheme

In this section, we provide the mathematical analysis of the proposed scheme. First, we show that the scheme (6) and (7) for (1) is uniquely solvable.

Theorem 3.1. *The proposed new scheme* (6)-(7) *has a unique solution.*

Proof. The linear system (6)-(7) which is expressed as in the following matrix form has unique solution if the matrix A is invertible. Let

$$AY^{n+1}=\sum_{l=0}^{n}B_lY^l+C^{n+\frac{1}{2}},$$

where $A=tridiag(L,D,U)$, with $L=\frac{1}{2}\left(\frac{\omega}{h^2}+\frac{a}{2h}\right)-(\mu-g_1)\left(\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right)$,
$D=\frac{\omega}{h^2}-(\mu-g_1)\left(1-\frac{2\omega_2}{h^2}\right)$ and $U=\frac{1}{2}\left(\frac{\omega}{h^2}-\frac{a}{2h}\right)-(\mu-g_1)\left(\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right)$.

Note that

$$\frac{\omega}{h^2}\pm\frac{a}{2h}=\frac{d}{h^2}\left(\frac{ah}{2d}\coth\left(\frac{ah}{2d}\right)\pm\frac{ah}{2d}\right)\ge 0.$$

Using Lemmas 1 and 3 in [19], we get

$$\left|\frac{\omega}{h^2}+\frac{a}{2h}\right|+\left|\frac{\omega}{h^2}-\frac{a}{2h}\right|=\left|\frac{2\omega}{h^2}\right|=\frac{2\omega}{h^2}.$$

Also since, $1-\frac{2\omega_2}{h^2}=\left|1-\frac{2\omega_2}{h^2}\right|>\left|\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right|+\left|\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right|$, we obtain

$$\begin{aligned}|L|+|U|&\le|\mu-g_1|\left(\left|\frac{\omega_2}{h^2}+\frac{\omega_1}{2h}\right|+\left|\frac{\omega_2}{h^2}-\frac{\omega_1}{2h}\right|\right)+\frac{1}{2}\left(\left|\frac{\omega}{h^2}+\frac{a}{2h}\right|+\left|\frac{\omega}{h^2}-\frac{a}{2h}\right|\right)\\&<|\mu-g_1|\left|\left(1-\frac{2\omega_2}{h^2}\right)\right|+\left|\frac{2\omega}{h^2}\right|=\left|(\mu-g_1)\left(1-\frac{2\omega_2}{h^2}\right)+\frac{2\omega}{h^2}\right|=|D|.\end{aligned} \tag{8}$$

Hence the matrix A is strictly diagonaly dominant. This completes the proof. $\square$

3.1. **Stability analysis.** In this section, we provide the stability analysis of the proposed scheme (6)-(7). In the beginning we give some preliminaries and notations required in the subsequent analysis. We denote

$$\begin{aligned}
s_1 &= \frac{1}{2}\left(\frac{\omega}{h^2} - \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} + \frac{\omega_1}{2h}\right),\ s_2 = -\frac{\omega}{h^2} - \mu\left(1 - \frac{2\omega_2}{h^2}\right),\\
s_3 &= \frac{1}{2}\left(\frac{\omega}{h^2} + \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} - \frac{\omega_1}{2h}\right), z_1 = -\frac{1}{2}\left(\frac{\omega}{h^2} - \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} + \frac{\omega_1}{2h}\right),\\
z_2 &= \frac{\omega}{h^2} - \mu\left(1 - \frac{2\omega_2}{h^2}\right),\ z_3 = -\frac{1}{2}\left(\frac{\omega}{h^2} + \frac{a}{2h}\right) - \mu\left(\frac{\omega_2}{h^2} - \frac{\omega_1}{2h}\right),\\
w_1 &= \left(\frac{\omega_2}{h^2} + \frac{\omega_1}{2h}\right),\ w_2 = \left(1 - \frac{2\omega_2}{h^2}\right),\ w_3 = \left(\frac{\omega_2}{h^2} - \frac{\omega_1}{2h}\right).
\end{aligned} \tag{9}$$

