TWMS J. App. and Eng. Math. V.14, N.4, 2024, pp. 1655-1667

REDUCTION OF W-TRANSITIVE AND S-TRANSITIVE INTUITIONISTIC FUZZY MATRICES AND THEIR APPLICATIONS

R. A. PADDER[1,*], P. MURUGADAS[2], S. A. GANAI[1], §

Abstract. The collection of s-transitive and w-transitive intuitionistic fuzzy matrices comprise properly the collection of the transitive intuitionistic fuzzy matrices for which reduction models have already been proved. We have proved that basic properties of these models also holds for s-transitive and w-transitive intuitionistic fuzzy matrices

Keywords: Intuitionistic Fuzzy Matrix, s-transitive intuitionistic fuzzy matrix , w-transitive intuitionistic fuzzy matrix.

AMS Subject Classification:03E72, 15B15.

1. Introduction

The problems in engineering, economics, social sciences and environmental sciences, which cannot be solved by the well known methods of traditional Mathematics, pose a great difficulty in today's practical world (different types of uncertainties are presented in these problems). To handle situations like these, many tools have been recommended. Some of them are probability theory, rough set theory [32], fuzzy set theory [1], etc,.
The traditional fuzzy set is characterized by the grade of membership value. Some times it may be very hard to assign the membership value for fuzzy sets. In current scenario of practical problems in belief system, information fusion, expert systems and so on, we must consider the falsity-membership as well as the truth membership for proper description of an object in imprecise and doubtful environment. As a result, Intuitionistic Fuzzy Set (IFS) was introduced by Atanassov [2] and expressed it as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in E\}$, where E denotes a universal set in which $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ denote membership and non-membership functions of A respectively and its sum is less than or equal to one. In short we write the elements of IFS as $\langle x, x'\rangle$ such that $x + x' \leq 1$. The ideas of IFS were developed later in [3, 4, 5, 6, 7]. Mondal and Samanta [13] have

[1] Department of Mathematics School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, 144411, India.
e-mail: riyaz.28709@lpu.co.in; ORCID: https://orcid.org/0000-0002-3543-1375.
[*] Corresponding author.
e-mail: suhailnissar96@gmail.com; ORCID: https://orcid.org/0000-0003-0335-3605.
[2] Department of Mathematics, Govt. Arts and Science College, Veerapandi Theni, Dt-625534, India.
e-mail:muruga@yahoo.com; ORCID: https://orcid.org/0000-0002-9523-0559
§ Manuscript received: December 03, 2022; accepted: April 22, 2023.

developed the another concept of IFSs known as generalized IFSs. Bhowmik and Pal [14] studied generalized interval-valued intuitionistic fuzzy set. In 1977, Thomoson [31] studied the behavior of powers of Fuzzy Matrices (FMs) using max-min operation. Rageb and Emann [8] studied adjoint of a fuzzy matrix. Hashimoto [19, 20, 21] introduced implication operator in fuzzy matrices and derived various results. Hashimoto [20, 25] studied the reduction of retrieval and nilpotent fuzzy matrices. Antonion. et.al [26] studied reduction of transitive fuzzy matrices. The notion Intuitionistic Fuzzy Matrix (IFM) was introduced by Atanssov [34]. After that Pal and Shyamal [9, 10] have given the idea of intuitionistic fuzzy matrix and defined distance between intuitionistic fuzzy matrices. Bhowmik and Pal [11, 12] studied properties of intuitionistic fuzzy matrices, generalized intuitionistic fuzzy matrices and intuitionistic circulant fuzzy matrices. Pal. et.al [15] discussed intuitionistic fuzzy matrices. In [16] intuitionistic fuzzy relational equations has been discussed . Sriram and Murugadas [17, 18] studied semiring and sub-inverse of intuitionistic fuzzy matrices. In [22, 23, 24] implication operators have been introduced and defined g-inverse, decomposition and sub-inverse of intuitionistic fuzzy matrices. Several authors [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45] have worked on IFMs and found various interesting results, which are very helpful in handling uncertainty problems in our daily life.

1.1. **Research Gap.** The reduction is an interesting problem in the theory of IFM. Using implication operators[27, 30] we have studied reduction, of rectangular intuitionistic fuzzy matrix and nilpotent intutionistic fuzzy matrix, the intutionistic fuzzy matrix to be reduced using max-min transitive operation. In this article we look at about the reduction of w-transitive intuitionistic fuzzy matrices and s-transitive intuitionistic fuzzy matrices and its applications. We also provide some illustrations, so that theoretical contents of this paper can be understood easily.

2. BASIC DEFINITIONS

Definition 2.1. [2] *An Intuitionistic Fuzzy Set (IFS) A in X (universal set) is defined as an object of the following form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$, where the functions: $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the membership function and non-membership function of the element $x \in X$ respectively and for every $x \in X : 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.*

Definition 2.2. [28]*Let $X = \{x_1, x_2, ...x_m\}$ be a set of alternatives and $Y = \{y_1, y_2, ...y_n\}$ be the attribute set of each element of X. An Intuitionistic Fuzzy Matrix (IFM) is defined by $A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j)\rangle)$ for $i = 1, 2...m$ and $j = 1, 2, ...n$, where $\mu_A : X \times Y \to [0,1]$ and $\nu_A : X \times Y \to [0,1]$ satisfy the condition $0 \leq \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \leq 1$. For simplicity we denote an intuitionistic fuzzy matrix (IFM) as a matrix of pairs $A = (\left\langle a_{ij}, a'_{ij} \right\rangle)$ of a non negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all i, j. We denote the set of all IFM of order $m \times n$ by $\mathscr{F}_{mn}$.*

Atanassov introduced operations $\langle x, x'\rangle \vee \langle y, y'\rangle = \langle max\{x, y\}, min\{x', y'\}\rangle$ and $\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle min\{x, y\}, max\{x', y'\}\rangle$. Moreover, the operation $\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle$ defined by

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle = \begin{cases} \langle x, x'\rangle & \text{if } \langle x, x'\rangle > \langle y, y'\rangle, \\ \langle 0, 1\rangle & \text{if } \langle x, x'\rangle \leq \langle y, y'\rangle. \end{cases} \quad (1)$$

The operation in (1) is defined only for comparable elements.

Let $A = [a_{ij}, a'_{ij}]_{m\times n}$, $B = [b_{ij}, b'_{ij}]_{m\times n}$, $T = [t_{ij}, t'_{ij}]_{n\times n}$, $S = [s_{ij}, s'_{ij}]_{m\times m}$ and $Q = [q_{ij}, q'_{ij}]_{n\times p}$ be IFMs. Following operations are defined as:

$A \vee B = (\langle a_{ij} \vee b_{ij}, a'_{ij} \wedge b'_{ij} \rangle)$,

$AQ = [\bigvee_{k=1}^{n} (\langle a_{ik}, a'_{ik} \wedge \langle q_{kj}, q'_{kj} \rangle)]_{m \times p}$ (max-min composition)

$A \overset{c}{\leftarrow} B = [(\langle a_{ij}, a'_{ij} \rangle \overset{c}{\leftarrow} \langle b_{ij}, b'_{ij} \rangle)]_{m \times n}$ (Component wise)

$A - Q = [(\langle a_{ik}, a_{ik} \rangle - \langle q_{kj}, q'_{kj} \rangle)]_{m \times p}$

where

$$\langle x, x' \rangle - \langle y, y' \rangle = \begin{cases} \langle x - y, y' - x' \rangle & \text{if } \langle x, x' \rangle > \langle y, y' \rangle, \\ \langle 0, 1 \rangle & \text{if } \langle x, x' \rangle \leq \langle y, y' \rangle. \end{cases} \tag{2}$$

$\Delta T = [(\langle t_{ij}, t_{ij} \rangle) \overset{c}{\leftarrow} (\left\langle t_{ji}, t'_{ji} \right\rangle)]_{n \times n}$

$A/T = A \overset{c}{\leftarrow} (AT)$;

$A//(S, T) = A \overset{c}{\leftarrow} (SAT)$;

$T^T = [t_{ji}, t'_{ji}]_{n \times n}$ (Transpose of T):

$T^1 = T$, $T^{k+1} = T^k T$, $k = 1, 2, ...$;

The entries of T^k, are represented by $\langle t^k_{ij}, t'^k_{ij} \rangle$ i.e; $T^k = [\langle t^k_{ij}, t'^k_{ij} \rangle]_{n \times n}$;

$T^+ = T \vee T^3 \vee T^3 \vee ... \vee T^n$ (max-min transitive closure of T.

