

**CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE  
(p, q)–DERIVATIVE OPERATOR FOR GENERALIZED DISTRIBUTION  
SATISFYING SUBORDINATE CONDITION**

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ABSTRACT. The main object of this paper is to study classes of analytic function associated with generalized Struve functions and using (p, q)–Jackson derivative. Furthermore, the bounds for generalized distribution using subordination principle involving modified sigmoid functions wear we estimate.

Keywords: Hadamard (convolution) product, subordination and quasi-subordination, Jackson (p, q)-derivative, generalized Struve functions, modified sigmoid functions.

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1. INTRODUCTION

Let  $\mathcal{H}(\mathbb{D})$  represent the class of all functions analytic in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , and also denote by  $\mathcal{A}$  the subclass of  $\mathcal{H}(\mathbb{D})$  comprising of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1}$$

which are normalized by the condition  $f(0) = f'(0) - 1 = 0$ .

For  $f_r(z) = \sum_{n=0}^{\infty} a_{n,r} z^n$ ,  $r = 1, 2$ , which are two analytic functions in  $\mathbb{D}$ , the *Hadamard (or convolution) product* of  $f_1$  and  $f_2$  is defined by

$$(f_1 * f_2)(z) := z + \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n, \quad z \in \mathbb{D}. \tag{2}$$

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For two functions  $h, g \in \mathcal{H}(\mathbb{D})$  we say  $h$  is *subordinate* to  $g$ , carved as  $h(z) \prec g(z)$ , if there exists a *Schwarz function*  $\psi(z) = \sum_{n=1}^{\infty} b_n z^n$  analytic in  $\mathbb{D}$  with  $\psi(0) = 0$  and  $|\psi(z)| < 1, z \in \mathbb{D}$ , such that  $h(z) = g(\psi(z))$  for all  $z \in \mathbb{D}$ . Further, if the function  $g$  is univalent in  $\mathbb{D}$ , then the next equivalence holds:

$$h(z) \prec g(z) \Leftrightarrow h(0) = g(0) \text{ and } h(\mathbb{D}) \subset g(\mathbb{D}).$$

The perception of *quasi-subordination* for two analytic functions is due to Robertson [28]. Thus, for the functions  $h, g \in \mathcal{H}(\mathbb{D})$ , the function  $h$  is said to be *quasi-subordinate* to  $g$ , written as  $h(z) \prec_{\varrho} g(z)$ , if there exist the analytic functions  $\varrho$  and  $w$ , with  $w(0) = 0$  such that  $|\varrho(z)| \leq 1, |w(z)| < 1$ , and  $h(z) = \varrho(z)g(w(z))$  for all  $z \in \mathbb{D}$ .

If we fix  $\varrho(z) \equiv 1$ , in the above definition then  $h(z) = g(w(z))$ , that is  $h(z) \prec g(z)$ . Also, if  $w(z) \equiv z$ , then  $h(z) = \varrho(z)g(z)$ , and in this case we say that  $h$  is *majorized* by  $g$ , written as  $h(z) \ll g(z)$ . Thus, the quasi-subordination is a generalization of the usual subordination as well as of the majorization (for details, see [2, 13, 29]).

Recall that if  $f \in \mathcal{A}$ , with  $zf'(z)/f(z) \prec \phi(z)$  and  $(1 + zf''(z))/f'(z) \prec \phi(z)$ , where  $\phi(z) = (1+z)/(1-z)$ , we acquire the two standard subclasses of (1) which are starlike and convex functions, respectively, and the classes comprising of starlike and convex functions are denoted by  $\mathcal{S}^*$  and  $\mathcal{K}$ , correspondingly.

