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ON EMBEDDING FAMILY OF NUMERICAL SCHEME FOR SOLVING NON-LINEAR EQUATIONS WITH ENGINEERING APPLICATIONS

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Abstract. The solution of non-linear equations is one of the most important and frequent problems in numerous engineering and scientific disciplines. Numerous real-world issues can be described using non-linear equations in a variety of scientific fields, including natural science, social science, electrical, chemical, and mechanical engineering, economics, statistics, weather forecasting, and particularly biomedical engineering. Iterative techniques must be used in order to solve such nonlinear problems. The majority of numerical methods required the first or higher derivative of the functions, whose computational cost is large and diverges if the slope of the functions at the beginning or some intermediate points approaches zero. To prevent this, we develop numerical methods that utilize the parameter embedding, also known as Homotopy methods, to find the root of nonlinear equations. Convergence analysis shows that the proposed family of methods' order of convergence is two. To determine the error equation of the proposed technique, the computer algebra system CAS-Maple is employed. To illustrate the accuracy, validity, and usefulness of the proposed technique, we consider a few real-world applications from the fields of civil and chemical engineering. In terms of residual error, computational time, computational order of convergence, efficiency, and absolute error, the test examples' acquired numerical results demonstrate that the newly proposed algorithm performs better than the other classical methods already existing in literature.

Keywords: Embedding parameter; Roots; Homotopy; Dynamical Plane; Convergence order.

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1. Introduction

Numerical iterative algorithms for computing approximate roots of nonlinear scalar equations are of significant importance in computational and applied mathematics due to their widespread applications in many fields of modern science, including engineering, mathematical chemistry, bio-mathematics, biomedical engineering, economics, linear programming, physics, and statistics. Most of the engineering problems turns out to solve nonlinear equations. We required an iterative algorithm to solve these types of technical and scientific problems because, in most situations, analytical techniques are ineffective. The Newton's method, a well-known and classical iterative algorithm, was presented by Newton [1] in the last decades of the fifteenth century and has been widely utilized for many years to solve non-linear equations. Recently, by utilizing a variety of mathematical procedures, researchers have presented a new variety of single-step and multi-step iterative methods and modified the Newton's iteration schemes. In 2007, Noor et al. [2] developed second derivative-free algorithm by modifying generalized one parameter Halley method. Iterative techniques requiring the higher derivative of the functions were developed by Gutierrez et al. [3] in 1997, Hernández et al. [4] in 2001, Kang et al. [5] in 2013, and many others.

These iterative numerical algorithms have the drawback of increasing computational costs, since higher-order derivatives are utilized. It is quite challenging to maintain a balance between computation cost and convergence rate because doing so would definitely cause the other to decrease. Second, the numerical technique diverges if the slope of the function at the starting or intermediate points approaches zero. Due to these restrictions, we seek for numerical techniques that consistently converge on the starting values while other methods fail. The main goal of this study is to develop a reliable numerical method that converges as the function or its derivative gets closer to zero at the initial values or some other intermediate values.

The main contributions of this research works are

- Construction of family of numerical scheme for finding roots of scalar nonlinear equations.
- Using CAS-Maple to verify the convergence order of the proposed method.
- Construction of numerical iterative scheme using embedding parameter method.
- Utilizing basins of attraction, perform a dynamic study of the proposed embedding approach.
- Computational tools are used to analyze the convergence rate, efficiency, stability, and applicability of the proposed technique.
- Analyzing how the suggested technique is applied in various research disciplines.

In 2007, Wu et al.[6] consider secant homotopy continuation method and in 2008 Wu [7] modified ancient Chinese algorithm into Homotopy continuation method. Among such methods, the most famous is classical Newton's method [8] has a local quadratic convergence for finding roots of

$$f(x) = 0, \tag{1}$$

given as:

$$y^{(u)} = x^{(u)} - \frac{f(x^{(u)})}{f'(x^{(u)})}. \tag{2}$$

Method (2) faces some difficulty when the first derivative $f'(x)$ is zero or near zero which rises a divergent iterative sequence. To overcome this problem we look towards homotopy continuation methods (HCM). Homotopy continuation method is very effective and a powerful tool to approximate the root of (1).

In 2005, Wu et al.[9] modify method (2) into homotopy method (abbreviated as NHCM) given as:

$$y^{(u)} = x^{(u)} - \frac{\text{Ħ}(x^{(u)}, t)}{\text{Ħ}'(x^{(u)}, t)}, \tag{3}$$

where $H(x^{(u)}, t)$ is known as homotopy function and continuous refer to an embedded $f(p)$ with auxilary homotopy function $g(p)$ in a single parameter family, say t, over the interval $t \in [0, 1]$ such that

$$\text{Ħ}(x^{(u)}, t) = tf(x^{(u)}) + (1 - t)g(x^{(u)}), \tag{4}$$

satisfying the following conditions

$$(i) \qquad \text{Ħ}(x^{(u)}, 0) = g(x^{(u)}), \tag{5}$$

$$(ii) \qquad \text{Ħ}(x^{(u)}, 1) = f(x^{(u)}), \tag{6}$$

where divergence of (3) occurs if

$$\text{Ħ}'(x^{(u)}, t) = 0 \text{ or } \text{Ħ}'(x^{(u)}, t) \to 0. \tag{7}$$

To avoid this, we have to assume that $\text{Ħ}'(x^{(u)}, t) \neq 0$ or $\text{Ħ}'(x^{(u)}, t) \nrightarrow 0$. Thus, we guaranteed that

$$tf'(x^{(u)}) + (1 - t)g'(x^{(u)}) \neq 0 \text{ or } \nrightarrow 0. \tag{8}$$

Therefore, we assume that

$$tf'(x^{(u)}) + (1 - t)g'(x^{(u)}) = C \tag{9}$$

or $Ce^{x^{(u)}}$ or $C\sin(x^{(u)})+D, ...,$ where $C, D, ...,$ are constant and $C = D, ..., \neq 0$. Integrating $tf'(x^{(u)}) + (1 - t)g'(x^{(u)})$, we have

$$tf(x^{(u)}) + (1 - t)g(x^{(u)}) = Cx^{(u)} + k, \tag{10}$$

or $Ce^{(u)}+k, ...,$ where k is a parameter and may be zero. For the removal of the divergence we have the choice to choose auxiliary function $g(x)$ such that $g(x)$ satisfies the following rules i.e.,

$$(i) For\ t = 0 \implies g(x) = Cx + k \text{ or } Ce^x + k \tag{11}$$

$$(ii) For\ t = 1 \implies f(x) = Cx + k \text{ or } Ce^x + k \tag{12}$$

$$(iii) For\ t \in (0, 1) \implies g(x) = C_1 f(x) + C_2 x + k \tag{13}$$

$$\text{or } C_1 f(x) + C_2 e^x + k, \ldots \tag{14}$$

where $C_1, C_2, \ldots$ are constants and not equal to zero.