Let $\tilde{Y}_j^n$ be the approximate solution of (6)-(7) and denote, $\rho_j^n = Y_j^n - \tilde{Y}_j^n,\ 1 \le j \le M-1,\ 0 \le n \le N$, with corresponding vector $\rho^n = \left(\rho_1^n, \rho_2^n, \dots, \rho_{M-1}^n\right)^T$. Define $\rho^n(x) : [0, L] \to \mathbb{R}$ as

$$\rho^n(x) = \begin{cases} \rho_j^n, & \text{in } x_{j-\frac{h}{2}} < x \le x_{j+\frac{h}{2}},\ 1 \le j \le M-1,\\ 0, & \text{in } 0 \le x \le h/2 \text{ and } L - h/2 < x \le L. \end{cases}$$

The Fourier series for $\rho^n(x)$ can be expressed as $\rho^n(x) = \sum_{l=-\infty}^{\infty} \xi^n(l) e^{2\pi i l x/L}$. For any vector $y \in \mathbb{R}^{M-1}$, the discrete l^2 norm is considered as $\|y\|_{l^2} = \left(h \sum_{j=1}^{M-1} y_j^2\right)^{\frac{1}{2}}$. The discrete Fourier coefficients are given by $\xi^n(l) = \frac{1}{L}\int_0^L \rho^n(x) e^{-2\pi i l x/L} dx$. The Parseval's equality for the discrete Fourier transform is

$$\|\rho^n\|_{l^2}^2 = \sum_{j=1}^{M-1} h\left|\rho_j^n\right|^2 = \int_0^L |\rho^n(x)|^2\, dx = \sum_{l=-\infty}^{\infty} |\xi^n(l)|^2.$$

Suppose $\rho_j^n = \xi^n e^{i\sigma j h}$, where $\sigma = 2\pi l/L$. Then the approximate error equation for (6) can be obtained as

$$s_1\rho_{j+1}^1 + s_2\rho_j^1 + s_3\rho_{j-1}^1 = z_1\rho_{j+1}^0 + z_2\rho_j^0 + z_3\rho_{j-1}^0, \quad \text{for } n = 1, \tag{10}$$

is transformed to

$$\xi^1\left((s_1 + s_3)\cos\sigma h + s_2 + i\,(s_1 - s_3)\sin\sigma h\right) = \left((z_1 + z_3)\cos\sigma h + z_2 + i\,(w_1 - z_3)\sin\sigma h\right)\xi^0. \tag{11}$$

Similarly the error equation for (7) can be obtained as

$$\begin{aligned}
s_1\rho_{j+1}^{n+1} + s_2\rho_j^{n+1} + s_3\rho_{j-1}^{n+1} &= \left(z_1\rho_{j+1}^n + z_2\rho_j^n + z_3\rho_{j-1}^n\right) + \ a_1\left(w_1\rho_{j+1}^n + w_2\rho_j^n + w_3\rho_{j-1}^n\right)\\
+ \sum_{l=1}^{n-1}(g_{n-l+1} - g_{n-l})\left(w_1\rho_{j+1}^l + w_2\rho_j^l + w_3\rho_{j-1}^l\right) &- \ a_n\left(w_1\rho_{j+1}^0 + w_2\rho_j^0 + w_3\rho_{j-1}^0\right),
\end{aligned} \tag{12}$$

for $n \geq 1$, is transformed to

$$
\begin{aligned}
&((s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h)\xi^{n+1} \\
&\qquad = \Bigg[\left[(z_1+z_3)\cos\sigma h + z_2 + i\,(z_1-z_3)\sin\sigma h\right]\xi^n \\
&\qquad + g_1\left[(w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h\right]\xi^n \\
&\qquad - g_n\left[(w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h\right]\xi^0 \\
&\qquad + \sum_{l=1}^{n-1}(g_{n-l+1}-g_{n-l})\left[(w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h\right]\xi^l\Bigg].
\end{aligned} \tag{13}
$$

Theorem 3.2. *The proposed new scheme* (6)-(7) *is unconditionally stable.*

Proof. Denote $v = \dfrac{ah}{2d}$, then $\omega = dv\coth(v)$, $\omega_2 = \left(\dfrac{d}{a}\right)^2(1 - v\coth(v)) + \dfrac{h^2}{6}$ and consider