Ma and Minda [21] unified various subclasses  $\mathcal{S}^*(\varphi)$  and  $\mathcal{K}(\varphi)$  of starlike and convex functions sustaining the subordination  $zf'(z)/f(z) \prec \varphi(z)$  and  $1 + zf''(z)/f'(z) \prec \varphi(z)$  respectively, where  $\varphi(z) = 1 + L_1z + L_2z^2 + \dots$ , with  $L_1 \in \mathbb{R}, L_1 > 0$ , and note that many results connected with these classes were obtained by several authors. For example, if  $\varphi(z) = (1 + Az)/(1 + Bz), -1 \leq B < A \leq 1$ , the classes  $\mathcal{S}^*(\varphi)$  and  $\mathcal{K}(\varphi)$  represent, respectively, to the classes  $\mathcal{S}^*[A, B]$  and  $\mathcal{K}[A, B]$  of *Janowski starlike and convex functions* [16], and in fact,  $\mathcal{S}^* := \mathcal{S}^*[1, -1]$  and  $\mathcal{K} := \mathcal{K}[1, -1]$ .

Let us recall some basic notations of  $(p, q)$ -calculus. Letting  $0 < q < p \leq 1$ , then the *Jackson  $(p, q)$ -derivative* of the function  $f$  is defined as

$$\mathfrak{D}_{p,q}f(z) := \begin{cases} \frac{f(pz) - f(qz)}{(p-q)z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0. \end{cases} \quad (3)$$

For functions  $f$  of the form (1), from (3) we have

$$\mathfrak{D}_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}, \quad z \in \mathbb{D}, \quad (4)$$

where the symbol  $[n]_{p,q}$  denotes the so-called  $(p, q)$ -*bracket* or *twin-basic number*, that is

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}.$$

We note that  $[n]_{p,q} = [n]_{q,p}$ , and for  $p = 1$  the Jackson  $(p, q)$ -derivative reduces to the *Jackson  $q$ -derivative*

$$\mathfrak{D}_q f(z) := \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0. \end{cases}$$

Also, the *twin-basic number* is a natural generalization of the  $q$ -*number*, that is

$$\lim_{p \rightarrow 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1.$$

One can easily show that  $\mathfrak{D}_{p,q}f(z) \rightarrow f'(z)$  as  $p \rightarrow 1^-$  and  $q \rightarrow 1^-$ , and  $\mathfrak{D}_{p,q}f(0) = f'(0) = 1$ . The operator  $\mathfrak{D}_{p,q}$  provide is an important tool that has been used to investigate the various subclasses of analytic functions of the form given by (3) (see [1, 5] and [25]).

Let  $S$  denote the sum of the convergent series of the form

$$S = \sum_{n=0}^{\infty} a_n,$$

where  $a_n \geq 0$ , for all  $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Recently, Porwal [27] introduced the *generalized discrete probability distribution*, whose probability mass function is given by

$$p(n) = \frac{a_n}{S}, \quad n \in \mathbb{N}_0,$$

where  $p(n)$  is a probability mass function because  $p(n) \geq 0$  and  $\sum_{n \in \mathbb{N}_0} p_n = 1$ . Moreover, if we let

$$\wp(w) = \sum_{n=0}^{\infty} a_n w^n,$$

then the above series is convergent for  $|w| < 1$  and  $w = 1$ . Further, the sum of the power series whose coefficients are probabilities of the generalized distribution is given by

$$K(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{S} z^n, \quad z \in \mathbb{D}, \quad (5)$$

where  $S = \sum_{n=0}^{\infty} a_n$ .

Recently, Srivastava et al. [31] discussed the function  $H_\sigma$  associated with the *generalized Struve function* as follows:

$$H_\sigma(z) := z + \sum_{n=2}^{\infty} \frac{(1+n\sigma-\sigma)(-c/4)^{n-1}}{(3/2)_{n-1}(k)_{n-1}} z^n, \quad z \in \mathbb{D}, \quad 0 \leq \sigma \leq 1, \quad c \in \mathbb{C}, \quad (6)$$

where  $(x)_n$  denotes the *Pochhammer symbol* given by  $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\dots(x+n-1)$ ,  $(x)_0 = 1$ , and  $(1)_n = n!$ .