The main aim of this article to construct two parameter family of iterative method which is more efficient as compared to classical methods. Dynamical planes provide better convergence region as compared to classical methods which contains derivatives and HCM methods. Using the computer program Maple 18.0, we compare the draw basins of attraction of the suggested embedded Homotopy numerical algorithm to those of the currently used methods for the graphical analysis. The graphical results show how quickly the created method converges, confirming its supremacy to other techniques.

2. Construction of HCM

In this section, first we proposed the following Newton type single root finding method:

$$y^{(u)} = x^{(u)} - \frac{f(x^{(u)})}{f'(x^{(u)})}\left(\frac{1}{1-\alpha\left(\frac{f(x^{(u)})}{\beta-\left(f(x^{(u)})\right)^2}\right)}\right), \tag{15}$$

where α, β are real parameters.

Convergence Analysis

Here, we prove the following theorem for iterative schemes (15).

Theorem 2.1. *Let $\zeta \in I$ be a simple root of a sufficiently differential functionf $: I \subseteq \Re \longrightarrow \Re$ is an open interval I. If x_0 is sufficiently close to ζ then the convergence order of the family of (15) is 2 and satisfying the error equation*

$$e_{i+1} = \left(c_2 - \frac{\alpha}{\beta}\right) e_i^2 + O(e_i^3), \tag{16}$$

where $c_m = \frac{f^m(\zeta)}{m!f'(\zeta)}; m \geq 2$.

Proof. Let ζ be a simple root of f and $x^{(u)} = \zeta + e_i$. By Taylor's series expansion of $f(x^{(u)})$ around $x = \zeta$, taking $f(\zeta) = 0$, we get:

$$f(x^{(u)}) = f^{'}(\zeta)(e_i + c_2e_i^2 + c_3e_i^3 + O(e_i^4), \tag{17}$$

and

$$f'(x^{(u)}) = f^{'}(\zeta)(1 + 2c_2e_i + 3c_3e_i^2 + ...). \tag{18}$$

Dividing (17) by (18), we have:

$$\frac{f(x^{(u)})}{f'(x^{(u)})} = e_i + c_2e_i^2 + (2c_2^2 - 4c_3)e_i^3 + ... \tag{19}$$

$$\left(f'(x^{(u)})\right)^2 = 1 + 4c_2e_i + (4c_2^2 + 6c_3)e_i^2 + ... \tag{20}$$

$$\begin{aligned} &\frac{\alpha f(x^{(u)})}{\beta + (f'(x^{(u)}))^2} \\ = \; &\frac{\alpha}{\beta}e_i + \frac{\alpha c_2}{\beta}e_i^2 + \left(\frac{\alpha c_3}{\beta} - \frac{\alpha}{\beta^2}\right) e_i^3 + ... \end{aligned} \tag{21}$$

$$\frac{f(x^{(u)})}{f'(x^{(u)})}\frac{\alpha f(x^{(u)})}{\beta + (f'(x^{(u)}))^2} = e_i + \left(-c_2 + \frac{\alpha}{\beta}\right) e_i^2 + ... \tag{22}$$

Thus,

$$y^{(u)} = \zeta + \left(c_2 - \frac{\alpha}{\beta}\right) e_i^2 + O(e_i^3), \tag{23}$$

$$e_{i+1} = \left(c_2 - \frac{\alpha}{\beta}\right) e_i^2 + O(e_i^3). \tag{24}$$

Hence, this proves the theorem. $\square$

Thus, homotopic form of the method (abbreviated as MHCM) is given by:

$$y^{(u)} = x_i^{(u)} - \frac{\text{Ħ}(x_i^{(u)}, t)}{\text{Ħ}'(x_i^{(u)}, t)} \left(\frac{1}{1 - \alpha \left(\frac{\text{Ħ}(x_i^{(u)}, t)}{\beta - \left(\text{Ħ}(x_i^{(u)}, t) \right)^2} \right)} \right), \tag{25}$$

where α, β are real parameters.

3. Dynamics Analysis

Here, we discuss the dynamical study of iterative methods MHCM and NHCM. For details on the dynamical behavior of the iterative methods, one can consult [10], [11], [12]. To generate basins of attraction, we take grid 2000×2000 of square $[-2.5 \times 2.5]^2 \in \mathbb{C}$. To each root of (1), we assign a color to which the corresponding orbit of the iterative methods starts and converges to a fixed point. Taking, color map as Jet. We use $\left| f\left(x_i^{(u)}\right) \right| < 10^{-3}$ as a stopping criteria and maximum number of iteration is taken as 25. Dark black regions are colored if the orbit of the iterative methods does not converge to root after 25 iterations. As the dynamical planes of HCM depends upon the value of parameter t chosen. For different value $t \in [0, 1]$, we draw dynamical plane of MCHM. The method MHCM further contains two real parameter α, β. For different values of α and β, the dynamical planes of MHCM and NHCM graphed for the following non-linear function with auxilary equation are $f_1(x) = \sin(x) - x + 1, g_1(x) = \sin(x) - x$, $f_2(x) = \cos(x) - 4, g_2(x) = x^5 + x + 10$ and $f_3(x) = \cos(x) + \sin(x) - 2x, g_3(x) = x^5 + x + 10$ respectively.

The basin of attraction of the MHCM is depicted in Figure 1(A-M) for various values of the homotopic parameter t using the nonlinear functions $f_1(x) = \sin(x) - x + 1, g_1(x) = \sin(x) - x$ respectively. We increase the t values from 0 to 1 by 0.1, which results in better convergence behavior as color brightness grows. Figure also clearly illustrates how, as the homotopic parameter values vary from 0 to 1, one function gradually transforms into another and, at time t=0.5, obtains the perfect homotopic function picture in which neither $f_1(x)$ nor $g_1(x)$ contribute. In Figure 1(A-T), we take parameter value $\alpha = 10, \beta = -5$ while in Figure 1(G-L), we take $\alpha = 0.3$ and $\beta = -1.005$.

The basin of attraction of the MHCM is depicted in Figure 2(A-M) for various values of the homotopic parameter t using the nonlinear functions $f_2(x) = \cos(x) - 4, g_2(x) = x^5 + x + 10$ respectively. We increase the t values from 0 to 1 by 0.1, which results in better convergence behavior as color brightness grows. Figure also clearly illustrates how, as the homotopic parameter values vary from 0 to 1, one function gradually transforms into another and, at time t=0.5, obtains the perfect homotopic function picture in which neither $f_1(x)$ nor $g_1(x)$ contribute. In Figure 2(A-T), we take parameter value $\alpha = 9, \beta = -5.01$ while in Figure 2(G-L), we take $\alpha = 0.3$ and $\beta = -1.005$.

