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DISTANCE MAGIC LABELING FOR SOME PRODUCT GRAPHS

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ABSTRACT. In this paper, we prove $W_t \,\square\, G$ and $W_t \boxtimes G$ are not distance magic if graph G contains two vertices with the same neighborhood. And we also prove $W_t \times C_4$ $(t \geq 5)$, $W_3 \square P_n$ and $W_4 \times P_n$ $(n \geq 3)$ are not distance magic graphs.

Keywords: Cartesian product, direct product, distance magic labeling, strong product.

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1. INTRODUCTION

We consider here, all graphs G are simple with vertex set $V(G)$ and edge set $E(G)$ are finite. We adopt Gross and Yeelen [4] for various graphs and its theoretic notation and Burton [2] for number theoretic results. For acquiring the latest update, we follow a dynamic survey on graph labeling by Gallian [3].

A distance magic labeling of a graph G is a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ such that $\sum_{p \in N(q)} f(p) = \gamma$, for all $q \in V(G)$, where $N(q)$ is the set of all vertices of $V(G)$ which are adjacent to q. The constant γ is called the *magic constant* of the distance magic labeling of f. A graph which admits a distance magic labeling is called distance magic graph. For any vertex $q \in V(G)$, the neighbor sum $\sum_{p \in N(q)} f(p)$ is called the weight of the vertex $q \in V(G)$ and is denoted by $w(q)$.

Different terminologies are being used for this concept by various authors. Like, Vilfred [9] used the term sigma labeling; Miller et al. [6] used the term 1-vertex magic labeling [1-VML]; Acharya et al. [1] used the term neighborhood magic labeling, and Sugeng et

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al. [8] used the term distance magic labeling.
Among them, Miller et al. [6] have proved the following Lemmas.

Lemma 1.1. [6] *A necessary condition for the existence of a distance magic labeling f of a graph G is*

$$\gamma v = \sum_{p \in N(q)} d(p) f(p)$$

where $d(p)$ is the degree of vertex p and v is the number of vertices.

Lemma 1.2. [6] *If G contains two vertices p and q such that $|N(p) \cap N(q)| = d(p) - 1 = d(q) - 1$, then G has no distance magic labeling.*

Miller et al. [6] have proved: P_n admits distance magic labeling if and only if $n = 1, 3$; C_n admits distance magic labeling if and only if $n = 4$; K_n admits distance magic labeling if and only if $n = 1$, and W_n admits distance magic labeling if and only if $n = 4$.

Definition 1.1. [5] A Cartesian product*, denoted by $G \square H$ is a graph with vertex set $V(G) \times V(H)$. Vertices (p, q) and (p', q') in $G \square H$ are adjacent if and only if $p = p'$ and q is adjacent to q' in H or $q = q'$ and p is adjacent to p' in G.*

Definition 1.2. [5] A direct product*, denoted by $G \times H$ is a graph with vertex set $V(G) \times V(H)$. Vertices (p, q) and (p', q') in $G \times H$ are adjacent if and only if p is adjacent to p' in G and q is adjacent to q' in H.*

Definition 1.3. [5] A strong product*, denoted by $G \boxtimes H$ is a graph with vertex set $V(G) \times V(H)$. Vertices (p, q) and (p', q') in $G \boxtimes H$ are adjacent if and only if $p = p'$ and q is adjacent to q' in H or $q = q'$ and p is adjacent to p' in G or p is adjacent to p' in G and q is adjacent to q' in H.*

Shrimali and Parmar [7] have proved $C_3^t \square C_4$, $C_4 \times C_3^t$ are not distance magic graphs. In the present paper again we deal with some product graphs.