Using (5) and (6), we define the function

$$\mathcal{F}(z) := (K * H_\sigma)(z) = z + \sum_{n=2}^{\infty} \frac{(1+n\sigma-\sigma)(-c/4)^{n-1} a_{n-1}}{(3/2)_{n-1}(k)_{n-1} S} z^n, \quad z \in \mathbb{D}. \quad (7)$$

From (4) and (7) it easily follows that

$$\mathfrak{D}_{p,q}\mathcal{F}(z) = 1 + \sum_{n=2}^{\infty} \frac{(1+n\sigma-\sigma)[n]_{p,q}(-c/4)^{n-1} a_{n-1}}{(3/2)_{n-1}(k)_{n-1} S} z^{n-1}, \quad z \in \mathbb{D}.$$

In [15] Fadipe-Joseph et al. studied some geometrical properties of the modified sigmoid function (see Figure 1)

$$\begin{aligned} \varpi(z) &= \frac{2}{1+e^{-z}} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} z^n \right)^m \\ &= 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{17}{40320} z^7 + \dots, \quad z \in \mathbb{D}. \end{aligned} \quad (8)$$

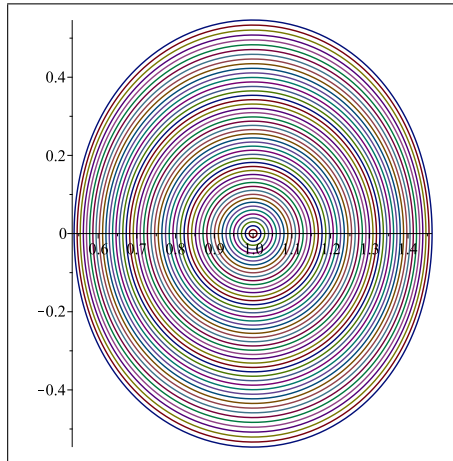


FIGURE 1. The image of  $\varpi(\mathbb{D})$

Special functions and quantum calculus have important applications in almost every field of engineering and science (see, for details, [3, 6, 7, 8, 9, 10, 11, 14, 20, 26, 30, 31, 32, 34, 35]). Nowadays, specially applications of *Struve functions* occur in water-wave and surface-wave problems, unsteady aerodynamics, resistive MHD instability theory and optical diffraction. Further sigmoid function has recently taken the attention of many researchers these days because of its wide application in neural network, artificial intelligence, nonlinear approximation theory, statistics, and so on. By using the aforementioned concepts and recent studies on analytic functions, in this work the concept of subordinate principle involving modified sigmoid functions with  $(p, q)$ -derivative operator is used to define new classes of analytic functions as given below:

**Definition 1.1.** Let  $0 < q < p \leq 1$ ,  $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and suppose that  $\vartheta(z) = 1 + B_1z + B_2z^2 + \dots$ , with  $B_1 \in \mathbb{R}$ ,  $B_1 > 0$ , and  $\varrho(z) = \varrho_0 + \varrho_1z + \varrho_2z^2 + \dots$ , with  $|\varrho(z)| \leq 1$  for all  $z \in \mathbb{D}$ , are two analytic functions in the unit disc  $\mathbb{D}$ .

(i) We say that the function  $f \in \mathcal{A}$  is in the class  $\psi_\varrho S_{\beta, \gamma}^{p, q}(\vartheta)$  if

$$\frac{1}{\gamma} \left( (1 + i \tan \beta) \frac{z \mathfrak{D}_{p, q} \mathcal{F}(z)}{\mathcal{F}(z)} - i \tan \beta - 1 \right) \prec_\varrho \vartheta(z) - 1.$$

(ii) The function  $f \in \mathcal{A}$  belongs to the class  $\hat{\psi}_\varrho C_{\beta, \gamma}^{p, q}(\vartheta)$  if

$$\frac{1}{\gamma} \left( (1 + i \tan \beta) \frac{z \mathfrak{D}_{p, q} (\mathfrak{D}_{p, q} \mathcal{F}(z))}{\mathfrak{D}_{p, q} \mathcal{F}(z)} - i \tan \beta - 1 \right) \prec_\varrho \vartheta(z) - 1.$$