The basin of attraction of the MHCM is depicted in Figure 3(A-M) for various values of the homotopic parameter t using the nonlinear functions $f_3(x) = \cos(x) + \sin(x) - 2x, g_3(x) = x^5 + x + 10$ respectively. We increase the t values from 0 to 1 by 0.1, which results in better convergence behavior as color brightness grows. Figure also clearly illustrates how, as the homotopic parameter values vary from 0 to 1, one function gradually transforms into another and, at time t=0.5, obtains the perfect homotopic function picture in which neither $f_1(x)$ nor $g_1(x)$ contribute. In Figure 3(A-T), we take parameter value $\alpha = 10, \beta = -5$ while in Figure 3(g-l), we take $\alpha = 0.3$ and $\beta = -1.005$.

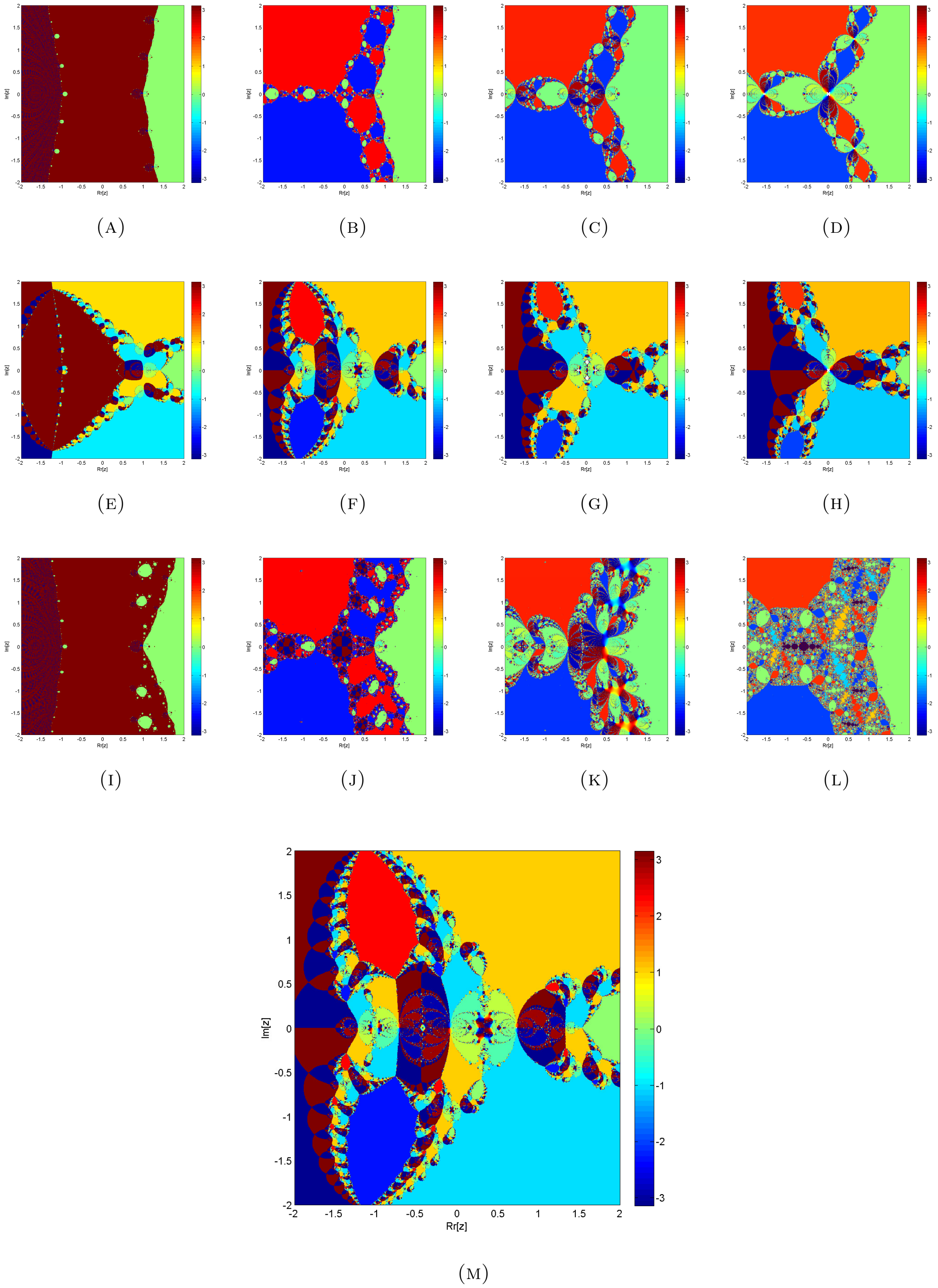


FIGURE 1. (A) $t = 0$. (B) $t = 0.1$. (C) $t = 0.2$. (D) $t = 0.3$. (E) $t = 0.4$. (F) $t = 0.5$. (G) $t = 0.6$. (H) $t = 0.7$. (I) $t = 0.8$. (J) $t = 0$. (K) $t = 0.1$. (L) $t = 0.2$. (M) Perfect homotopic basins of attraction picture in which both nonlinear functions i.e., $f_1(x)$ and $g_1(x)$ contribute equally. Clearly shows the basins of attraction for generated by MCHM iterative scheme for nonlinear functions $f_1(x) = \sin(x) - x + 1, g_1(x) = \sin(x) - x$ respectively by employing different values of t.