2. Main Results

To prove all results in this section, we have used necessary condition for the existence of a distance magic labeling of a graph. Here we establish some sufficient conditions for the non-existence of distance magic product graphs.
Let W_t be a wheel graph obtained by joining all vertices $u_1, u_2, \ldots, u_t$ of a cycle C_t to central vertex r and let G be a graph with n vertices $v_1, v_2, \ldots, v_n$.
Consider $W_t \square G$, $W_t \boxtimes G$ are graphs with vertex set, $\{u_k^i, r^i / 1 \leq k \leq t, 1 \leq i \leq n\}$, where $u_k^i = (u_k, v_i)$, $r^i = (r, v_i)$.

Lemma 2.1. *Let G be a graph of order n. If G contains two vertices v_i and v_j such that $N(v_i) = N(v_j)$, then $W_t \square G$ is not a distance magic graph.*

Proof. Assume that $G^* = W_t \square G$ is a distance magic graph under a distance magic labeling f.
Let $N(v_i) = N(v_j) = \{v_{i_1}, v_{i_2}, \ldots v_{i_p}\}$, $1 \leq i_1, i_2, \ldots, i_p \leq n$.
Now,

$$w(r^i) = \sum_{k=1}^{t} f(u_k^i) + \sum_{\beta=i_1}^{i_p} f(r^\beta) \tag{1}$$

and

$$w(r^j) = \sum_{k=1}^{t} f(u_k^j) + \sum_{\beta=i_1}^{i_p} f(r^\beta). \tag{2}$$

Since, G^* is a distance magic graph, $w(r^i) = w(r^j)$.
So by (1) and (2), we get

$$\sum_{k=1}^{t} f(u_k^i) = \sum_{k=1}^{t} f(u_k^j). \tag{3}$$

Now,

$$\sum_{k=1}^{t} w(u_k^i) = tf(r^i) + 2\sum_{k=1}^{t} f(u_k^i) + \sum_{k=1}^{t}\sum_{\beta=i_1}^{i_p} f(u_k^\beta) \tag{4}$$

and

$$\sum_{k=1}^{t} w(u_k^j) = tf(r^j) + 2\sum_{k=1}^{t} f(u_k^j) + \sum_{k=1}^{t}\sum_{\beta=i_1}^{i_p} f(u_k^\beta). \tag{5}$$

Since, G^* is a distance magic graph, by Equations (4) and (5), we get

$$tf(r^i) + 2\sum_{k=1}^{t} f(u_k^i) = tf(r^j) + 2\sum_{k=1}^{t} f(u_k^j). \tag{6}$$

So, by Equations (3) and (6), $f(r^i) = f(r^j)$.
It gives us a contradiction to our statement that G^* is a distance magic graph.
Hence, G^* is not a distance magic graph. □

Lemma 2.2. *Let G be a graph of order n. If G contains two vertices v_i and v_j such that $N(v_i) = N(v_j)$, then $W_t \boxtimes G$ is not a distance magic graph.*

Proof. Assume that $G^* = W_t \boxtimes G$ is a distance magic graph under a distance magic labeling f.
Let $N(v_i) = N(v_j) = \{v_{i_1}, v_{i_2}, \ldots v_{i_p}\}$, $1 \le i_1, i_2, \ldots, i_p \le n$.
Since, G^* is a distance magic graph,

$$w(r^i) = w(r^j).$$

Therefore,

$$\sum_{k=1}^{t} f(u_k^i) + \sum_{k=1}^{t}\sum_{\beta=i_1}^{i_p} f(u_k^\beta) + \sum_{\beta=i_1}^{i_p} f(r^\beta) = \sum_{k=1}^{t} f(u_k^j) + \sum_{k=1}^{t}\sum_{\beta=i_1}^{i_p} f(u_k^\beta) + \sum_{\beta=i_1}^{i_p} f(r^\beta)$$

which implies,

$$\sum_{k=1}^{t} f(u_k^i) = \sum_{k=1}^{t} f(u_k^j). \tag{7}$$

Now,

$$w(u_1^i) = f(u_2^i) + f(u_n^i) + f(r^i) + \sum_{\beta=i_1}^{i_p} [f(u_1^\beta) + f(u_2^\beta) + f(u_n^\beta) + f(r^\beta)]$$

and

$$w(u_1^j) = f(u_2^j) + f(u_n^j) + f(r^j) + \sum_{\beta=i_1}^{i_p} [f(u_1^\beta) + f(u_2^\beta) + f(u_n^\beta) + f(r^\beta)].$$