$$
\begin{aligned}
&|(s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h| - |(z_1+z_3)\cos\sigma h + z_2 + i\,(z_1-z_3)\sin\sigma h| \\
&= \left[\left(\frac{\omega}{h^2} - \frac{2\omega_2}{h^2}\mu\right)\cos\sigma h - \left(\frac{\omega}{h^2} + \left(1 - \frac{2\omega_2}{h^2}\right)\mu\right)\right]^2 + \left(\frac{a}{2h} + \frac{\omega_1}{h}\mu\right)^2\sin\sigma^2 h \\
&- \left[\left[-\left(\frac{\omega}{h^2} + \frac{2\omega_2}{h^2}\mu\right)\cos\sigma h + \left(\frac{\omega}{h^2} - \left(1 - \frac{2\omega_2}{h^2}\right)\mu\right)\right]^2 + \left(\frac{a}{2h} - \frac{\omega_1}{h}\mu\right)^2\sin\sigma^2 h\right] \\
&= 8\left(\frac{\omega\omega_2}{h^4}\mu\cos\sigma h\right)(1-\cos\sigma h) + 4\mu\frac{\omega}{h^2}\left(1 - \frac{2\omega_2}{h^2}\right)(1-\cos\sigma h) + 2\frac{a}{h^2}\omega_1\mu\sin^2\sigma h \\
&= 2\sin^2\frac{\sigma h}{2}\mu\left(8\frac{\omega\omega_2}{h^4}\cos\sigma h + 4\frac{\omega}{h^2}\left(1 - \frac{2\omega_2}{h^2}\right)\right) + 8\mu\frac{d-\omega}{h^2}\sin^2\frac{\sigma h}{2}\cos^2\frac{\sigma h}{2} \\
&= \frac{8\mu}{h^2}\sin^2\frac{\sigma h}{2}\left(\omega\left(1 - \frac{4\omega_2}{h^2}\sin^2\frac{\sigma h}{2}\right) + (d-\omega)\cos^2\frac{\sigma h}{2}\right) \\
&= \frac{8\mu}{h^2}\sin^2\frac{\sigma h}{2}\left(\omega\left(1 - 4\frac{\omega_2}{h^2}\right)\sin^2\left(\frac{\sigma h}{2}\right) + d\cos^2\left(\frac{\sigma h}{2}\right)\right) \\
&\geq \frac{8\mu}{h^2}\sin^2\frac{\sigma h}{2}\left(du\coth(v)\left(\frac{1}{3} + 4\left(\frac{d}{ah}\right)^2(v\coth(v) - 1)\right)\sin^2\left(\frac{\sigma h}{2}\right)\right) \geq 0.
\end{aligned}
$$

Hence

$$
\frac{|(z_1+z_3)\cos\sigma h + z_2 + i\,(z_1-z_3)\sin\sigma h|}{|(s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h|} \leq 1. \tag{14}
$$

Therefore by (11), we have $|\xi^1| \leq |\xi^0|$. With the assumption, $|\xi^n| \leq \left|\xi^0\right|,\ n = 1, 2, \ldots, k$ and equation (13), we can obtain

$$
\begin{aligned}
\left|\xi^{k+1}\right| \leq &\frac{1}{|(s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h|}\Bigg[|\,(z_1+z_3)\cos\sigma h + z_2 \\
&+ i\,(z_1-z_3)\sin\sigma h||\xi^k| + \Bigg|g_1((w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h)(\xi^k) \\
&+ \sum_{l=1}^{k-1}(g_{k-l+1}-g_{k-l})\,((w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h)(\xi^l) \\
&- g_k((w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h)(\xi^0)\Bigg|\Bigg]
\end{aligned}
$$

$$=\left|\frac{(z_1+z_3)\cos\sigma h+z_2+i\,(z_1-z_3)\sin\sigma h}{(s_1+s_3)\cos\sigma h+s_2+i\,(s_1-s_3)\sin\sigma h}\right|\left|\xi^0\right|\le\left|\xi^0\right|.$$

Thus by induction, we can say that $|\xi^n|\le|\xi^0|$ for all n. Hence, we obtain

$$\|\rho^n\|_{l^2}^2=h\sum_{j=1}^{M-1}\left|\rho_j^n\right|^2=\frac{h}{L}\sum_{j=1}^{M-1}\left|\xi^n e^{i\sigma jh}\right|^2\le\frac{h}{L}\sum_{j=1}^{M-1}\left|\xi^0\right|^2=\frac{h}{L}\sum_{j=1}^{M-1}\left|\xi^0 e^{i\sigma jh}\right|^2=\left\|\rho^0\right\|_{l^2}^2,$$