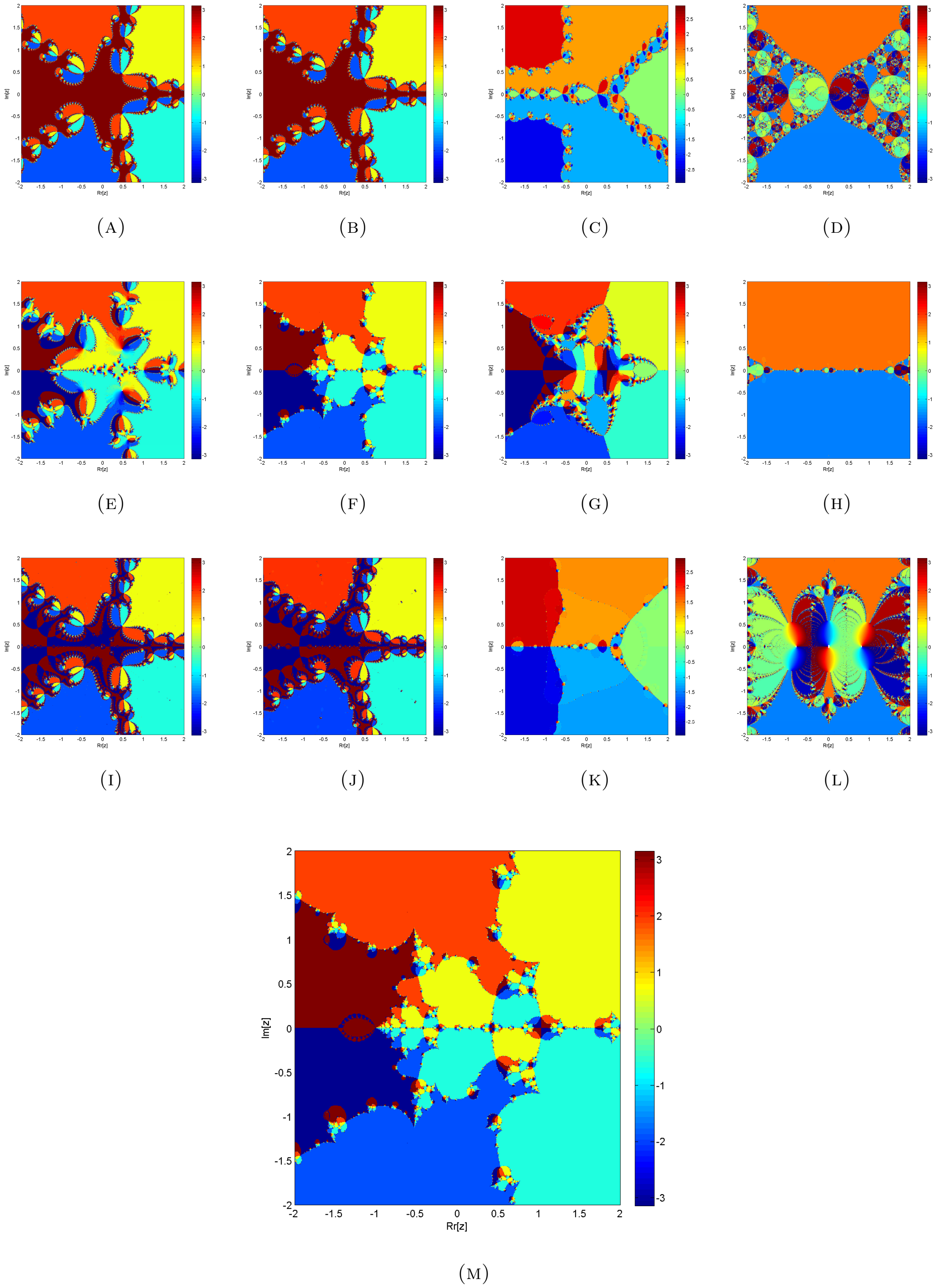


FIGURE 2. (A) $t = 0$. (B) $t = 0.1$. (C) $t = 0.2$. (D) $t = 0.3$. (E) $t = 0.4$. (F) $t = 0.5$. (G) $t = 0.6$. (H) $t = 0.7$. (I) $t = 0.8$. (J) $t = 0$. (K) $t = 0.1$. (L) $t = 0.2$. (M) Perfect homotopic basins of attraction picture in which both nonlinear functions i.e., $f_2(x)$ and $g_2(x)$ contribute equally. Clearly shows the basins of attraction for generated by MCHM iterative scheme for nonlinear functions $f_2(x) = \cos(x) - 4, g_2(x) = x^5 + x + 10$ respectively by employing different values of t.

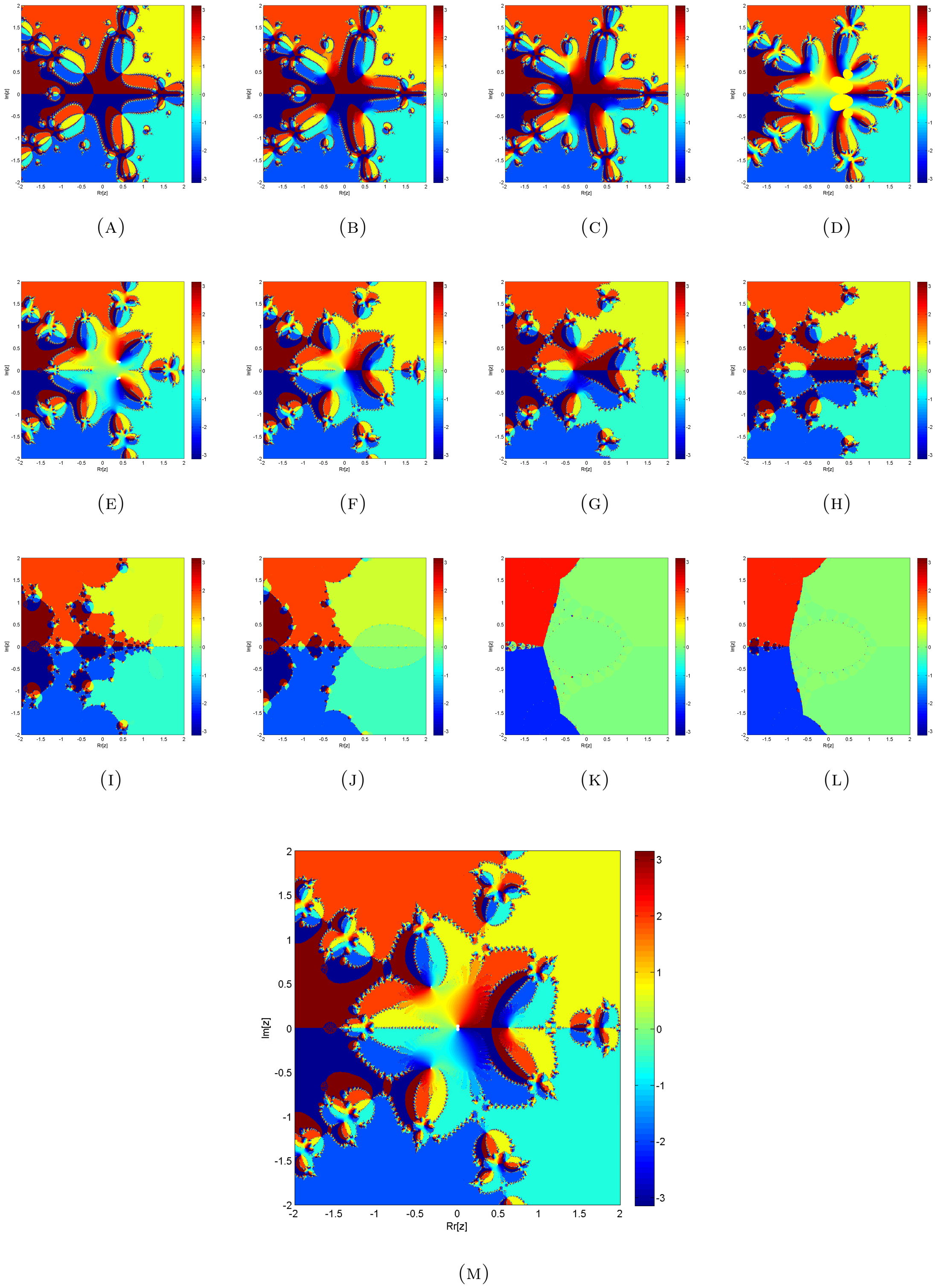
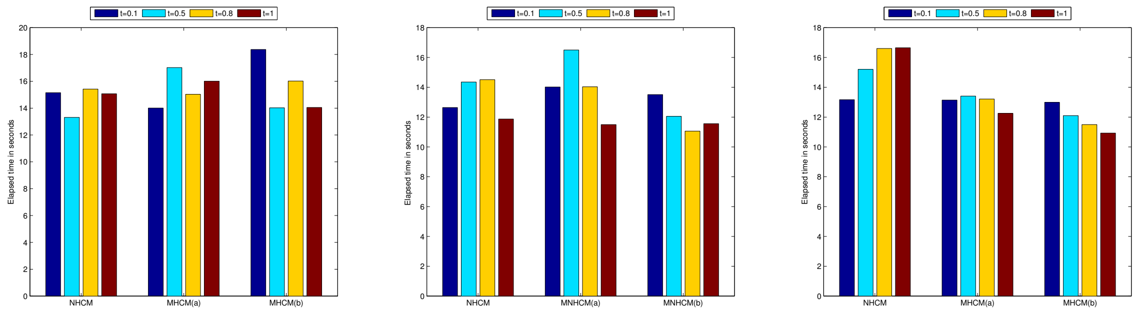
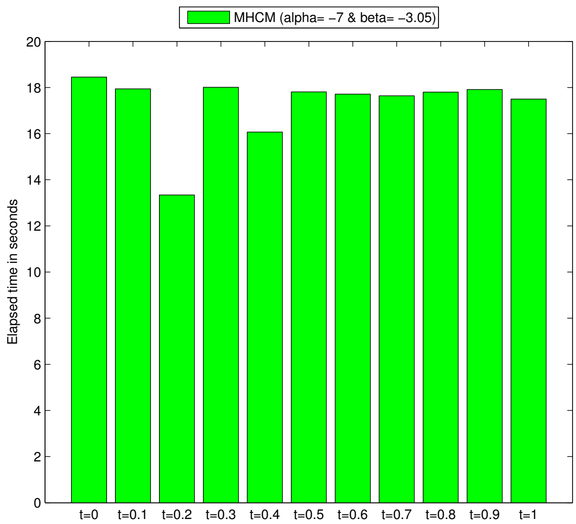