$A \leq B$ iff $\langle a_{ij}, a'_{ij} \rangle \leq \langle b_{ij}, b'_{ij} \rangle$ for all i,j;

$A \prec B$ iff for all i,j such that$\langle b_{ij}, b'_{ij} \rangle = \langle 0, 1 \rangle$, then $\langle a_{ij}, a'_{ij} \rangle = \langle 0, 1 \rangle$;

$A \approx B$ iff $A \prec B$ and $B \prec A$.

It can be easily predicted that $'' \approx ''$ is an equivalence relation on all $m \times n$ IFMs. Let , $A \approx B$ means that A and B have the same number of zero-entries placed correspondingly. We say that a matrix T is reflexive iff $\langle t_{ii}, t'_{ii} \rangle = \langle 1, 0 \rangle$ for all i, irreflexive iff $\langle t_{ii}, t'_{ii} \rangle = \langle 0, 1 \rangle$ for all i, (perfectly) antisymmetric iff $\langle t_{ij}, t'_{ij} \rangle > \langle 0, 1 \rangle$ implies $\langle t_{ji}, t'_{ji} \rangle = \langle 0, 1 \rangle$ for all i,j with $i \neq j$, nilpotent iff $T^n = \langle 0, 1 \rangle$ (here $\langle 0, 1 \rangle$ stands for the zero matrix), max-min transitive iff $T^2 \leq T$, w-transitive iff $(\langle t_{ik} \wedge t_{kj}, t'_{ik} \vee t'_{kj} \rangle) > \langle 0, 1 \rangle$: implies $\langle t_{ij}, t'_{ij} \rangle > \langle 0, 1 \rangle$ for all i,j,k or equivalently iff $T^2 \equiv T$, s-transitive iff $\langle t_{ik}, t'_{ik} \rangle > \langle t_{ki}, t'_{ki} \rangle$ and $\langle t_{kj}, t'_{kj} \rangle > \langle t_{jk}, t'_{jk} \rangle$ implies $\langle t_{ij}, t'_{ij} \rangle > \langle t_{ji}, t'_{ji} \rangle$ for any i,j,k such that $i \neq j$, $j \neq k$, $i \neq k$ or equivalently iff $(\Delta T)^2 \prec \Delta T$

It is obvious that always positive matrix T (i.e $\langle t_{ij}, t'_{ij} \rangle > \langle 0, 1 \rangle$ for all i,j) is w-transitive.

Definition 2.3. [29] *Let A and Q be $m \times n$ and $n \times q$ intuitionistic fuzzy matrices respectively, then*

$A - Q = (\bigwedge_{k=1}^{n} \langle a_{ik} - q_{kj}, a'_{ik} - q'_{kj} \rangle)$

where

$$a_{ik} - q_{kj} = \begin{cases} a_{ik} & \text{if } a_{ik} \geq q_{kj}, \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$a'_{ik} - q'_{kj} = \begin{cases} a'_{ik} & \text{if } a'_{ik} < q'_{kj}, \\ 1 & \text{otherwise} \end{cases}$$

Theorem 2.4. [27] *Let T and S be transitive IFMs. If P is an $n \times n$ nilpotent IFM such that $P \prec T$, then*

$$S(A//(S, P))T = SAT \tag{4}$$

for any IFM A.

Corollary 2.5. [27] *Let T and P be $n \times n$ w-transitive IFMs. If P is irreflexive IFM and $P \prec T$, then*

$$(A/P)T = AT \tag{5}$$

for any $m \times n$ IFM A.

3. Reduction of an w-transitive and s-transitive intuitionistic fuzzy matrices

In this section we examine the general reduction system of IFM concerning a product of three IFM. If A is an $m \times n$, T is an $n \times n$ and S is an $m \times m$ IFM respectively. Also we prove some properties of reduction of nilpotent IFMs of [30] remain valid for w-transitive intuitionistic fuzzy matrices and s-transitive intuitionistic fuzzy matrices.

Lemma 3.1. *: Let T be antisymmetric IFM then T is w-transitive IFM iff T is s-transitive IFM.*

Proof. Let T be w-transitive,
then $\langle t_{ik}, t'_{ik}\rangle \wedge \langle t_{kj}, t'_{kj}\rangle > \langle 0, 1\rangle \Rightarrow \langle t_{ij}, t'_{ij}\rangle > \langle 0, 1\rangle$
Since T is antisymmetric $\langle t_{ij}, t'_{ij}\rangle > \langle 0, 1\rangle \Rightarrow \langle t_{ji}, t'_{ji}\rangle = \langle 0, 1\rangle$
Now let ,
$\langle t_{ik}, t'_{ik}\rangle > \langle t_{ki}, t'_{ki}\rangle$ and $\langle t_{kj}, t'_{kj}\rangle > \langle t_{jk}, t'_{jk}\rangle$.
To prove : $\left\langle t_{ij}, t'_{ij}\right\rangle > \left\langle t_{ji}, t'_{ji}\right\rangle$
$\langle t_{ik}, t'_{ik}\rangle > \langle t_{ki}, t'_{ki}\rangle$ and $\left\langle t_{kj}, t'_{kj}\right\rangle > \left\langle t_{jk}, t'_{jk}\right\rangle$
$\Rightarrow \langle t_{ik}, t'_{ik}\rangle \wedge \langle t_{kj}, t'_{kj}\rangle > \langle 0, 1\rangle \Rightarrow \langle t_{ij}, t'_{ij}\rangle > \langle 0, 1\rangle \Rightarrow \langle t_{ji}, t'_{ji}\rangle > \langle 0, 1\rangle$
Therefore $\left\langle t_{ij}, t'_{ij}\right\rangle > \left\langle t_{ji}, t'_{ji}\right\rangle$
Conversely let T be s-transitive, then
$\langle t_{ik}, t'_{ik}\rangle > \langle t_{ki}, t'_{ki}\rangle$ and $\langle t_{kj}, t'_{kj}\rangle > \langle t_{jk}, t'_{jk}\rangle \Rightarrow \langle t_{ij}, t'_{ij}\rangle > \langle t_{ji}, t'_{ji}\rangle$
To prove : $\langle t_{ij}, t'_{ij}\rangle > \langle 0, 1\rangle$
Let $\langle t_{ik}, t'_{ik}\rangle \wedge \langle t_{kj}, t'_{kj}\rangle > \langle 0, 1\rangle$
$\Rightarrow \langle t_{ik}, t'_{ik}\rangle > \langle 0, 1\rangle$, $\langle t_{kj}, t'_{kj}\rangle > \langle 0, 1\rangle$
$\Rightarrow \langle t_{ki}, t'_{ki}\rangle = \langle 0, 1\rangle$, $\langle t_{jk}, t'_{jk}\rangle = \langle 0, 1\rangle$ (because T is antisymmetric)
$\langle t_{ik}, t'_{ik}\rangle > \langle t_{ki}, t'_{ki}\rangle$ and $\langle t_{kj}, t'_{kj}\rangle > \langle t_{jk}, t'_{jk}\rangle \Rightarrow \langle t_{ij}, t'_{ij}\rangle > \langle t_{ji}, t'_{ji}\rangle$
$\Rightarrow \langle t_{ij}, t'_{ij}\rangle > \langle 0, 1\rangle$ (By antisymmetric property) □