$n=1,2,\ldots,N$. This completes the proof. □

3.2. **Convergence analysis.** This section uses a Fourier analysis approach to verify the convergence of the suggested compact numerical technique. Use $e_j^n=y_j^n-Y_j^n$, $1\le j\le M-1$ and $1\le n\le N$, to denote the error in the numerical solution. The error equation at $t=t_1$ is

$$s_1e_{j+1}^1+s_2e_j^1+s_3e_{j-1}^1=\left(z_1e_{j+1}^0+z_2e_j^0+z_3e_{j-1}^0\right)+R_j^{\frac{1}{2}}, \tag{15}$$

and for $n\ge1$, we have

$$\begin{aligned}&s_1e_{j+1}^{n+1}+s_2e_j^{n+1}+s_3e_{j-1}^{n+1}\\&\quad=\left(z_1e_{j+1}^n+z_2e_j^n+z_3e_{j-1}^n\right)+\sum_{l=1}^{n-1}\left(g_{n-l+1}-g_{n-l}\right)\left(w_1e_{j+1}^l+w_2e_j^l+w_3e_{j-1}^l\right)\\&\quad+\ g_1\left(w_1e_{j+1}^n+w_2e_j^n+w_3e_{j-1}^n\right)-\ g_n\left(w_1e_{j+1}^0+w_2e_j^0+w_3e_{j-1}^0\right)+R_j^{n+\frac{1}{2}}.\end{aligned} \tag{16}$$

Due to the initial and boundary conditions, we have $e_j^0=0,\ 0\le j\le M$ and $e_0^n=e_M^n=0,\ 0\le n\le N$. Define $e^n(x):[0,L]\to\mathbb{R}$ and $R^{n+\frac{1}{2}}(x):[0,L]\to\mathbb{R}$ as

$$e^n(x)=\begin{cases}e_j^n, & x_{j-\frac{h}{2}}<x\le x_{j+\frac{h}{2}},\ 1\le j\le M-1,\\ 0, & 0\le x\le\frac{h}{2}\text{ and }L-\frac{h}{2}<x\le L,\end{cases} \tag{17}$$

and

$$R^{n+\frac{1}{2}}(x)=\begin{cases}R_j^{n+\frac{1}{2}}, & x_{j-\frac{h}{2}}<x\le x_{j+\frac{h}{2}},\ 1\le j\le M-1,\\ 0, & 0\le x\le\frac{h}{2}\text{ and }L-\frac{h}{2}<x\le L,\end{cases} \tag{18}$$

Consequently, the Fourier series for $e^n(x)$ and $R^{n+\frac{1}{2}}(x)$ can be described as

$$e^n(x)=\sum_{l=-\infty}^{\infty}\eta^n(l)e^{2\pi ilx/L}\quad\text{and}\quad R^{n+\frac{1}{2}}(x)=\sum_{l=-\infty}^{\infty}\zeta^{n+\frac{1}{2}}(l)e^{2\pi ilx/L}, \tag{19}$$

where

$$\eta^n(l)=\frac{1}{L}\int_0^L e^n(x)e^{-2\pi ilx/L}dx\quad\text{and}\quad\zeta^{n+\frac{1}{2}}(l)=\frac{1}{L}\int_0^L R^{n+\frac{1}{2}}(x)e^{-2\pi ilx/L}dx.$$

For $e^n\in\mathbb{R}^{M-1}$ and $R^{n+\frac{1}{2}}\in\mathbb{R}^{M-1}$, we have

$$\|e^n\|_{l^2}^2=\sum_{n=1}^{M-1}h\left|e_j^n\right|^2=\int_0^L\left|e^n(x)\right|^2dx=\sum_{l=-\infty}^{\infty}\left|\eta^n(l)\right|^2, \tag{20}$$

$$\left\|R^{n+\frac{1}{2}}\right\|_{l^2}^2=\sum_{n=1}^{M-1}h\left|R_j^{n+\frac{1}{2}}\right|^2=\int_0^L\left|R^{n+\frac{1}{2}}(x)\right|^2dx=\sum_{l=-\infty}^{\infty}\left|\zeta^{n+\frac{1}{2}}(l)\right|^2. \tag{21}$$