As $w(u_1^i) = w(u_1^j)$, we get,

$$f(u_2^i) + f(u_n^i) + f(r^i) = f(u_2^j) + f(u_n^j) + f(r^j).$$

Analogously, we get such $t-$equations for each u_k^i and $u_k^j, k = 1, 2, \ldots, t$. If we add all such $t-$equations, we get,

$$2\sum_{k=1}^{t} f(u_k^i) + tf(r^i) = 2\sum_{k=1}^{t} f(u_k^j) + tf(r^j). \tag{8}$$

So, by Equations (7) and (8), $f(r^i) = f(r^j)$, which contradicts our hypothesis.
Hence, $W_t \boxtimes G$ is not a distance magic graph. □

Theorem 2.1. *The graph $W_t \times C_4$, $t \geq 5$ is not a distance magic graph.*

Proof. Suppose that the cycle C_4 has vertex set $\{v_0, v_1, v_2, v_3\}$ and let $G^* = W_t \times C_4$ be a distance magic graph under a distance magic labeling f and magic constant γ. So, weights of each vertex are equal, called it γ.
Here, vertex set $V(G^*) = \{u_k^i, r^i / 1 \leq k \leq t, 0 \leq i \leq 3\}$, where $u_k^i = (u_k, v_i)$, $r^i = (r, v_i)$.
We have

$$\begin{aligned} \gamma = w(r^0) &= \sum_{k=1}^{t} [f(u_k^1) + f(u_k^3)] \\ &= w(r^1) = \sum_{k=1}^{t} [f(u_k^0) + f(u_k^2)]. \end{aligned} \tag{9}$$

Now,

$$\sum_{k=1}^{t} w(u_k^0) = 2\sum_{k=1}^{t} [f(u_k^1) + f(u_k^3)] + t[f(r^1) + f(r^3)]. \tag{10}$$

Since, G^* is a distance magic graph, by Equations (9) and (10),

$$t\gamma = \sum_{k=1}^{t} w(u_k^0) = 2w(r^0) + t(f(r^1) + f(r^3)). \tag{11}$$

Similarly we can derive,

$$t\gamma = \sum_{k=1}^{t} w(u_k^1) = 2w(r^1) + t(f(r^0) + f(r^2)). \tag{12}$$

By Equations (9), (11) and (12), we get

$$f(r^1) + f(r^3) = f(r^0) + f(r^2) = \delta \text{ (say)}. \tag{13}$$

Let us take $f(u_k^1) + f(u_k^3) = \mu_k, \forall k$ in Equation (9), we get

$$\gamma = \sum_{k=1}^{t} \mu_k \tag{14}$$

Now, by Equations (10), (13) and (14)

$$\sum_{k=1}^{t} \mu_k = \frac{t \cdot \delta}{t-2}. \tag{15}$$

Let us find the sum of labels of all vertices except vertices $r^k (k = 0, 1, 2, 3)$, here $|V(G^*)| = 4t + 4$

$$\sum_{k=1}^{t} \left[f(u_k^0) + f(u_k^1) + f(u_k^2) + f(u_k^3) \right] = \frac{(4t+4)(4t+5)}{2} - \left(f(r^0) + f(r^1) + f(r^2) + f(r^3) \right)$$

$$2\sum_{k=1}^{t} \mu_k = 2(t+1)(4t+5) - 2\delta \tag{16}$$

By Equations (15) and (16)

$$\delta = \frac{(t+1)(4t+5)(t-2)}{2(t-1)},$$

which implies that

$$f(r^0) + f(r^2) = f(r^1) + f(r^3) = \frac{(t+1)(4t+5)(t-2)}{2(t-1)}. \tag{17}$$