Lemma 3.2. *If T is max-min transitive IFM then T is w-transitive IFM.*

Proof. Let $T^2 \leq T$
$\Rightarrow \langle t_{ik}, t'_{ik}\rangle \wedge \langle t_{kj}, t'_{kj}\rangle \leq \langle t_{ij}, t'_{ij}\rangle$
If $\Rightarrow \langle t_{ik}, t'_{ik}\rangle \wedge \langle t_{kj}, t'_{kj}\rangle > \langle 0, 1\rangle$
$\Rightarrow \langle t_{ij}, t'_{ij}\rangle > \langle 0, 1\rangle \Rightarrow$ T is w-transitive □

Lemma 3.3. *If $T = \langle t_{ij}, t'_{ij}\rangle$ is max-min transitive IFM then $T = \langle t_{ij}, t'_{ij}\rangle$ is s-transitive IFM.*

Proof. We have to show that if T is max-min transitive IFM, $\langle t_{ik}, t'_{ik}\rangle > \langle t_{ki}, t'_{ki}\rangle$ and $\langle t_{ji}, t'_{ji}\rangle > \langle t_{ij}, t'_{ij}\rangle$ then $\langle t_{jk}, t'_{jk}\rangle > \langle t_{kj}, t'_{kj}\rangle$
Suppose if $\langle t_{jk}, t'_{jk}\rangle \leq \langle t_{kj}, t'_{kj}\rangle$
Now
$\langle t_{jk}, t'_{jk}\rangle \geq \langle t_{ji}, t'_{ji}\rangle \wedge \langle t_{ik}, t'_{ik}\rangle > \langle t_{ij}, t'_{ij}\rangle \wedge \langle t_{ki}, t'_{ki}\rangle$ (given)
$\Rightarrow \langle t_{jk}, t'_{jk}\rangle \geq \langle t_{kj}, t'_{kj}\rangle \wedge \langle t_{ji}, t'_{ji}\rangle \wedge \langle t_{ij}, t'_{ij}\rangle = \langle t_{kj}, t'_{kj}\rangle \wedge \langle t_{ij}, t'_{ij}\rangle$
$\Rightarrow \langle t_{kj}, t'_{kj}\rangle > \langle t_{ij}, t'_{ij}\rangle$
on the other hand ,
$\langle t_{ik}, t'_{ik}\rangle > \langle t_{ki}, t'_{ki}\rangle \geq \langle t_{kj}, t'_{kj}\rangle \wedge \langle t_{ij}, t'_{ij}\rangle$

Since $\langle t_{kj}, t'_{kj}\rangle > \langle t_{ij}, t'_{ij}\rangle$
$\Rightarrow \langle t_{ik}, t'_{ik}\rangle \wedge \langle t_{kj}, t'_{kj}\rangle > \langle t_{ij}, t'_{ij}\rangle$,
which contradicts the max-min transitive of T.
Hence $\langle t_{jk}, t'_{jk}\rangle > \langle t_{kj}, t'_{kj}\rangle$

□

Theorem 3.4. *If T antisymmetric and s-transitive IFM, implies ΔT w-transitive and nilpotent IFM .*

Proof. Let T be antisymmetric $\Rightarrow \Delta T = T$.
Since T is s-transitive $\Rightarrow T$ is w-transitive by Lemma 3.1.
i.e $T^2 \approx T \Rightarrow T^2 \prec T$ and $T \prec T^2$
$\Rightarrow (\Delta T)^2 \approx \Delta T \Rightarrow (\Delta T)^2 \prec \Delta T$ and $\Delta T \prec (\Delta T)^2$. (By antisymmetric property).
Hence ΔT is w-transitive.
Let $(\Delta T)^n = [\langle t^{\Delta,n}_{ij}, t'^{\Delta,n}_{ij}\rangle]$. Let us consider that there exists indices $i, j \in \{1, 2, ..n\}$ so that $\langle t^{\Delta,n}_{ij}, t'^{\Delta,n}_{ij}\rangle > \langle 0, 1\rangle$. Then $\langle t^{\Delta,n}_{ij}, t'^{\Delta,n}_{ij}\rangle = \langle t^{\Delta}_{h_0h_1}, t'^{\Delta}_{h_0h_1}\rangle \wedge \langle t^{\Delta}_{h_1h_2}, t'^{\Delta}_{h_1h_2}\rangle \wedge ... \wedge \langle t^{\Delta}_{h_{n-1}h_n}, t'^{\Delta}_{h_{n-1}h_n}\rangle > \langle 0, 1\rangle$ for a few integers $h_0, h_1, h_2..., h_n \in \{1, 2, ..., n\}$ so that $h_0 = i$ and $h_n = j$.
Then $h_a = h_b$ for a and b $(a < b)$ and $\langle t^{\Delta}_{h_ah_{a+1}}, t'^{\Delta}_{h_ah_{a+1}}\rangle > \langle 0, 1\rangle = \langle t^{\Delta}_{h_{a+1}h_a}, t'^{\Delta}_{h_{a+1}h_a}\rangle$, $\langle t^{\Delta}_{h_{a+1}h_{a+2}}, t'^{\Delta}_{h_{a+1}h_{a+2}}\rangle > \langle 0, 1\rangle = \langle t^{\Delta}_{h_{a+2}h_{a+1}}, t'^{\Delta}_{h_{a+2}h_{a+1}}\rangle, ..., \langle t^{\Delta}_{h_{b-1}h_b}, t'^{\Delta}_{h_{b-1}h_b}\rangle > \langle 0, 1\rangle =$
$\langle t^{\Delta}_{h_bh_{b-1}}, t'^{\Delta}_{h_bh_{b-1}}\rangle$
By applying the s-transitivity of IFM ΔQ we get
$\langle t^{\Delta,n}_{h_ah_a}, t'^{\Delta,n}_{h_ah_a}\rangle = \langle t^{\Delta,n}_{h_ah_b}, t'^{\Delta,n}_{h_ah_b}\rangle > \langle t^{\Delta,n}_{h_bh_a}, t'^{\Delta,n}_{h_bh_a}\rangle = \langle t^{\Delta,n}_{h_ah_a}, t'^{\Delta,n}_{h_ah_a}\rangle$
which is not possible. □

Theorem 3.5. *Let T be any w-transitive and irreflexive IFM then $T^n = (\langle 0, 1\rangle)$.*