By equation (21), there exists a constant $K_2>0$, such that

$$|\zeta^{n+\frac{1}{2}}|=|\zeta^{n+\frac{1}{2}}(l)|\le\frac{1}{3}K_2|\zeta^{\frac{1}{2}}(l)|=\frac{1}{3}K_2|\zeta^{\frac{1}{2}}|,\ n=0,1,2,\ldots,N-1. \tag{22}$$

Let $\sigma = 2\pi l/L$, substituting $e_j^n = \eta^n e^{i\sigma jh},\ R_j^{n+\frac{1}{2}} = \zeta^{n+\frac{1}{2}} e^{i\sigma jh}$ in equations (15) and (16) and by noting that $\eta_0 = 0$, we get

$$\eta^1((s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h) = \zeta^{\frac{1}{2}},\ \text{for } t=t_1,\ \text{and}$$

$$\eta^{n+1}((s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h) = \Bigg[[(z_1+z_3)\cos\sigma h + z_2 + i\,(z_1-z_3)\sin\sigma h]\eta^n + \ g_1\,((w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h)\,\eta^n + \sum_{l=1}^{n-1}(g_{n-l+1}-g_{n-l})\,[(w_1+w_3)\cos\sigma h + w_2 + i\,(w_1-w_3)\sin\sigma h]\,\eta^l + \zeta^{n+\frac{1}{2}}\Bigg], \text{for } n \ge 1.$$

Lemma 3.1. *The following inequality holds for* $0 < \tau < 1$,

$$|\,(s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h| \ge \frac{1}{3}. \tag{23}$$

Proof. For $\tau \in (0,1)$, we have

$$\begin{aligned}
&|\,(s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h| \\
&= \left|\left(\frac{\omega}{h^2} - \frac{2\omega_2}{h^2}\mu\right)\cos\sigma h + \left(\frac{\omega}{h^2} - \left(1 - \frac{2\omega_2}{h^2}\right)\mu\right) - i\left(\frac{a}{2h} + \frac{\omega_1}{h}\mu\right)\sin\sigma h\right| \\
&\ge \left|\left(\frac{\omega}{h^2} - \frac{2\omega_2}{h^2}\mu\right)\cos\sigma h - \left(\frac{\omega}{h^2} + \left(1 - \frac{2\omega_2}{h^2}\right)\mu\right)\right| \\
&= \left|(1-\cos\sigma h)\left(\frac{2\omega_2}{h^2}\mu - \frac{\omega}{h^2}\right) - \mu\right| \\
&= \left|(1-\cos\sigma h)\left[\frac{2\mu}{h^2}\left(\left(\frac{d}{a}\right)^2(1 - v\coth v) + \frac{h^2}{6}\right) - \frac{\omega}{h^2}\right] - \mu\right| \\
&= \left|(1-\cos\sigma h)\frac{2\mu}{h^2}\left(\frac{d}{a}\right)^2(v\coth v - 1) + \mu\left(1 - \frac{(1-\cos\sigma h)}{3}\right) + \omega\frac{(1-\cos\sigma h)}{h^2}\right| \\
&\ge \left|\mu\left(1 - \frac{(1-\cos\sigma h)}{3}\right)\right| \ge \frac{1}{3\tau} > \frac{1}{3}.
\end{aligned}$$

This completes the proof. □

Lemma 3.2. *The following inequality holds*

$$|\eta^{n+1}| \le K_2((n+1)\tau)|\zeta^{\frac{1}{2}}|,\ n = 0,1,2,\ldots \tag{24}$$

Proof. For $n = 0$, by means of of Lemma 3.1 and the inequality (22), we have

$$|\eta^1| = \frac{|\zeta^{\frac{1}{2}}|}{|\,(s_1+s_3)\cos\sigma h + s_2 + i\,(s_1-s_3)\sin\sigma h|} \le 3|\zeta^{\frac{1}{2}}| \le K_2\tau|\zeta^{\frac{1}{2}}|. \tag{25}$$