Note that, if we consider the functions

$$f_*(z) = z + \frac{\gamma B_1 \varrho_0 w_1}{(1 + i \tan \beta) ([2]_{p, q} - 1)} z^2, \quad z \in \mathbb{D},$$

and

$$\hat{f}_*(z) = z + \frac{\gamma B_1 \varrho_0 w_1}{(1 + i \tan \beta) [2]_{p, q} ([2]_{p, q} - 1)} z^2, \quad z \in \mathbb{D},$$

then it is easy to check that  $f_* \in \psi_\varrho S_{\beta, \gamma}^{p, q}(\vartheta)$  and  $\hat{f}_* \in \hat{\psi}_\varrho C_{\beta, \gamma}^{p, q}(\vartheta)$ , that is  $\psi_\varrho S_{\beta, \gamma}^{p, q}(\vartheta) \neq \emptyset$  and  $\hat{\psi}_\varrho C_{\beta, \gamma}^{p, q}(\vartheta) \neq \emptyset$ .

**Remark 1.1.** (i) For  $p = 1$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$ , a function  $f \in \mathcal{A}$  belongs to the class  $\psi S_{\beta, \gamma}^q(\varpi) := \psi_{\varrho} S_{\beta, \gamma}^{1, q}(\vartheta)$  if

$$1 + \frac{1}{\gamma} \left( (1 + i \tan \beta) \frac{z \mathfrak{D}_q \mathcal{F}(z)}{\mathcal{F}(z)} - i \tan \beta - 1 \right) \prec \varpi(z),$$

and  $f \in \mathcal{A}$  is in the class  $\hat{\psi} C_{\beta, \gamma}^q(\varpi) := \hat{\psi}_{\varrho} C_{\beta, \gamma}^{1, q}(\vartheta)$  if

$$1 + \frac{1}{\gamma} \left( (1 + i \tan \beta) \frac{z \mathfrak{D}_q (\mathfrak{D}_q \mathcal{F}(z))}{\mathfrak{D}_q \mathcal{F}(z)} - i \tan \beta - 1 \right) \prec \varpi(z),$$

where  $0 < q < 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  is defined by (8).

(ii) For  $\gamma = 1$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$ , a function  $f \in \mathcal{A}$  belongs to the class  $\psi S_{\beta}^{p, q}(\varpi) := \psi_{\varrho} S_{\beta, 1}^{p, q}(\vartheta)$  if

$$(1 + i \tan \beta) \frac{z \mathfrak{D}_q \mathcal{F}(z)}{\mathcal{F}(z)} - i \tan \beta \prec \varpi(z),$$

and  $f \in \mathcal{A}$  is in the class  $\hat{\psi} C_{\beta}^q(\varpi) := \hat{\psi}_{\varrho} C_{\beta, 1}^{p, q}(\vartheta)$  if

$$(1 + i \tan \beta) \frac{z \mathfrak{D}_q (\mathfrak{D}_q \mathcal{F}(z))}{\mathfrak{D}_q \mathcal{F}(z)} - i \tan \beta \prec \varpi(z),$$

where  $0 < q < p \leq 1$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  as defined by (8).

(iii) For  $\beta \equiv 0$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$ , a function  $f \in \mathcal{A}$  belongs to the class  $\psi S_{\gamma}^{p, q}(\varpi) := \psi_{\varrho} S_{0, \gamma}^{p, q}(\vartheta)$  if

$$1 + \frac{1}{\gamma} \left( \frac{z \mathfrak{D}_{p, q} \mathcal{F}(z)}{\mathcal{F}(z)} - 1 \right) \prec \varpi(z),$$

and  $f \in \mathcal{A}$  is in the class  $\hat{\psi} C_{\gamma}^{p, q}(\varpi) := \hat{\psi}_{\varrho} C_{0, \gamma}^{p, q}(\vartheta)$  if

$$1 + \frac{1}{\gamma} \left( \frac{z \mathfrak{D}_{p, q} (\mathfrak{D}_{p, q} \mathcal{F}(z))}{\mathfrak{D}_{p, q} \mathcal{F}(z)} - 1 \right) \prec \varpi(z),$$

where  $0 < q < p \leq 1$ ,  $\gamma \in \mathbb{C}^*$ , and  $\varpi$  as defined by (8).