Proof. Assume that $\langle t^n_{ij}, t'^n_{ij}\rangle > \langle 0, 1\rangle$
Then there exists $l_i, l_2, ..., l_{n-1}$ such that
$\langle t_{il_1}, t'_{il_1}\rangle \wedge \langle t_{l_1l_2}, t'_{l_1l_2}\rangle \wedge ... \wedge \langle t_{l_{n-1}j}, t'_{l_{n-1}j}\rangle > \langle 0, 1\rangle$
Put $l_0 = i$ and $l_n = j$ for some a and b such that $(a < b)$
$\Rightarrow \langle t_{l_al_{a+1}}, t'_{l_al_{a+1}}\rangle \wedge ... \wedge \langle t_{l_al_{a+1}}, t'_{l_al_{a+1}}\rangle > \langle 0, 1\rangle$
$\Rightarrow \langle t_{l_al_a}, t'_{l_al_a}\rangle$ contradicts with fact that T is irreflexive. Therefore $T^n = (\langle 0, 1\rangle)$. □

Example 3.6. *Let* $A = \begin{pmatrix} \langle 0.6, 0.2\rangle & \langle 0.3, 0.2\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.4, 0.3\rangle \\ \langle 0.5, 0.2\rangle & \langle 0.6, 0.1\rangle \end{pmatrix}$,
$T = \begin{pmatrix} \langle 1.0, 0.0\rangle & \langle 0.6, 0.1\rangle \\ \langle 0.6, 0.1\rangle & \langle 1.0, 0.0\rangle \end{pmatrix}$
We assume T be a similarity matrix where $\left\langle t_{ij}, t'_{ij}\right\rangle$ denotes the degree.
Now, let
$P = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.6, 0.1\rangle & \langle 0.0, 1.0\rangle \end{pmatrix} \leq T$ *be nilpotent IFM by means of which we reduce A.*
$AP = \begin{pmatrix} \langle 0.3, 0.2\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.6, 0.1\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}$
Hence $A/P = A \overset{c}{\leftarrow} (AP)$
$A/P = \begin{pmatrix} \langle 0.6, 0.2\rangle & \langle 0.3, 0.2\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.4, 0.3\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.6, 0.1\rangle \end{pmatrix}$,

$(A/P)T = \begin{pmatrix} \langle 0.6, 0.2\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.6, 0.1\rangle \\ \langle 0.6, 0.1\rangle & \langle 0.6, 0.1\rangle \end{pmatrix}$, $AT = \begin{pmatrix} \langle 0.6, 0.2\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.6, 0.1\rangle \\ \langle 0.6, 0.1\rangle & \langle 0.6, 0.1\rangle \end{pmatrix}$.
$\Rightarrow (A/P)T = AT$

Theorem 3.7. *Let T and S be w-transitive IFMs. If P is an $n \times n$ nilpotent IFM such that $P \prec T$, then $S(A//(S,P))T = SAT$ for any $m \times n$ matrix A.*

Proof. Let the matrices $B = [\left\langle b_{ij}, b'_{ij}\right\rangle]_{m\times n} \approx S(A//(S,P))T$
and $C = [\left\langle c_{ij}, c'_{ij}\right\rangle]_{m\times n} = SAT$. Thus

$$\begin{aligned}\langle b_{ij}, b'_{ij}\rangle = \Bigg\langle & \bigvee_{k=1}^{m}\bigvee_{l=1}^{n}\left\{ s_{ik} \wedge \left[a_{kl} \leftharpoondown \left(\bigwedge_{f=1}^{m}\bigwedge_{g=1}^{n} (s_{kf} \wedge a_{fg} \wedge p_{gl})\right)\right] \wedge t_{ij}\right\}, \\ & \bigwedge_{k=1}^{m}\bigwedge_{l=1}^{n}\left\{ s'_{ik} \vee \left[a'_{kl} \leftharpoondown \left(\bigwedge_{f=1}^{m}\bigwedge_{g=1}^{n} (s'_{kf} \vee a'_{fg} \vee p'_{gl})\right)\right] \vee t'_{ij}\right\}\Bigg\rangle\end{aligned} \tag{6}$$

$$\langle c_{ij}, c'_{ij}\rangle = \left\langle \bigvee_{h=1}^{m}\bigvee_{l=1}^{n} (s_{ih} \wedge a_{hl} \wedge t_{lj}), \bigwedge_{h=1}^{m}\bigwedge_{l=1}^{n} (s'_{ih} \vee a'_{hl} \vee t'_{lj})\right\rangle \tag{7}$$

Where $P = [\left\langle p_{ij}, p'_{ij}\right\rangle]_{n\times n}$

we have to show, that $B \succ C$ since $B \leq C$ and hence $B \prec C$. Assume that $\left\langle b_{ij}, b'_{ij}\right\rangle = \langle 0, 1\rangle$ and $\left\langle c_{ij}, c'_{ij}\right\rangle > \langle 0, 1\rangle$ for some i, j. Then there exist h_0 and d_0 such that $\left\langle s_{ih_0}, s'_{ih_0}\right\rangle > \langle 0, 1\rangle$, $\left\langle a_{h_0d_0}, a'_{h_0d_0}\right\rangle > \langle 0, 1\rangle$ and $\left\langle t_{d_0j}, t'_{d_0j}\right\rangle > \langle 0, 1\rangle$. Since $\left\langle b_{ij}, b'_{ij}\right\rangle = \langle 0, 1\rangle$, we have that

$$\left\langle \bigvee_{f=1}^{m}\bigvee_{g=1}^{n} (s_{h_0f} \wedge a_{fg} \wedge p_{gd_0}), \bigwedge_{f=1}^{m}\bigwedge_{g=1}^{n} (s'_{h_0f} \vee a'_{fg} \vee p'_{gd_0})\right\rangle \geq \left\langle a_{h_0d_0}, a'_{h_0d_0}\right\rangle > \langle 0, 1\rangle \tag{8}$$

Then we get, for a few f_1 and g_1, that $\left\langle s_{h_0f_1}, s'_{h_0f_1}\right\rangle > \langle 0, 1\rangle$, $\left\langle a_{f_1g_1}, a'_{f_1g_1}\right\rangle > \langle 0, 1\rangle$, $\left\langle p_{g_1d_0}, p'_{g_1d_0}\right\rangle > \langle 0, 1\rangle$ and $\left\langle t_{g_1d_0}, t'_{g_1d_0}\right\rangle > \langle 0, 1\rangle$.Therefore $\left\langle s_{if_1}, s'_{if_1}\right\rangle > \langle 0, 1\rangle$,$\left\langle a_{f_1g_1}, a'_{f_1g_1}\right\rangle > \langle 0, 1\rangle$,$\left\langle t_{g_1j}, t'_{g_1j}\right\rangle > \langle 0, 1\rangle$ and $\left\langle p_{g_1d_0}, p'_{g_1d_0}\right\rangle > \langle 0, 1\rangle$. Since $\left\langle b_{ij}, b'_{ij}\right\rangle = \langle 0, 1\rangle$, we have

$$\left\langle \bigvee_{f=1}^{m}\bigvee_{g=1}^{n} (s_{f_1f} \wedge a_{fg} \wedge p_{gg_1}), \bigwedge_{f=1}^{m}\bigwedge_{g=1}^{n} (s'_{f_1f} \vee a'_{fg} \vee p'_{gg_1})\right\rangle \geq \left\langle a_{f_1g_1}, a'_{f_1g_1}\right\rangle > \langle 0, 1\rangle. \tag{9}$$