FIGURE 3. (A) $t = 0$. (B) $t = 0.1$. (C) $t = 0.2$. (D) $t = 0.3$. (E) $t = 0.4$. (F) $t = 0.5$. (G) $t = 0.6$. (H) $t = 0.7$. (I) $t = 0.8$. (J) $t = 0$. (K) $t = 0.1$. (L) $t = 0.2$. (M) Perfect homotopic basins of attraction picture in which both nonlinear functions i.e., $f_3(x)$ and $g_3(x)$ contribute equally. Clearly shows the basins of attraction for generated by MCHM iterative scheme for nonlinear functions $f_3(x) = \cos(x) + \sin(x) - x, g_3(x) = x^5 + x + 10$ respectively by employing different values of t.

The elapsed time for the generation of basin of attraction of the MHCM is depicted in Figure 4(A-C) for various values of the homotopic parameter t using the nonlinear functions $f_1(x) = \sin(x) - x + 1, g_1(x) = \sin(x) - x,\ f_2(x) = \cos(x) - 4, g_2(x) = x^5 + x + 10$ $f_3(x) = \cos(x) + \sin(x) - 2x, g_3(x) = x^5 + x + 10$ respectively. We increase the t values from 0 to 1 by 0.1, which results in better convergence behavior as color brightness grows.



(A) (B) (C)



(D)

Figure 4. (A) Elapsed time for the generation of basins of attraction for $f_1(x)$ and $g_1(x)$. (B) Elapsed time for the generation of basins of attraction for $f_2(x)$ and $g_2(x)$. (C) Elapsed time for the generation of basins of attraction for $f_3(x)$ and $g_3(x)$. (D) Elapsed time for the generation of basins of attraction for different value of homotopic parameter t. The elapsed time for the generation of basins of attraction of the nonlinear function using different homotopic parameter values t, as shown in Figures 1-3.

In all figures from Figure 1-3, we observe that as the value of parameter t increase from 0 to 0.5, the brightness in color increase which shows increase in convergence rate of the proposed numerical scheme MHCM. After $t = 0.5$, the rate of convergence gradually decreases. This behavior shows the homotopic chance of one function to another. At $t = 0.5$, we obtain such a function in which both functions either contributes equally or vice versa. Similar behavior also observe in the lapsed time for the generation of the basins of attractions for the nonlinear functions used in Figure 1-3. The parameter values used in numerical scheme MCHM are $\alpha = 3, \beta = -1.0005$.

4. Numerical Results

Here some numerical examples from engineering are considered to demonstrate the performance of NHCM, MHCM and NM method All the estimations are done by *Maple* 18 with 128 digits floating point arithmetic and use the following stopping criteria for estimating the roots:

$$(i) \qquad e^{(u)} = \left| f\left(x^{(u)}\right) \right| < \in,$$
$$(ii) \qquad e^{(u)} = \left| x^{(u+1)} - x^{(u)} \right| < \in,$$

where $e^{(u)}$ represents the absolute error in approximation.

Engineering Application 1: Fractional Conversion

An expression describe in [12]

$$f_{1*}(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.58x - 1.67 \tag{26}$$

is the fractional conversion of N and H feed at $250atm$ and $287K$, where N stands for nitrogen and H for hydrogen gas. Non-linear equation (26) has two real roots $-0.3481, 0.2778$ (as shown in figure 5) and two conjugate complex roots are $3.9485 \pm 0.3161i$. For fractional conversion our desired root is 0.2728 and we take $p_0 = 0.1$ as initial guesses.To approximates root of (26), we take two auxilary functions $g_{1,1}(x) = x + 1$ and $g_{1,2}(x) = sin(x) - x - 1$ with $\in = 10^{-10}$ with 128 digit floating point arithmetic.