Now, $f(r^0), f(r^2) \in \{1, 2, \ldots, 4t+4\}$. If we take $f(r^0) = 4t+4$, $f(r^1) = 4t+3$, $f(r^2) = 4t+1$ and $f(r^3) = 4t + 2$ then $f(r^1) + f(r^3) = f(r^0) + f(r^2) = \delta = 8t + 5$, which is less than $\frac{(t+1)(4t+5)(t-2)}{2(t-1)}$, for $t \geq 5$. So, equality in (17) cannot be possible.
Hence, $W_t \times C_4$, $t \geq 5$ is not a distance magic graph. □

Theorem 2.2. *The graph $W_3 \square P_n$ is not a distance magic graph.*

Proof. Let us assume that $G^* = W_3 \square P_n$ be a distance magic graph. So weights of every vertex are equal. Here, vertex set $V(G^*) = \{u_k^i, r^i / 1 \leq k \leq 3, 1 \leq i \leq n\}$, where $u_k^i = (u_k, v_i)$, $r^i = (r, v_i)$ and $\{v_i / 1 \leq i \leq n\}$ is the vertex set of path P_n.
Now,

$$w(u_1^1) = f(u_2^1) + f(u_3^1) + f(u_1^2) + f(r^1) \tag{18}$$

and

$$w(u_2^1) = f(u_1^1) + f(u_3^1) + f(u_2^2) + f(r^1). \tag{19}$$

Since, weights are equal, we obtain from (18) and (19),

$$f(u_2^1) + f(u_1^2) = f(u_1^1) + f(u_2^2). \tag{20}$$

Now,

$$w(u_1^2) = w(u_2^2) \tag{21}$$

$$\therefore f(u_2^2) + f(u_1^1) + f(u_1^3) = f(u_1^2) + f(u_2^1) + f(u_2^3). \tag{22}$$

From (20) and (21), $f(u_1^3) = f(u_2^3)$,
which is a contradict statement to our hypothesis.
Hence, $W_3 \square P_n$ is not a distance magic graph. □

Theorem 2.3. *The graph $W_4 \times P_3$ is not a distance magic graph.*

Proof. Let $G^* = W_4 \times P_3$ be a distance magic graph under the distance magic labeling f with magic constant γ. Here, vertex set $V(G^*) = \{u_k^i, r^i / 1 \leq k \leq 4, 1 \leq i \leq 3\}$, where $u_k^i = (u_k, v_i),\ r^i = (r, v_i)$ and $\{v_1, v_2, v_3\}$ is the vertex set of path P_3.
Here,

$$w(u_1^2) = f(u_2^1) + f(u_4^1) + f(u_2^3) + f(u_4^3) + f(r^1) + f(r^3) \tag{23}$$

$$w(u_2^2) = f(u_1^1) + f(u_3^1) + f(u_1^3) + f(u_3^3) + f(r^1) + f(r^3) \tag{24}$$

and

$$w(r^2) = f(u_1^1) + f(u_2^1) + f(u_3^1) + f(u_4^1) + f(u_1^3) + f(u_2^3) + f(u_3^3) + f(u_4^3). \tag{25}$$

From (23), (24) and (25), we get

$$w(u_1^2) + w(u_2^2) = w(r^2) + 2f(r^1) + 2f(r^3).$$

Thus,

$$f(r^1) + f(r^3) = \frac{\gamma}{2}.$$

Now,

$$w(u_1^1) = f(u_2^2) + f(u_4^2) + f(r^2) \tag{26}$$

$$w(u_2^1) = f(u_1^2) + f(u_3^2) + f(r^2) \tag{27}$$

and

$$w(r^1) = f(u_1^2) + f(u_2^2) + f(u_3^2) + f(u_4^2). \tag{28}$$

From (26), (27) and (28), we get

$$w(u_1^1) + w(u_2^1) = w(r^1) + 2f(r^2).$$

Thus,

$$f(r^2) = \frac{\gamma}{2}.$$

Here, $|V(G^*)| = 15$.
Thus, $\gamma \leq 30$ as $f(r^2) \in \{1, 2, \ldots, 15\}$.
Therefore we have,