We have f_2 and g_2 such that $\left\langle s_{f_1f_2}, s'_{f_1f_2}\right\rangle > \langle 0, 1\rangle$, $\left\langle a_{f_2g_2}, a'_{f_2g_2}\right\rangle > \langle 0, 1\rangle$, $\left\langle p_{g_2g_1}, p'_{g_2g_1}\right\rangle > \langle 0, 1\rangle$ and $\left\langle t_{g_2g_1}, t'_{g_2g_1}\right\rangle > \langle 0, 1\rangle$. Therefore $\left\langle s_{if_2}, s'_{if_2}\right\rangle > \langle 0, 1\rangle$, $\left\langle a_{f_2g_2}, a'_{f_2g_2}\right\rangle > \langle 0, 1\rangle$, $\left\langle t_{g_2j}, t'_{g_2j}\right\rangle > \langle 0, 1\rangle$ and $\left\langle p^n_{g_1d_0}, p'^n_{g_1d_0}\right\rangle > \langle 0, 1\rangle$, continuing the process, we get $\left\langle s_{if_n}, s'_{if_n}\right\rangle > \langle 0, 1\rangle$, $\left\langle a_{f_ng_n}, a'_{f_ng_n}\right\rangle > \langle 0, 1\rangle$, $\left\langle t_{g_nj}, t'_{g_nj}\right\rangle > \langle 0, 1\rangle$ and $\left\langle p^n_{g_nd_0}, p'^n_{g_nd_0}\right\rangle > \langle 0, 1\rangle$. This contradicts the fact that P is a nilpotent IFM. □

Corollary 3.8. *Let T and P be w-transitive square IFMs of order n. If P is irreflexive IFM and $P \prec T$, then*

$$(A/P)T \approx AT \tag{10}$$

for any $m \times n$ matrix A

Corollary 3.9. *Let T and S be w-transitive IFMs. If P is an nilpotent IFM of order n such that $P \prec S$, then $S(A//(P,T))T \approx SAT$ for any $m \times n$ matrix A. Let, $S(A \overset{c}{\leftarrow} (PA)) \approx SA$ if T is the identity matrix of order n.*

By applying Theorem 3.6 and Corollary 3.8 we find following corollaries which are very important for the reduction of s-transitive IFMs.

Corollary 3.10. *Let T and S be s-transitive IFMs. If P is nilpotent IFM of oder n such that $P \prec \Delta T$, then $\Delta S(A//(\Delta S, P))\Delta T \approx \Delta SA\Delta T$ for any $m \times n$ matrix A.*

Corollary 3.11. *Let T and S be s-transitive IFMs. If P is nilpotent IFM of oder m such that $P \prec \Delta S$, then $\Delta S(A//(P, \Delta T))\Delta T \approx \Delta SA\Delta T$ for any matrix A of oder m.*

In previous Example if we look at $\langle t_{ij}, t_{ij}\rangle$ as a degree in which term t_j has broader meaning than the term t_i. Implies that T is irreflexive, antisymmetric and w-transitive IFM. For instance, we write as:

Example 3.12. *Let* $A = \begin{pmatrix} ‘\langle 0.6, 0.2\rangle & \langle 0.3, 0.2\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.4, 0.3\rangle \\ \langle 0.5, 0.2\rangle & \langle 0.6, 0.1\rangle \end{pmatrix}$,

$T = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}$

be irreflexive, antisymmetric and w-transitive IFM. Now, let

$P = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.8, 0.1\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix} \prec T$

be nilpotent IFM by using this matrix we can reduce matrix A by applying (10) .

$AP = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.7, 0.1\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.5, 0.2\rangle \end{pmatrix}$

Hence $A/P = A \overset{c}{\leftarrow} (AP)$

$A/P = \begin{pmatrix} \langle 0.6, 0.2\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.5, 0.2\rangle & \langle 0.6, 0.1\rangle \end{pmatrix}$,

$(A/PT) = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \end{pmatrix}$. $AT = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \end{pmatrix}$.

$\Rightarrow (A/P)T = AT$

Theorem 3.13. *Let $m \times n$ IFM A, $S = A - A^T$ is irreflexive and transitive, hence nilpotent IFM.*

Proof. Let $S = [\langle s_{ij}, s'_{ij}\rangle]_{m\times m}$, that is $[\langle s_{ij}, s'_{ij}\rangle] = \langle \bigwedge_{k=1}^{n} (a_{ik} - a_{jk}), \bigvee_{k=1}^{n} (a'_{ik} - a'_{jk})\rangle$. The irreflexivity is obvious.

To prove S is max-min transitive IFM , let

$\langle (s_{il} \wedge s_{lj}), (s'_{il} \vee s'_{lj})\rangle > \langle s_{ij}, s'_{ij}\rangle \geq \langle 0, 1\rangle$ for some i, l, j.

Then $\langle s_{ij}, s'_{ij}\rangle = \langle (a_{ik} - a_{jk}), (a'_{ik} - a'_{jk})\rangle$ for some k,

$\langle (a_{ik} - a_{lk}), (a'_{ik} - a'_{lk})\rangle \geq \langle s_{il}, s'_{il}\rangle > \langle 0, 1\rangle$ and $\langle (a_{lk} - a_{jk}), (a'_{lk} - a'_{jk})\rangle \geq \langle s_{lj}, s'_{lj}\rangle > \langle 0, 1\rangle$, i.e, $\langle a_{ik}, a'_{ik}\rangle > \langle a_{lk}, a'_{lk}\rangle > \langle a_{jk}, a'_{jk}\rangle$. Thus $\langle (s_{il} - s_{lj}), (s'_{il} - s'_{lj})\rangle > \langle s_{lj}, s'_{lj}\rangle = \langle a_{ik}, a'_{ik}\rangle - \langle a_{jk}, a'_{jk}\rangle = (\langle a_{ik}, a'_{ik}\rangle - \langle a_{lk}, a'_{lk}\rangle) + (\langle a_{lk}, a'_{lk}\rangle - \langle a_{jk}, a'_{jk}\rangle) \geq \langle s_{il}, s'_{il}\rangle + \langle s_{lj}, s'_{lj}\rangle$, contradiction.

Let A_i be the i-th row of A. If$A_i > A_j$, i.e, $\langle a_{id}, a'_{id}\rangle > \langle a_{jd}, a'_{jd}\rangle$ for any d=1,2,3...n, then $\langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle$, Where $\langle s_{ij}, s'_{ij}\rangle$ is the(i,j) entry of $S = A - A^T$. $\langle s_{ij}, s'_{ij}\rangle$ denotes, in accordance to the definition of the operation $-$ the strict preference of A_i over A_j. So, S provides the pecking order of the rows of A. If A is a term-document matrix, S represents the fuzzy strict preference amoung the terms. □

Theorem 3.14. *Let R is irreflexive and w-transitive IFM, then* $(T/T)^+ \approx T$