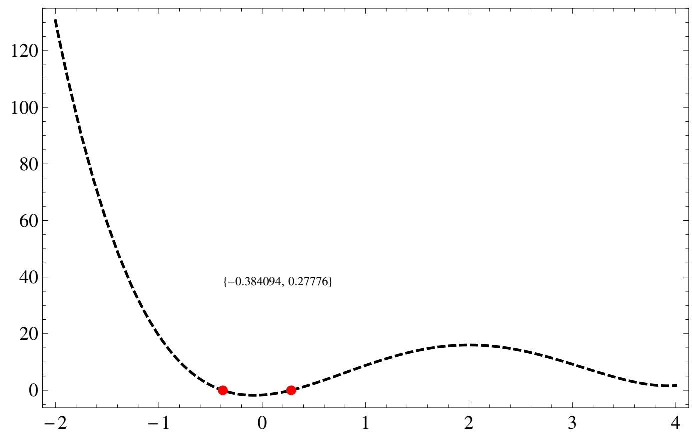


Figure 5. Location of the exact roots of the nonlinear function used in engineering application 1

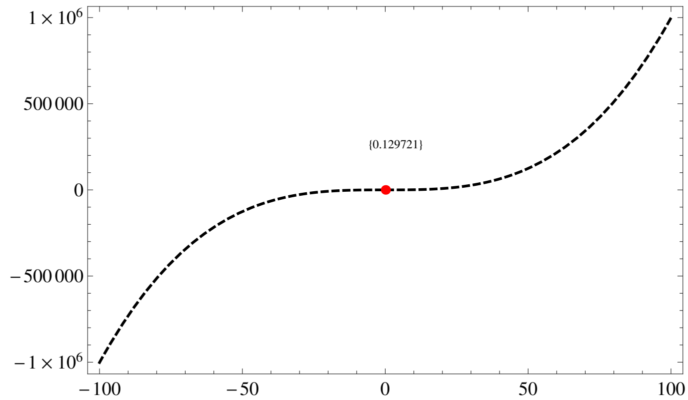


Figure 6. Location of the exact roots of the nonlinear function used in engineering application 2

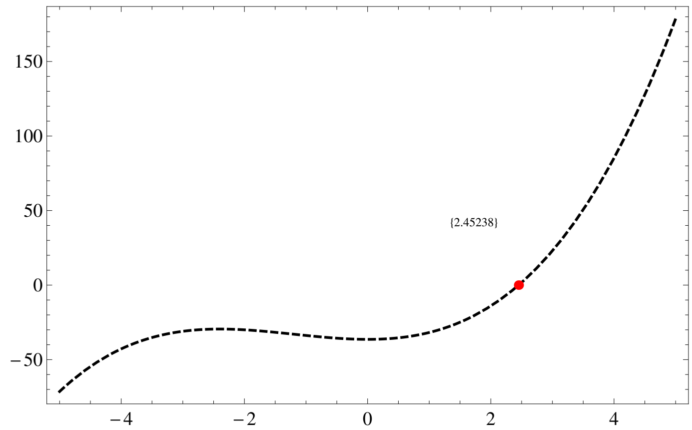


Figure 7. Location of the exact roots of the nonlinear function used in engineering application 3

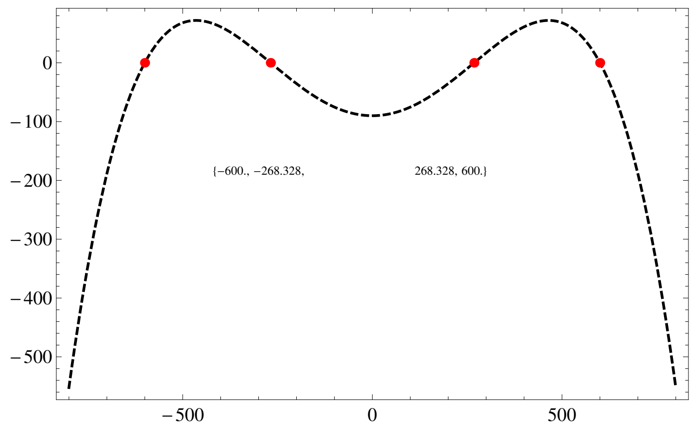


Figure 8. Location of the exact roots of the nonlinear function used in engineering application 4

Table.1: Comparison of HCMs for $f_1(x)$

Method	$x^{(u+1)} - x^{(u)}$	$f_{1*}\left(x^{(u)}\right)$	CPU	ρ
$g_{1,1}(x) = x+1,\ u = 4, \alpha = 1.1, \beta = 5.5$				
NHCM	2.1e-4	0.1e-9	0.034	1.99
MHCM	0.6e-11	0.0	0.016	2.1
NM	1.2e-5	1.2e-9	0.031	2.0
$g_{1,2}(x) = \sin(x) - x - 1, u = 4, \alpha = 1.1, \beta = 5.5$				
NHCM	1.2e-5	0.1e-8	0.017	2.0
MHCM	0.6e-11	0.5e-18	0.015	2.0
NM	1.2e-5	1.2e-9	0.031	2.0

Table 1 shows the numerical results of engineering application 1. The dominance convergence behavior and efficiency of MHCM over NHCM and NM are clearly illustrated in Table 1. The CPU time and residual error of MCHM are both better than those of NCHM and NM.

Figure 9(A,B) shows residual error of iterative schemes MHCM, NHCM, NM for non-linear function with $g_{1,1}(x)$ and $g_{1,2}(x)$ respectively. Figure 9(a,b) clearly shows good convergence behavior of MHCM over NHCM and NM on same number of iteration.

Engineering Application 2: Civil Engineering Problem

A worker in "Down to the Toilet Company" that makes floats for ABC commodes as shown [13]. Specific gravity of $0.6N$ and a radios of $5.5cm$ is taking for the floating ball. To find the depth x in meter for which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

or

$$f_{2*}(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4} \tag{27}$$

Non-linear equation (27) has three real roots $0.04374, 0.1464, 0.0624$ as shown in figure 6. Our desire root is 0.0624 for which $x_0 = 0.08$ is taken as initial guess and two auxilary function $g_{2,1}(x)$ =x+1 and $g_{2,2}(x) = \cos(x) - 1$ are taken.

Table.2: Comparison of HCMs for $f_1(x)$

Method	$x^{(u+1)} - x^{(u)}$	$f_{2*}\left(x^{(u)}\right)$	CPU	ρ
$g_{2,1}(x)$ =x+1, $k = 4, \alpha = -0.01, \beta = -5.5$				
NHCM	0.5e-9	5.6e-16	0.03	2.0
MHCM	0.7e-10	5.7e-18	0.02	2.1
NM	3.1e-5	2.5e-12	0.01	2.0
$g_{2,2}(x) = \cos(x) - 1$,$u = 4, \alpha = -0.01, \beta = -5.5$				
NHCM	0.3e-9	0.5e-17	0.02	2.0
MHCM	0.3e-9	0.1e-18	0.01	2.0
NM	3.1e-5	2.5e-12	0.01	2.0

Table 2 shows the numerical results of engineering application 2. The dominance convergence behavior and efficiency of MHCM over NHCM and NM are clearly illustrated in Table 2. The CPU time and residual error of MCHM are both better than those of NCHM and NM.

Figure 10(A,B) shows residual error of iterative schemes MHCM, NHCM, NM for non-linear functions with $g_{2,1}(x)$ and $g_{2,2}(x)$ respectively. Figure 10(A,B) clearly shows good convergence behavior of MHCM over NHCM and NM on same number of iteration.