For sufficiently large n, assume $\left|\eta^j\right| \leq jK_2\tau\left|\zeta^{\frac{1}{2}}\right|$ for $j = 1, 2, , \cdots, n$. Then by using (2.1), Lemma 3.1 and the inequality (22), we get

$$\begin{aligned}
\left|\eta^{n+1}\right| \leq & \frac{1}{\left|\left(s_1+s_3\right)\cos\sigma h+s_2+i\left(s_1-s_3\right)\sin\sigma h\right|}\Bigg[\left|\left(z_1+z_3\right)\cos\sigma h+z_2\right. \\
& \left.+i\left(z_1-z_3\right)\sin\sigma h\right|\left|\eta^n\right|+\Bigg|g_1\left(\left(w_1+w_3\right)\cos\sigma h+w_2+i\left(w_1-w_3\right)\sin\sigma h\right)\eta^n \\
& +\sum_{l=1}^{n-1}\left(g_{n-l+1}-g_{n-l}\right)\left(\left(w_1+w_3\right)\cos\sigma h+w_2+i\left(w_1-w_3\right)\sin\sigma h\right)\eta^l\Bigg|+\left|\zeta^{n+\frac{1}{2}}\right|\Bigg] \\
& \leq \frac{K_2(n\tau)\left|\zeta^{\frac{1}{2}}\right|}{\left|\left(s_1+s_3\right)\cos\sigma h+s_2+i\left(s_1-s_3\right)\sin\sigma h\right|}\Big[\left|\left(z_1+z_3\right)\cos\sigma h+z_2+i\left(z_1-z_3\right)\sin\sigma h\right|\Big] \\
& +\frac{\left|\zeta^{n+\frac{1}{2}}\right|}{\left|\left(s_1+s_3\right)\cos\sigma h+s_2+i\left(s_1-s_3\right)\sin\sigma h\right|} \\
& \leq \frac{K_2(n\tau)\left|\zeta^{\frac{1}{2}}\right|}{\left|\left(s_1+s_3\right)\cos\sigma h+s_2+i\left(s_1-s_3\right)\sin\sigma h\right|}\Big[\left|\left(z_1+z_3\right)\cos\sigma h+z_2+i\left(z_1-z_3\right)\sin\sigma h\right|\Big] \\
& \quad +\frac{\frac{1}{3}K_2\tau\left|\zeta^{\frac{1}{2}}\right|}{\left|\left(s_1+s_3\right)\cos\sigma h+s_2+i\left(s_1-s_3\right)\sin\sigma h\right|} \\
& \leq K_2(n\tau)\left|\zeta^{\frac{1}{2}}\right|+K_2\tau\left|\zeta^{\frac{1}{2}}\right|=K_2((n+1)\tau)\left|\zeta^{\frac{1}{2}}\right|.
\end{aligned}$$

The proof is completed. □

Theorem 3.3. *The proposed exponential numerical scheme* (6)-(7) *is convergent of* $\mathcal{O}\left(\tau^{2-\gamma}+h^4\right)$.

Proof. Local truncation error of (6)-(7) is $R_j^{n+\frac{1}{2}} = \mathcal{O}\left(\tau^{2-\gamma}+h^4\right)$, hence there exists a constant $K_3 > 0$ satisfying

$$\left\|R^{n+\frac{1}{2}}\right\|_{l^2}^2=\sum_{j=1}^{M-1} h\left|R_j^{n+\frac{1}{2}}\right|^2 \leq LK_3^2\left(\tau^{2-\gamma}+h^4\right)^2,\ n=0,1,\ldots,N-1.$$

Since $n\tau \leq T$ and using Lemma 3.2, we obtain

$$\begin{aligned}
\left\|e^n\right\|_{l^2}^2 &= \sum_{l=-\infty}^{\infty}\left|\eta^n(l)\right|^2 \leq \sum_{l=-\infty}^{\infty} K_2^2(n\tau)^2\left|\zeta^{\frac{1}{2}}\right|^2 = K_2^2(n\tau)^2\sum_{l=-\infty}^{\infty}\left|\zeta^{\frac{1}{2}}\right|^2 \\
&= K_2^2(n\tau)^2\left\|R^{\frac{1}{2}}\right\|_{l^2}^2 \leq K_2^2(n\tau)^2 LK_3^2\left(\tau^{2-\gamma}+h^4\right)^2 \leq K^2\left(\tau^{2-\gamma}+h^4\right)^2.
\end{aligned}$$

where $K = K_2K_3T\sqrt{L}$, that is

$$\left\|e^n\right\|_{l^2} \leq K\left(\tau^{2-\gamma}+h^4\right),$$

which completes the proof. □

4. Numerical observations

This section provides some numerical examples to demonstrate the accuracy and reliability of the proposed scheme (6)-(7) for solving equation (1). We compare the accuracy of the proposed scheme with that of the methods in [11, 16].