Proof. Let $N = [\langle n_{ij}, n_{ij}\rangle]_{n\times n} = T/T$,
$\left\langle n_{ij}, n'_{ij}\right\rangle = \left\langle t_{ij}, t'_{ij}\right\rangle \leftharpoondown \left\langle \bigvee_{k=1}^{n} (t_{ik}\wedge t'_{kj}), \bigwedge_{k=1}^{n} (t'_{ik}\vee t'_{kj})\right\rangle$.
To prove $N^+ \prec T$, suppose $\left\langle t_{ij}, t'_{ij}\right\rangle = \langle 0,1\rangle$ and $\left\langle n^k_{ij}, n'^k_{ij}\right\rangle > \langle 0,1\rangle$ for some k,
i.e.,$\left\langle n_{ih_1}, n'_{ih_1}\right\rangle > \langle 0,1\rangle$, $\left\langle n_{h_1h_2}, n'_{h_1h_2}\right\rangle > \langle 0,1\rangle$,...,$\left\langle n_{h_{k-1}j}, n'_{h_{k-1}j}\right\rangle > \langle 0,1\rangle$ for a few indices $h_0 = i, h_1, h_2, ..., h_{k-1},\ h_k = j$, which implies $\left\langle t_{ih_1}, t'_{ih_1}\right\rangle > \langle 0,1\rangle$,$\left\langle t_{h_1h_2}, t'_{h_1h_2}\right\rangle > \langle 0,1\rangle$,...,$\left\langle t_{h_{k-1}j}, t'_{h_{k-1}j}\right\rangle > \langle 0,1\rangle$. By w-transitivity of T, we get $\left\langle t_{ij}, t'_{ij}\right\rangle > \langle 0,1\rangle$, a contradiction, now we have to prove that, $T \prec N^+$. By theorem 2.5, T and N is nilpotent, since $N \le T$. Assume that $\left\langle t_{ij}, t'_{ij}\right\rangle > \langle 0,1\rangle$ and $\left\langle n^k_{ij}, n'^k_{ij}\right\rangle = \langle 0,1\rangle \Rightarrow \left\langle n^k_{ij}, n'^k_{ij}\right\rangle > \langle 0,1\rangle$ for every $k = 1, 2, ..., n-1$. Since $\left\langle n_{ij}, n'_{ij}\right\rangle > \langle 0,1\rangle$.
We obtain $\left\langle t_{ih_1}, t'_{ih_1}\right\rangle \ge \left\langle t_{ij}, t'_{ij}\right\rangle > \langle 0,1\rangle$, $\left\langle t_{h_1j}, t'_{h_1j}\right\rangle > \langle 0,1\rangle$ for some h_1 and consequently $\left\langle t^2_{ij}, t'^2_{ij}\right\rangle > \langle 0,1\rangle$.
Now we have to prove that $\left\langle n_{ip}, n'_{ip}\right\rangle > \langle 0,1\rangle$ and $\left\langle t_{pj}, t'_{pj}\right\rangle > \langle 0,1\rangle$ for a few p. If $\left\langle n_{ih_1}, n'_{ih_1}\right\rangle = \langle 0,1\rangle$ then $\left\langle t_{ih_2}, t'_{ih_2}\right\rangle \ge \left\langle t_{ih_1}, t'_{ih_1}\right\rangle > \langle 0,1\rangle$, $\left\langle t_{h_2h_1}, t'_{h_2h_1}\right\rangle \ge \left\langle t_{ih_1}, t'_{ih_1}\right\rangle > \langle 0,1\rangle$, for a few h_2 and consequently $\left\langle t^3_{ij}, t'^3_{ij}\right\rangle > \langle 0,1\rangle$. Of course, $\left\langle t_{h_2j}, t'_{h2j}\right\rangle > \langle 0,1\rangle$. If $\left\langle n_{ih_2}, n'_{ih_2}\right\rangle = \langle 0,1\rangle$, then $\langle t_{ih_3}, t'_{ih3}\rangle \ge \left\langle t_{ih_2}, t'_{ih_2}\right\rangle > \langle 0,1\rangle$, $\left\langle t_{h_3h_2}, t'_{h_3h_2}\right\rangle \ge \langle t_{ih_2}, t'_{ih2}\rangle > \langle 0,1\rangle$ for a few h_3 and consequently $\left\langle t^4_{ij}, t'^4_{ij}\right\rangle > \langle 0,1\rangle$. Of course, $\left\langle t_{h_3j}, t'_{h_3j}\right\rangle > \langle 0,1\rangle$. By repeating the same process, we have $\left\langle t^n_{ij}, t'^n_{ij}\right\rangle > \langle 0,1\rangle$ which is impossible because T is nilpotent. So we get $\left\langle n_{ih_p}, n'_{ih_p}\right\rangle > \langle 0,1\rangle$ and $\left\langle t_{h_p}, t'_{h_pj}\right\rangle > \langle 0,1\rangle$ for a few p. Since $\left\langle n^2_{ij}, n'^2_{ij}\right\rangle = \langle 0,1\rangle$,we obtain $\left\langle n_{h_pj}, n'_{h_pj}\right\rangle = \langle 0,1\rangle$ and consequently $\left\langle t_{h_pk}, t'_{h_pk}\right\rangle \ge \left\langle t_{h_pj}, t'_{h_pj}\right\rangle > \langle 0,1\rangle$,$\left\langle t_{kj}, t'_{kj}\right\rangle \ge \left\langle t^2_{h_pj}, t'^2_{h_pj}\right\rangle > \langle 0,1\rangle$. By repeating the above process , we get $\left\langle n_{h_pl_2}, n'_{h_pl_2}\right\rangle > \langle 0,1\rangle$ for a few l_2 and consequently $\langle n^2_{il_2}, n'^2_{il_2}\rangle > \langle 0,1\rangle$ $(l_1 = h_p)$. By continuing this procedure, we would have $\langle n^n_{il_n}, n'^n_{il_n}\rangle > \langle 0,1\rangle$, which contradicts the fact that N is nilpotent. □

Corollary 3.15. *Let R is s-transitive, then* $(\Delta T/\Delta T)^+ \approx \Delta T$.

Theorem 3.16. *If U be an w-transitive IFM of oder n and T be such that* $U^+ \approx T$. *Then* $T/T \prec U^2 \prec U \prec T$

Proof. By the given conditions, it follows that, since $U^k \le U^+$, $U^k \prec T$ for $k = 1, 2, 3, ..., n$. In particular, we have $U \prec T$. Then we have show that $T/T \prec U^2$. Let $N = [\left\langle n_{ij}, n'_{ij}\right\rangle] = T/T$,i.e., $\left\langle n_{ij}, n'_{ij}\right\rangle = \left\langle t_{ij}, t'_{ij}\right\rangle \overset{c}{\leftharpoondown} \left\langle \bigvee_{k=1}^{n} (t_{ik}\wedge t_{kj}), \bigwedge_{k=1}^{n} (t'_{ik}\vee t'_{kj})\right\rangle$.

Suppose that $\left\langle n_{ij}, n'_{ij} \right\rangle > \langle 0,1 \rangle$ and so, $\left\langle u^{+}_{ij}, u'^{+}_{ij} \right\rangle > \langle 0,1 \rangle$.
On the other side, we have for some h, $1 \leq h \leq n$:

$$\langle u_{l_0 l_1}, u'_{l_0 l_1} \rangle \wedge \langle u_{l_1 l_2}, u'_{l_1 l_2} \rangle \wedge ... \wedge \left\langle u_{l_{h-1} l_h}, u'_{l_{h-1} l_h} \right\rangle = \left\langle u^{h}_{ij}, u'^{h}_{ij} \right\rangle = \left\langle u^{+}_{ij}, u'^{+}_{ij} \right\rangle > \langle 0,1 \rangle$$

for suitable indices $l_1, l_2, ..., l_{h-1} (h \geq 1)$,
Where $l_o = i$ and $l_h = j$. $\left\langle u_{il_{h-1}}, u'_{il_{h-1}} \right\rangle > \langle 0,1 \rangle$
from the w-transitivity of U and $\left\langle u_{l_{h-1} l_h}, u'_{l_{h-1} l_h} \right\rangle$ i.e.,
$\left\langle u^{2}_{ij}, u'^{2}_{ij} \right\rangle \geq \left\langle u_{il_{h-1}}, u'_{il_{h-1}} \right\rangle \wedge \left\langle u_{l_{h-1} j}, u'_{l_{h-1} j} \right\rangle > \langle 0,1 \rangle$.