Engineering Application 3: Chemical Engineering Problem

For a $[H_3O^+]$, the acidity of a solution $MgOH$ in Hcl is given by [14]

$$\frac{3.64 \times 10^{-11}}{[H_3O^+]} = \left[H_3O^+\right] + 3.6 \times 10^{-4}, \tag{28}$$

where $[H_3O^+]$ is hydronium ion concentration, Hcl is hydrocloric acid. we get the following non-linear model by taking $x = 10^4 [H_3O^+]$ i.e.,

$$f_{3*}(x) = x^3 + 3.6x^2 - 36.4, \tag{29}$$

(29) has two complex conjugate roots $-3 \pm 2.3i$ and one real root 2.4 shown in Figure 7. We want to approximate our desired result real root 2.4 with $x_0 = 2.3$ and two auxiliary functions $g_{3,1}(x) = (x+1)^3$ and $g_{3,2}(x) = e^x + 1$ by taking $\in= 10^{-10}$ with 128 digit floating

point arithmetic.

Table.3: Comparison of HCMs for $f_1(x)$				
Method	$\left\|x^{(u+1)} - x^{(u)}\right\|$	$\left\|f_{3*}\left(x^{(u)}\right)\right\|$	CPU	ρ
$g_{3,1}(x) = (x{+}1)^3$, $u = 4, \alpha = -0.01, \beta = -5.5$				
NHCM	0.9e-9	0.1e-18	0.016	2.0
MHCM	0.4e-10	0.5e-21	0.015	2.1
NM	1.2e-11	1.2e-19	0.031	2.0
$g_{3,2}(x) = e^x + 1, u = 4, \alpha = -0.01, \beta = -5.5$				
NHCM	0.9e-10	6.1e-20	0.017	1.9
MHCM	0.9e-10	0.5e-23	0.013	2.0
NM	1.2e-9	1.2e-19	0.031	1.9

Table 3 shows the numerical results of engineering application 3. The dominance convergence behavior and efficiency of MHCM over NHCM and NM are clearly illustrated in Table 3. The CPU time and residual error of MCHM are both better than those of NCHM and NM.

Figure 11(A,B) shows residual error of iterative schemes MHCM, NHCM, NM for non-linear function with $g_{3,1}(x)$ and $g_{3,2}(x)$ respectively. Figure 11(A,B) clearly shows good convergence behavior of MHCM over NHCM and NM on same number of iteration.

Engineering Application 4: Civil Engineering Problem

In [15], [16] indicate a beam subject to a linearly increasing distributed load. The non-linear equation for the resulting elastic curve is

$$f(x) = \frac{\omega_o}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right). \tag{30}$$

Taking $f'(x) = 0$, to determine the point of maximum deflection i.e.,

$$\frac{\omega_o}{120EIL}\left(-5z^4 + 6L^2z^2 - L^4\right) = 0,$$

$$f_{4*}(x) = \frac{\omega_o}{120EIL}\left(-5x^4 + 6L^2x^2 - L^4\right). \tag{31}$$

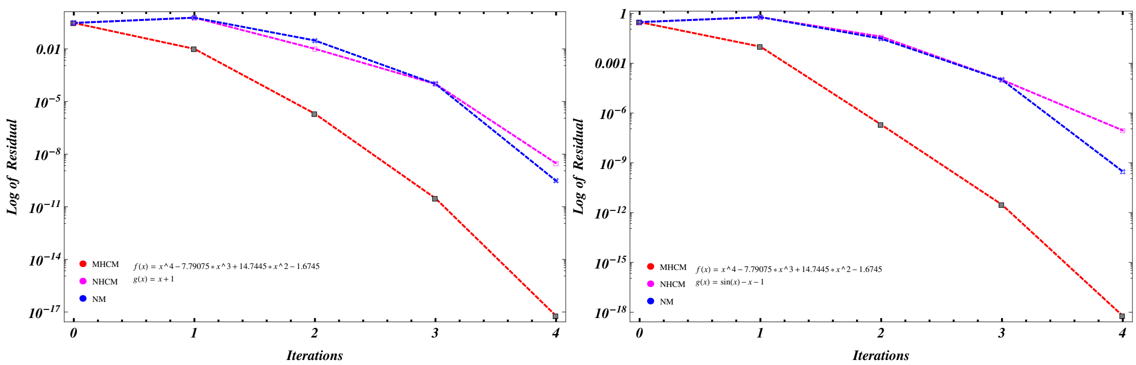
Then, substitute this value in (31) to determine the value of maximum deflection. Use the following values in computation $L = 600cm, E = 50,000KN/cm^3, I = 30,000cm^4$ and $\omega_o = 2.5KN/cm$ The equation (31) has four exact roots $\zeta_1 = -599.999$, $\zeta_2 = -268.328$, $\zeta_3 = 268.328$, $\zeta_4 = 599.999$ are shown in Figure 8. We have approximate our desire real root 2.4 with $x_0 = -575$ and two auxilary function $g_{4,1}(x) = \cos(x^2) + 1$ and $g_{4,2}(x) = e^x - 5$

by taking $\in = 10^{-10}$ with 128 digit floating point arithmetic.

Table.4: Comparison of HCMs for $f_1(x)$

Method	$\lvert x^{(u+1)} - x^{(u)} \rvert$	$\lvert f_{4*}\left(x^{(u)}\right) \rvert$	CPU	ρ
$g_{1,1}(x) = \cos(x^2) + 1,\ u = 4, \alpha = -0.01, \beta = -5.5$				
NHCM	1.3e-5	9.5e-15	0.014	1.9
MHCM	2.5e-5	0.1e-15	0.013	2.0
NM	2.5e-4	1.2e-15	0.015	2.0
$g_{1,2}(x) = e^x - 5,\ u = 4, \alpha = -0.01, \beta = -5.5$				
NHCM	2.5e-4	3.5e-14	0.017	2.0
MHCM	2.5e-5	0.1e-16	0.016	2.1
NM	1.2e-4	1.2e-15	0.020	2.0

Table 4 shows the numerical results of engineering application 4. The dominance convergence behavior and efficiency of MHCM over NHCM and NM are clearly illustrated in Table 4. The CPU time and residual error of MCHM are both better than those of NCHM and NM.