4.1. **Example 1.** Consider the equation (1) with $a = 1,\ d = 1$ and $f = \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\sin(\pi x)$ $+\sin(\pi x) + \pi t\cos(\pi x) + \pi^2 t\sin(\pi x)$, subject to the initial and boundary conditions $y(x, 0) = 0,\ 0 \le x \le 1,\ y(0, t) = 0,\ y(1, t) = 0,\ t > 0$. The example above has an analytical solution, which is $y(x, t) = t\sin(\pi x)$. Table 1 displays the maximum absolute error for different fractional orders. This table gives a clear illustration that the results with proposed compact exponential method are better than spline approximation method given in [16] with $\alpha = \frac{1}{12}, \beta = \frac{5}{12}$ which is the best among the tabulated values of Table 1 in [16]. Fig. 1 presents the plots of the numerical and analytical solutions of Example 1 at different time levels and the space-time graph framed with $T = 1$ for $\gamma = 0.5$.

h	k	$\gamma = 0.75$			$\gamma = 0.95$		
		Ravi [16]	Present	Order	Ravi[16]	Present	Order
1/8	1/12	8.7632e-04	4.8407e-05		4.3792e-03	4.8305e-05	
1/16	1/36	2.1182e-04	3.0195e-06	4.0028	9.3521e-04	3.0131e-06	4.0029
1/32	1/108	5.2614e-05	1.8862e-07	4.0008	2.2423e-04	1.8822e-07	4.0008
1/64	1/324	1.3189e-05	1.1787e-08	4.0002	5.5504e-05	1.1762e-08	4.0002

TABLE 1. Maximum absolute error for Example 1 with $T = 1$.

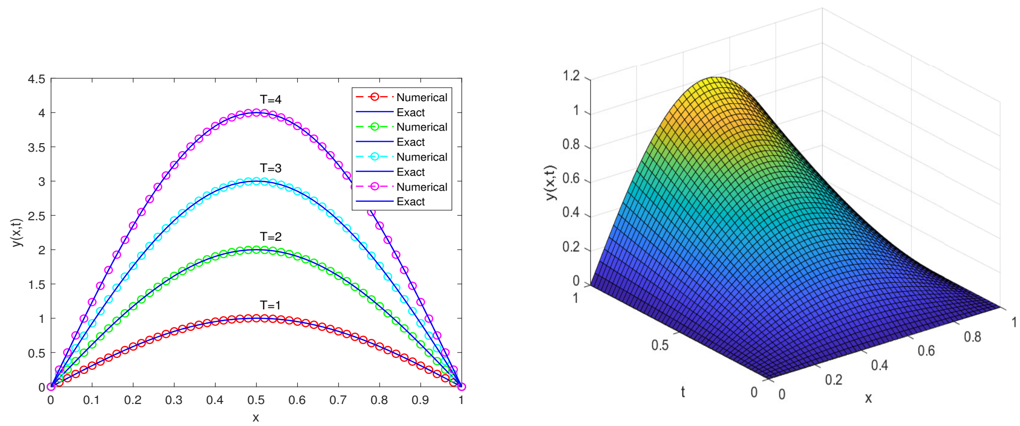

FIGURE 1. Numerical and exact solutions for various T (left) and space-time graph with $T = 1$ (right) of Example 1 with $h = 1/50, \tau = 1/50$ and $\gamma = 0.5$.

4.2. **Example 2.** Consider the equation (1) with $a = 1,\ d = 1$ and $f = 10(x - x^2)^2 + \dfrac{10(x - x^2)^2 t^{1-\gamma(x,t)}}{\Gamma(2 - \gamma(x, t))} + 20(t + 1)(x - 3x^2 + 2x^3) - 20(t + 1)(1 - 6x + 6x^2)$, subject to conditions $y(x, 0) = 10(x - x^2)^2,\ 0 \le x \le 1$ and $y(0, t) = 0,\ y(1, t) = 0,\ t > 0$. Let $\gamma(x, t) = 0.8 + 0.005\cos(xt)\sin(x)$, then the analytical solution is $y(x, t) = 10(x-x^2)^2(t+1)$. The discrete l^2 error for example 2 at time $T = 1$ with $h = \tau$ is displayed in Table 2. This table reveals that the proposed scheme betters the method used in [11]. The left of Fig. 2 depicts the solution behaviour at different time levels and the right shows the space-time graph with $T = 1$.

h	Liu [11]	Present	Order
1/8	5.1370e-02	7.7726e-05	
1/16	1.2871e-02	4.8588e-06	3.9997
1/32	3.2194e-03	3.0374e-07	3.9997
1/64	8.0498e-04	1.9019e-08	3.9973
1/128	2.0124e-04	1.2145e-09	3.9690

TABLE 2. Discrete l^2 error at $T = 1$ of Example 2.