It has already proved that from $U^{+} = T$ [30], it follows that $T/T \leq U \leq T$ while, If T is an irreflexive and max-min transitive matrix, then $T/T \leq U \leq T$ implies $U^{+} = U \wedge U^{2} \wedge ... \wedge U^{n-1} = T$. □

Theorem 3.17. *if T be irreflexive and w-transitive IFM and U be an IFM of oder n such that $T/T \prec U \prec T$.Then $U^{+} \approx T$.*

Proof. Let

$$E \prec G \text{ and } F \prec H \tag{11}$$

It follows that

$$(EF) \prec (GH) \tag{12}$$

for any $n \times n$ matrices E, F, G, H. By (10) we have
$\left\langle e_{ij}, e'_{ij} \right\rangle > \langle 0,1 \rangle$ implies $\left\langle g_{ij}, g'_{ij} \right\rangle > \langle 0,1 \rangle$ and $\left\langle f_{ij}, f'_{ij} \right\rangle > \langle 0,1 \rangle$ implies $\left\langle h_{ij}, h'_{ij} \right\rangle > \langle 0,1 \rangle$. Thus, if $(\left\langle e_{ik} \wedge f_{kj}, e'_{ik} \vee f'_{kj} \right\rangle) > \langle 0,1 \rangle$ for some k(i.e., the entry (i,j) of $E \circ F$ is positive). This is equivelent to (11).Then from $R/R \prec U$, we obtain $(T/T)^{k} \prec U^{k}$ for all k, which means that $(T/T)^{+} \prec U^{+}$ implies $T \prec U^{+}$. On the other hand, using the w-transitivity of T, we have $U^{2} \prec T^{2} \prec T$, $U^{3} \prec T^{2} \prec T$,$U^{4} \prec T^{2} \prec T$, and so on. Thus $U^{+} \prec T$, □

3.1. **Example.** Consider irreflexive and w-transitive IFM, whose graph is depicted easily.

$$T = \begin{pmatrix} \langle 0.0,1,0 \rangle & \langle 0.3,0.6 \rangle & \langle 0.5,0.4 \rangle & \langle 0.6,0.3 \rangle \\ \langle 0.0,1,0 \rangle & \langle 0.0,1,0 \rangle & \langle 0.4,0.5 \rangle & \langle 0.0,1,0 \rangle \\ \langle 0.0,1,0 \rangle & \langle 0.0,1,0 \rangle & \langle 0.0,1,0 \rangle & \langle 0.0,1,0 \rangle \\ \langle 0.0,1,0 \rangle & \langle 0.4,0.5 \rangle & \langle 0.8,0.1 \rangle & \langle 0.0,1,0 \rangle \end{pmatrix}$$ Then reduction of T and its transitive

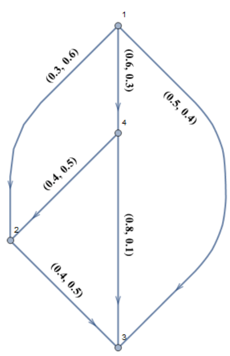

FIGURE 1. Matrix T

closure are, are given below

$$\Rightarrow T/T = T \overset{c}{\leftharpoondown} (TT) \tag{13}$$

$$TT = \begin{pmatrix} \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.6,0.3\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.0,1.0\rangle \end{pmatrix}$$

Then

$$T/T = \begin{pmatrix} \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.6,0.3\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.8,0.1\rangle & \langle 0.0,1.0\rangle \end{pmatrix}$$

Here $(T/T)^+ = T/T \vee (T/T)^2 \vee (T/T)^3 \vee (T/T)^4$

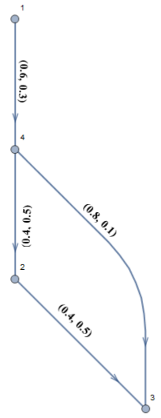

FIGURE 2. Matrix (T/T)

$$(T/T)^+ = \begin{pmatrix} \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1,0\rangle & \langle 0.6,0.3\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.8,0.1\rangle & \langle 0.0,1.0\rangle \end{pmatrix} \vee$$

$$\begin{pmatrix} \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.6,0.3\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.0,1.0\rangle \end{pmatrix}$$

$$\vee \begin{pmatrix} \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \end{pmatrix} \vee \begin{pmatrix} \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle \end{pmatrix}$$

$$\Rightarrow (T/T)^+ = \begin{pmatrix} \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.3\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.4,0.5\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle v\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.8,0.1\rangle & \langle 0.0,1.0\rangle \end{pmatrix}$$

At last we observe that $(T/T)^+ \approx T$ and we set $M = T/T$. It has been observed that $\left\langle m_{ij}^+, m_{ij}'^+ \right\rangle \geq \left\langle m_{ij}, m_{ij}' \right\rangle$ for arcs (i,j) removed after the reduction and $\left\langle m_{ij}^+, m_{ij}'^+ \right\rangle \leq \left\langle m_{ij}, m_{ij}' \right\rangle$ for the other transitive arcs(i,j) of the digraph.

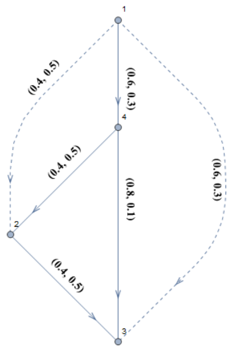

FIGURE 3. Matrix $(T/T)^+$

4. Conclusion

In this paper, we have studied the properties of w-transitive and s-transitive intuitionistic fuzzy matrices and their are applications like resolution of certain decision making problem and design intuitionistic fuzzy controller. In future these properties can be extended to index matrices.

Acknowledgement

The authors would like to thank the Editor and reviewer's for their valuable suggestions and comments for improving this paper in the present form.

References

[1] Zadeh, L. A., (1965), Fuzzy Sets,Journal of Information and Control, 8, pp. 338-353.

[2] Atanassov, K., (1983), Intuitionistic Fuzzy Sets , VII ITKR's Session, Sofia.

[3] Atanassov, K., (1999), Intuitionistic Fuzzy Sets ; Theory and Applications , Physica Verlag, 35.

[4] Atanassov, K.,(2005), Intuitionistic Fuzzy Implications and Modus Ponens, Notes on Intuitionistic Fuzzy Sets, 11(1), pp. 1-5.

[5] Atanassov, K., (2005), On Some Types of Fuzzy Negations, Notes on Intuitionistic Fuzzy Sets, 11(4), pp. 170-172.

[6] Atanassov, K., (2006), A New Intuitionistic Fuzzy Implication from a Modal Type Advance Studies in Contemporary Mathematics, 12(1), pp. 117-122.

[7] Atanassov, K., Gargov, G., (1998), Elements of Intuitionistic Fuzzy Logic.Part I , Fuzzy Sets and Systems, 95, pp. 39-52.

[8] Ragab, M. Z., Emam, E. G., (1994), The determinant and adjoint of a square fuzzy matrix, fuzzy Sets and Systems,61, pp. 297-307.

[9] Shyamal, A. K., Pal, M., (2002), Distances between intuitionistics fuzzy matrices, V. U. J. Physical Sciences 8, pp. 81-91.

[10] Shyamal, A. K., Pal, M., (2004), Two new operators on fuzzy matrices, Applied Mathematics and Computing, 15, pp.91-107.

[11] Bhowmik, M., Pal, M., (2008), Some results on intuitionistic fuzzy matrices and intuitionistic circulant fuzzy matrices, International Journal of Mathematical Sciences, 7(1-2), pp. 177-192.