Motivated by several earlier investigation in connections between various subclasses of analytic functions by using special functions [12, 17, 19, 33], and inspired by the recent work of Altinkaya and Olatunji [4] and Oladipo [22, 23, 24], now we obtain the bounds for the first three coefficients of the above defined subclasses of functions.

**Lemma 1.1.** [18] Let  $w(z) = w_1 z + w_2 z^2 + \dots$ ,  $z \in \mathbb{D}$ , be an analytic function in  $\mathbb{D}$ , with  $|w(z)| < 1$  in  $\mathbb{D}$ . Then, for any complex number  $t$  we have

$$|w_2 + t w_1^2| \leq \max\{1; |t|\}, \quad t \in \mathbb{C}.$$

The inequality is sharp for the functions  $w(z) = z$  or  $w(z) = z^2$ .

## 2. COEFFICIENTS BOUNDS

We begin this section by finding the estimates on the first three coefficients bounds for the functions that belongs to the classes  $\psi_{\varrho} S_{\beta, \gamma}^{p, q}(\varpi)$  and  $\hat{\psi}_{\varrho} C_{\beta, \gamma}^{p, q}(\varpi)$ .

Unless otherwise mentioned, we assume throughout the paper that

- (i)  $w(z) = w_1 z + w_2 z^2 + \dots$ , with  $|w(z)| < 1$ ,  $z \in \mathbb{D}$ ,
- (ii)  $\varrho(z) = \varrho_0 + \varrho_1 z + \varrho_2 z^2 + \dots$ , with  $|\varrho(z)| \leq 1$ ,  $z \in \mathbb{D}$ ,
- (iii)  $\vartheta(z) = 1 + B_1 z + B_2 z^2 + \dots$ , with  $B_1 \in \mathbb{R}$ ,  $B_1 > 0$ .

**Theorem 2.1.** Let  $0 < q < p \leq 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . Set

$$\begin{aligned} \kappa_1 &= \frac{2(1 + \sigma)}{3k} \left(-\frac{c}{4}\right), & \kappa_2 &= \frac{4(1 + 2\sigma)}{15k(k + 1)} \left(-\frac{c}{4}\right)^2, \\ \kappa_3 &= \frac{4(1 + \sigma)^2}{9k^2} \left(-\frac{c}{4}\right)^2, & \kappa_4 &= \frac{8(1 + 3\sigma)}{105k(k^2 + 3k + 2)} \left(-\frac{c}{4}\right)^3, \\ \kappa_5 &= \frac{8(1 + \sigma)(1 + 2\sigma)}{45k^2(k + 1)} \left(-\frac{c}{4}\right)^3, & \kappa_6 &= \frac{8(1 + \sigma)^3}{27k^3} \left(-\frac{c}{4}\right)^3. \end{aligned}$$

If  $f \in \psi_\varrho S_{\beta,\gamma}^{p,q}(\vartheta)$  is given by (1), then

$$\begin{aligned} \left| \frac{a_1}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([2]_{p,q} - 1) \kappa_1}, \\ \left| \frac{a_2}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([3]_{p,q} - 1) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta ([2]_{p,q} - 1) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\}, \\ \left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([3]_{p,q} - 1) \kappa_2} \\ &\times \max \left\{ 1; \left| \frac{B_2 \zeta ([2]_{p,q} - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma [(2]_{p,q} - 1) \kappa_3 - \mu ([3]_{p,q} - 1) \kappa_2}{B_1 \zeta ([2]_{p,q} - 1)^2 \kappa_1^2} \right| \right\}, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{a_3}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([4]_{p,q} - 1) \kappa_4} \\ &\times \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta ([2]_{p,q} - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma \kappa_3}{B_1 \zeta ([2]_{p,q} - 1) \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) \right. \right. \\ &\quad \left. \left. + \left( 1 + \frac{2B_2}{B_1 \xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\}, \end{aligned}$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_0^2 \varrho_0^2 \gamma^2 \kappa_6}{\zeta^2 ([2]_{p,q} - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{B_1 \varrho_0 \gamma [(2]_{p,q} - 1) + ([3]_{p,q} - 1) \kappa_5}{\zeta ([2]_{p,q} - 1) ([3]_{p,q} - 1) \kappa_1 \kappa_2}$ .