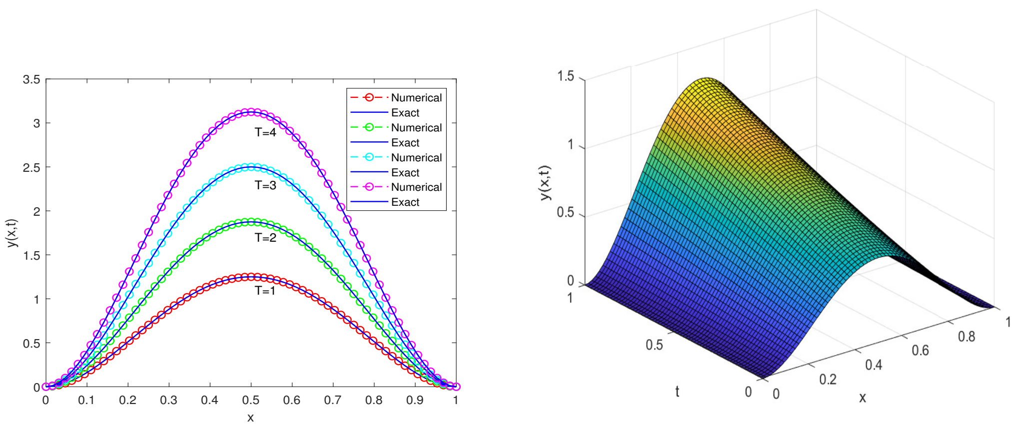

FIGURE 2. Numerical and exact solutions of Example 2 for various values of time T with $h = 1/16, \tau = 1/16$ (left). Space-time graph of Example 1 with $h = 1/32, \tau = 1/32$ and $T = 1$ (right).

4.3. **Example 3.** Consider the equation (1) with $a = 1$, $d = 1$ and $f = 5x(1-x) + \dfrac{5x(1-x)t^{1-\gamma(x,t)}}{\Gamma(2-\gamma(x,t))} + 5(t+1)(1-2x) + 10(t+1)$, subject to the boundary conditions $y(0,t) = 0$, $y(1,t) = 0$, $t > 0$ and initial condition $y(x,0) = 5x(1-x)$, $0 \leq x \leq 1$. Let $\gamma(x,t) = 0.8 + 0.005\cos(xt)\sin(x)$, then the analytical solution is $y(x,t) = 5(t+1)x(1-x)$. Table 3 shows the absolute error of example 3 with different space and time steps. This table provides a good demonstration of the accuracy of the proposed scheme.

	$h = \tau = 1/10$		$h = \tau = 1/100$	
x	$T = 0.1$	$T = 1$	$T = 0.1$	$T = 1$
0.1	8.2919e-07	1.0851e-07	5.2888e-09	9.3762e-10
0.2	9.3501e-07	9.8422e-08	6.5150e-09	1.0375e-09
0.3	6.4491e-07	6.2466e-08	4.7893e-09	6.2910e-10
0.4	1.7048e-07	4.9039e-09	1.2742e-09	1.6469e-11
0.5	3.4518e-07	5.2978e-08	2.8907e-09	5.4839e-10
0.6	7.9731e-07	8.9393e-08	6.6454e-09	8.8806e-10
0.7	1.0979e-06	9.2427e-08	9.0285e-09	9.1808e-10
0.8	1.1560e-06	6.5343e-08	9.1814e-09	6.6191e-10
0.9	8.5230e-07	3.5247e-08	6.3652e-09	2.6367e-10

TABLE 3. Absolute error for Example 3.

5. Conclusions

A new numerical scheme with fourth order accuracy in space based on Crank-Nicolson approach is derived in this work to solve the time fractional variable order mobile-immobile advection-dispersion model. We proved that the scheme is uniquely solvable, unconditionally stable and performed the convergence analysis. Numerical examples reveals the effectiveness of the proposed scheme and the compatibility with the theoretical results.

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