[12] Bhowmik, M., Pal, M., (2008), Generalized intuitionistic fuzzy matrices, Far-East Journal of Mathematical Sciences 29(3), pp. 533-554.

[13] Mondal, T .K., Samanta, S. K., (2002), Generalized intuitionistic fuzzy sets, The Journal of Fuzzy Mathematics, 10(4), pp.839-862.

[14] Khan, S. K., Pal, M., and Amiya, K. S., (2002), Intuitionistic Fuzzy Matrices, Notes on Intuitionistic Fuzzy Sets, 8 (2), pp. 51-62.

[15] Meenakshi, A. R., Gandhimathi, T., (2010), Intuitionistic Fuzzy Relational Equations, Advances in Fuzzy Mathematics, 5(3), pp. 239-244.

[16] Sriram, S., Murugadas, P., (2010), On Semi-ring of Intuitionistic Fuzzy Matrices, Applied Mathematical Science, 4(23), pp.1099-1105.
[17] Sriram, S., Murugadas, P., (2011), Sub-inversesof Intuitionistic Fuzzy Matrices, Acta Ciencia Indica Mathematics, 1, pp. 41-56.
[18] Hashimoto, H., (2005), Traces of Fuzzy Relations Under Dual Operations, Journal of Advanced Computational Intelligence and Intelligent Informatics, 9(5), pp. 563-569.
[19] Hashimoto, H., (1984), Sub-inverses of Fuzzy Matrices, Fuzzy Sets and Systems, 12, pp. 155-168.
[20] Hashimoto, H., (1982), Reduction of Retrieval Models, Information science, 27, pp. 133-140.
[21] Murugadas, P and Laitha, K., (2014), Bi-implication Operator on Intuitionistic Fuzzy Set, Journal of advances in Mathematics, 6(2), pp. 961-969.
[22] Murugadas, P., Lalitha, K., (2012), Dual implication Operator in Intuitionistic Fuzzy Matrices, Int.Conference on Mathematical Modelling and its Applications, Organized by Department of Mathematics, Annamalai University.
[23] Murugadas, P., Lalitha, K., (2014), Sub-inverse and g-inverse of an Intuitionistic Fuzzy Matrix Using Bi-implication Operator, Int.Journal of Computer Application, 89(1), pp. 1-5.
[24] Murugadas, P., Lalitha, K., (2014), Implication Operator on Intuitionistic Fuzzy Tautological Matrix, Int.Journal of Fuzzy Mathematical Archive, 5(2), pp. 79-87.
[25] Hashimoto, H., (1982), Reduction of a nilpotent fuzzy matrix, Fuzzy Sets and System, 27, pp. 233-243.
[26] Antonion, D. N., Waldemar, K., Salvatore, S., (1993), On Reduction of an Transitive Fuzzy Matrices and its Applications, International Journal of Approximate Reasoning, 9, pp. 249-261.
[27] Murugadas, P., Padder, R. A., (2015), Reduction of Rectangular Intuitionistic fuzzy Matrix, Science Journal of Annamalai University.
[28] Pal, M., Khan, S. K., (2002), Intuitionistic fuzzy matrices, Notes on Instuitionistic Fuzzy Sets,8(2), pp. 51-62.
[29] Adak, A. K., Bhowmik, M., Pal, M., (2012), Some Properties of Generalized Intuitionistic Fuzzy Nilpotent Matrices over Distributive Lattice, Fuzzy Inf. Engineing, 4, pp. 371-387.
[30] Padder, R. A., Murugadas, P., (2016), Reduction of a nilpotent intuitionistic fuzzy matrix using implication operator, Applications and Applied Mathematics, 11(2), pp. 614 – 631.
[31] Thomason, M. G., (1977), Convergence of powers of a fuzzy matrix, J. Math. Anal. Appl. 57, pp. 476-480.
[32] Pawlak, Z., (1982), Rough sets, International Journal of computing and information sciences, 11, pp. 341-356.
[33] Zadeh, L. A., (1975), The concept of a linguistic variable and its application to approximate reasoning-I, Information Sciences, 8(3), pp. 199-249.
[34] Atanassov, K., (1987), Generalized index matrices, C. R. Acad Bulgare Sci., 40 (11), pp. 15-18.
[35] Barman, D., Das, B., (2018), An imprecise-inventory model with PEND and SEND policy, Int. J. Mathematics in Operational Research, 13(2), pp. 243-268.
[36] Barman, D., Das, B., (2012), A volume flexible fuzzy production inventory model under interactive and simulation approach, International Journal of Mathematics in Operational Research, 4 (4), pp. 243-268.
[37] Kuiri, A., Das, B., (2020), An application of FISM and TOPSIS to a multi-objective multi-item solid transprtation problem.OPSEARCH . https://doi.org/10.1007/s12597-020-00456-7. 57, pp. 1299–1318.
[38] Padder, R. A., Murugadas, P., (2017), Convergence of powers and Canonical form of s-transitive intuitionistic fuzzy matrix, New Trends in Mathematical Sciences, 5, pp. 229-236.
[39] Padder, R. A., Murugadas, P., (2018), On Convergence of the min- max Composition of Intuitionistic Fuzzy Matrices, International Journal of Pure and Applied Mathematics, 119, pp. 233-241.
[40] Muthuraji, T., Sriram, S., Murugadas, P., (2016), Decomposition of intuitionistic fuzzy matrices, Fuzzy Information and Engg, 8, pp. 345-354.
[41] Muthuraji, T., Sriram, S., (2017), Reduction of IFM to fuzzy matrix with some algebraic properties, Annals of fuzzy mathematics, 13(4), pp. 475-483.
[42] Muthuraji, T., (2021), Commutative monoids on symmetrical difference over IFMs, Application and Applied mathematics, 16(1), pp. 320-331..
[43] Muthuraji, T., (2020), Semigroups on some new operations over IFMs, Malaya journal of Matematik, 8(4), pp. 2273-2276.
[44] Padder, R. A., Murugadas, P., (2022), Algorithm for controllable and nilpotent intuitionistic fuzzy matrices, Afr. Mat. 33(84) . https://doi.org/10.1007/s13370-022-01019-3
[45] Padder, R. A., Murugadas, P., (2019), Determinant theory for intuitionistic fuzzy matrices, Afr. Mat. 30, 943–955, https://doi.org/10.1007/s13370-019-00692-1

Riyaz Ahmad Padder received his M.Sc. in mathematics from Barkatullah Vishwavidylaya University in 2012 and Ph.D. from the Department of Mathematics in Annamalai University (2017). He is an assistant professor in the Department of Mathematics, Lovely Professional University. His area of interest include fuzzy matrix theory and intuitionistic fuzzy matrix theory, Numerical Analysis, Decision Making Problems.

P. Murgadas received his M.S., M.Phil in mathematics from Maduraj Kamaraj University, Ph.D. in Mathematics from Annamalai University in 1987, 1988 and 2011 respectively. He is working as an assistant professor in the Department of Mathematics at Govt. Arts and Science College, Veerapandi, Theni Dt-625534. His research interests include fuzzy algebra, fuzzy set theory, fuzzy matrix theory, intuitionistic fuzzy set theory and intuitionistic fuzzy matrix.

Suhail Ahmad Ganai received his M.Sc. in mathematics from Lovely Professional University in 2019. Since 2020, he has been pursuing his Ph.D. in the department of Mathematics at the same university . His area of interest include fuzzy set theory, intuitionistic fuzzy theory and decision Making Problems.