*Proof.* If  $f \in \psi_\varrho S_{\beta,\gamma}^{p,q}(\vartheta)$  then there exists analytic functions in  $\mathbb{D}$  that are  $\varrho$  and  $w$ , with  $|\varrho(z)| \leq 1$ ,  $w(0) = 0$  and  $|w| < 1$ , such that

$$\frac{1}{\gamma} \left( (1 + i \tan \beta) \frac{z \mathfrak{D}_{p,q} \mathcal{F}(z)}{\mathcal{F}(z)} - i \tan \beta - 1 \right) = \varrho(z) (\vartheta(w(z)) - 1). \tag{9}$$

We note that

$$\begin{aligned} \frac{z \mathfrak{D}_{p,q} \mathcal{F}(z)}{\mathcal{F}(z)} &= 1 + \frac{([2]_{p,q} - 1) \kappa_1 a_1}{S} z + \left( \frac{([3]_{p,q} - 1) \kappa_2 a_2}{S} - \frac{([2]_{p,q} - 1) \kappa_3 a_1^2}{S^2} \right) z^2 \\ &+ \left( \frac{([4]_{p,q} - 1) \kappa_4 a_3}{S} - \frac{[(2]_{p,q} - 1) + ([3]_{p,q} - 1) \kappa_5 a_1 a_2}{S^2} + \frac{([2]_{p,q} - 1) \kappa_6 a_1^3}{S^3} \right) z^3 + \dots, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \varrho(z) (\vartheta(w(z)) - 1) &= B_1 \varrho_0 w_1 z + [B_1 \varrho_1 w_1 + \varrho_0 (B_1 w_2 + B_2 w_1^2)] z^2 \\ &+ [B_1 \varrho_2 w_1 + \varrho_1 (B_1 w_2 + B_2 w_1^2) + \varrho_0 (B_1 w_3 + 2B_1 w_1 w_2 + B_3 w_1^3)] z^3 + \dots \end{aligned} \tag{11}$$

From (9), (10) and (11) we obtain

$$\frac{a_1}{S} = \frac{B_1 \varrho_0 \gamma w_1}{\zeta ([2]_{p,q} - 1) \kappa_1}, \tag{12}$$

$$\frac{a_2}{S} = \frac{B_1 \varrho_0 \gamma}{\zeta ([3]_{p,q} - 1) \kappa_2} \left[ w_2 + \left( \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta ([2]_{p,q} - 1) \kappa_1^2} + \frac{B_2}{B_1} \right) w_1^2 + \frac{\varrho_1}{\varrho_0} w_1 \right], \tag{13}$$

and using Lemma 1.1 we deduce

$$\left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([3]_{p,q} - 1) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta ([2]_{p,q} - 1) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\}.$$

According to (12) and (13) it follows that

$$\begin{aligned} \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} &= \frac{B_1 \varrho_0 \gamma}{\zeta ([3]_{p,q} - 1) \kappa_2} \\ &\times \left[ w_2 + \frac{B_2 \zeta ([2]_{p,q} - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma [( [2]_{p,q} - 1) \kappa_3 - \mu ([3]_{p,q} - 1) \kappa_2]}{B_1 \zeta ([2]_{p,q} - 1)^2 \kappa_1^2} w_1^2 + \frac{\varrho_1}{\varrho_0} w_1 \right], \end{aligned}$$