$$\begin{aligned}
&f(r^1) + f(r^3) \leq 15,\\
&f(u_1^2) + f(u_3^2) \leq 15,\\
&f(u_2^2) + f(u_4^2) \leq 15,\\
&f(u_2^1) + f(u_4^1) + f(u_2^3) + f(u_4^3) \leq 15,\\
&f(u_1^1) + f(u_3^1) + f(u_1^3) + f(u_3^3) \leq 15,
\end{aligned}$$

which is not possible.
Hence, $W_4 \times P_3$ is not a distance magic graph. □

Theorem 2.4. *The graph $W_4 \times P_n, n \geq 4$ is not a distance magic graph.*

Proof. Let $G^* = W_4 \times P_n, n \geq 4$ be a distance magic graph under the distance magic labeling f with magic constant γ. Here, vertex set $V(G^*) = \{u_k^i, r^i/1 \leq k \leq 4, 1 \leq i \leq n\}$, where $u_k^i = (u_k, v_i),\ r^i = (r, v_i)$ and $\{v_i/1 \leq i \leq n\}$ is the vertex set of path P_n
Here,

$$w(u_1^1) = f(u_2^2) + f(u_4^2) + f(r^2) \tag{29}$$

and

$$w(u_1^3) = f(u_2^2) + f(u_4^2) + f(u_2^4) + f(u_4^4) + f(r^2) + f(r^4) \tag{30}$$

Since, G^* is a distance magic graph, by Equations (29) and (30), we get

$$f(u_2^4) + f(u_4^4) + f(r^4) = 0,$$

which is not possible.
Hence, the graph $W_4 \times P_n,\ n \geq 4$ is not a distance magic. $\square$

3. Conclusions

Here, we have proved $W_t \,\square\, G$ and $W_t \boxtimes G$ are not distance magic if graph G contains two vertices with same neighborhood. Then, we have proved $W_t \times C_4$ $(t \geq 5)$, $W_3 \square P_n$ and $W_4 \times P_n$ $(n \geq 3)$ graphs are not distance magic. To explore some new distance magic graphs is an open problem.

References

[1] Acharya, B. D., Rao, S. B., Singh, T. and Parameswaran, V., (2004), Neighborhood magic graphs, National Conference on Graph Theory, Combinatorics and Algorithm.

[2] Burton, D. M., (2007), Elementary Number Theory, Tata McGrow-Hill, New Delhi.

[3] Gallian, J. A., (2022), A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 25, DS#6.

[4] Gross, J. and Yellen, J., (2005), Graph Theory and its Applications, CRC Press, New York.

[5] Hammack, R. , Imrich,W. and Klavžar, S., (2011), Handbook of Product Graph, CRC Press, Boca Raton, FL.

[6] Miller, M., Rodger, C. and Simanjuntak, R., (2003), Distance magic labeling of graphs, Australasian Journal of Combinatorics, 28, pp. 305–315.

[7] Shrimali, N. P. and Parmar, Y. M., (2020), Distance magic and distance antimagic labeling of some product graphs, Recent Advancements in Graph Theory, CRC Press, Taylor's and Francis Group, pp. 181-191.

[8] Sugeng, K. A., Fronček, D., Miller, M., Ryan, J. and Walker, J., (2009), On distance magic labeling of graphs, Journal of Combinatorial Mathematics and Combinatorial Computing, 71, pp. 39–48.

[9] Vilfred, V., (1994), $\Sigma-$ labelled graphs and circulant graphs, Ph.D. thesis, University of Kerala, Trivandrum, India.

[10] Hamidov S.J., (2023) Effective trajectories of economic dynamics models on graphs. Appl. Comput. Math., V.22, N.2, pp.215-224

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