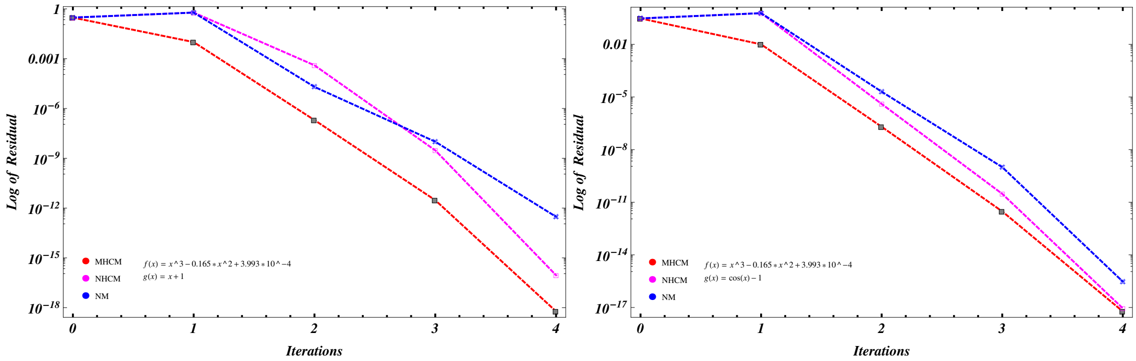
(A) (B)

FIGURE 9. (A) Error graph of MCHM, NCHM and NM method for $g_{1,1}(x)$. (B) Error graph of MCHM, NCHM and NM method for $g_{1,2}(x)$. Demonstrates the residual error for the iterative MHCM, NHCM, and NM schemes for non-linear functions with $g_{1,1}(x)$ and $g_{1,2}(x)$, respectively. Figure 9(A,B) unambiguously illustrates the improved convergence behavior of the MHCM over the NHCM and NM on the same number of iterations.

5. Results and Discussion

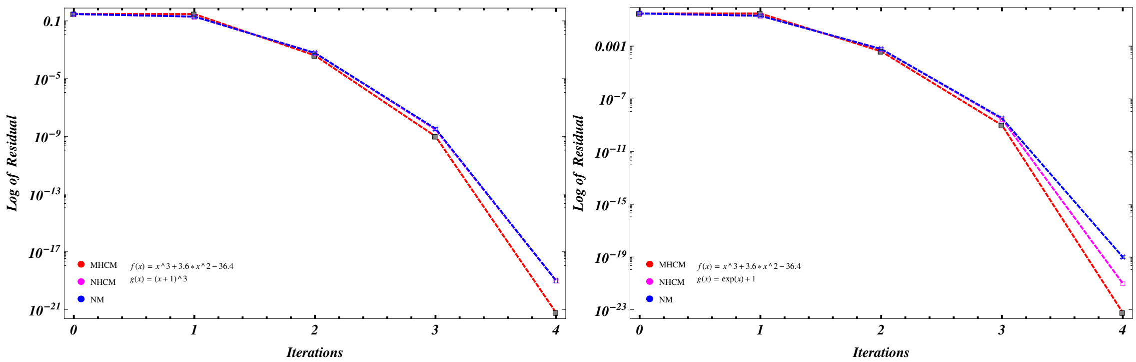
Here, we discuss the superiority of the numerical method MHCM over existing methods in the literature in terms of stability, residual errors and computational time.

- Figures 1(A-M)-3(A-M) depict the dynamical behavior of the suggested numerical techniques for various nonlinear functions with varying homotopic parameter values. Figure 1-3 clearly shows the improved rate of convergence for various t values.



(A) (B)

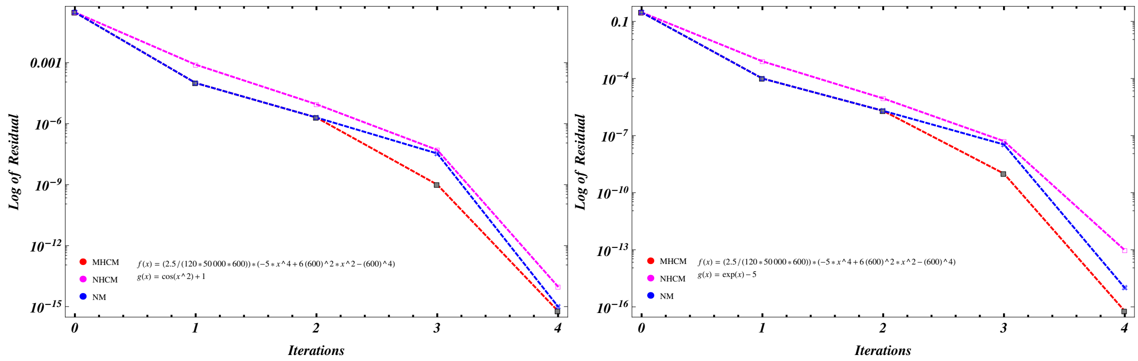
FIGURE 10. (A) Error graph of MCHM, NCHM and NM method for $g_{2,1}(x)$. (b) Error graph of MCHM, NCHM and NM method for $g_{2,2}(x)$. Demonstrates the residual error for the iterative MHCM, NHCM, and NM schemes for non-linear functions with $g_{2,1}(x)$ and $g_{2,2}(x)$, respectively. Figure 10(A,B) unambiguously illustrates the improved convergence behavior of the MHCM over the NHCM and NM on the same number of iterations.



(A) (B)

FIGURE 11. (A) Error graph of MCHM, NCHM and NM method for $g_{3,1}(x)$. (B) Error graph of MCHM, NCHM and NM method for $g_{3,2}(x)$. Demonstrates the residual error for the iterative MHCM, NHCM, and NM schemes for non-linear functions with $g_{3,1}(x)$ and $g_{3,2}(x)$, respectively. Figure 11(A,B) unambiguously illustrates the improved convergence behavior of the MHCM over the NHCM and NM on the same number of iterations.

- Figure 4(A-D) indicates that the suggested numerical scheme MCHM has a better elapsed time than NHCM and NM.
- Figure 5-8 shows the exact locations of the roots of the nonlinear functions, which were used in engineering applications with a four decimal places precision.
- The residual error graph of the numerical solutions to the engineering problems utilized is shown in Figures 9(A,B) and 12(A,B). In Figure 9-12, the MHCM's third iteration result error is clearly better than that of the NHCM and NM.



(A) (B)

Figure 12. (A) Error graph of MCHM, NCHM and NM method for $g_{4,1}(x)$. (B) Error graph of MCHM, NCHM and NM method for $g_{4,2}(x)$. Demonstrates the residual error for the iterative MHCM, NHCM, and NM schemes for non-linear functions with $g_{4,1}(x)$ and $g_{4,2}(x)$, respectively. Figure 12(A,B) unambiguously illustrates the improved convergence behavior of the MHCM over the NHCM and NM on the same number of iterations.

- The numerical results of the engineering application are shown in Tables 1-4. The numerical results of all Tables clearly demonstrate that, for various homotopic functions, our newly proposed homotopy method outperforms NHCM and NM in terms of residual error, computational CPU time, errors in absolute functional values, and computational order of convergence.

6. Conclusion

We have developed here a new two parametric family of quadratic convergence MHCM for approximating all roots of (1). Numerical results from non-linear engineering model clearly indicate the dominance efficiency and convergence behavior of our method MHCM over NHCM and NM. Dynamical planes, CPU and error graph also support our the dominance efficiency of MHCM over these classical methods.

Data Availability: No data were used to support this study.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions: All authors contributed equally to the preparation of this manuscript.

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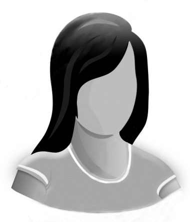

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