and moreover, from (9), (10) and (11) we get

$$\begin{aligned} \frac{a_3}{S} &= \frac{B_1 \varrho_0 \gamma}{\zeta ([4]_{p,q} - 1) \kappa_4} \times \left\{ w_3 + \xi \left[ \left( \frac{B_2 \zeta ([2]_{p,q} - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma \kappa_3}{B_1 \zeta ([2]_{p,q} - 1) \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} \right) w_1^3 \right. \right. \\ &\quad \left. \left. + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) w_1^2 + \left( 1 + \frac{2B_2}{B_1 \xi} \right) w_1 w_2 + \frac{1}{\varrho_0 \xi} (\varrho_1 w_2 + \varrho_2 w_1) \right] \right\}, \end{aligned}$$

where

$$\zeta = 1 + i \tan \beta, \eta = \frac{B_0^2 \varrho_0^2 \gamma^2 \kappa_6}{\zeta^2 ([2]_{p,q} - 1)^2 \kappa_1^3}, \text{ and } \xi = \frac{B_1 \varrho_0 \gamma [( [2]_{p,q} - 1) + ([3]_{p,q} - 1)] \kappa_5}{\zeta ([2]_{p,q} - 1) ([3]_{p,q} - 1) \kappa_1 \kappa_2}. \quad \square$$

If we take  $p = 1$  in Theorem 2.1 we obtain the following result:

**Corollary 2.1.** *Let  $0 < q < 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho S_{\beta, \gamma}^q(\vartheta)$  is given by (1), then*

$$\begin{aligned} \left| \frac{a_1}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([2]_q - 1) \kappa_1}, \\ \left| \frac{a_2}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\}, \\ \left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([3]_q - 1) \kappa_2} \\ &\times \max \left\{ 1; \left| \frac{B_2 \zeta ([2]_q - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma [( [2]_q - 1) \kappa_3 - \mu ([3]_q - 1) \kappa_2]}{B_1 \zeta ([2]_q - 1)^2 \kappa_1^2} \right| \right\}, \end{aligned}$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| ([4]_q - 1) \kappa_4} \times \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta ([2]_q - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma \kappa_3}{B_1 \zeta ([2]_q - 1) \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2B_2}{B_1 \xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_0^2 \varrho_0^2 \gamma^2 \kappa_6}{\zeta^2 ([2]_q - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{B_1 \varrho_0 \gamma [(2]_q - 1) + ([3]_q - 1) \kappa_5}{\zeta ([2]_q - 1) ([3]_q - 1) \kappa_1 \kappa_2}$ .

For  $p = 1$  and  $\gamma = 1$  Theorem 2.1 yields to the next result:

**Corollary 2.2.** Let  $0 < q < 1$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho S_\beta^q(\vartheta)$  is given by (1), then

$$\left| \frac{a_1}{S} \right| \leq \frac{B_1 \varrho_0}{|\zeta| ([2]_q - 1) \kappa_1},$$

$$\left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0}{|\zeta| ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{B_1 \varrho_0}{|\zeta| ([3]_q - 1) \kappa_2} \times \max \left\{ 1; \left| \frac{B_2 \zeta ([2]_q - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 [(2]_q - 1) \kappa_3 - \mu ([3]_q - 1) \kappa_2}{B_1 \zeta ([2]_q - 1)^2 \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{B_1 \varrho_0}{|\zeta| ([4]_q - 1) \kappa_4} \times \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta ([2]_q - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \kappa_3}{B_1 \zeta ([2]_q - 1) \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2B_2}{B_1 \xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_0^2 \varrho_0^2 \kappa_6}{\zeta^2 ([2]_q - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{B_1 \varrho_0 [(2]_q - 1) + ([3]_q - 1) \kappa_5}{\zeta ([2]_q - 1) ([3]_q - 1) \kappa_1 \kappa_2}$ .

Considering the special case  $p = 1$  and  $q \rightarrow 1^-$  in Theorem 2.1 we obtain:

**Corollary 2.3.** Let  $\gamma \in \mathbb{C}^*$  and  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho S_{\beta, \gamma}(\vartheta)$  is given by (1), then

$$\left| \frac{a_1}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| \kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{2|\zeta| \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{2|\zeta| \kappa_2} \max \left\{ 1; \left| \frac{B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 \gamma [\kappa_3 - 2\mu \kappa_2]}{B_1 \zeta \kappa_1^2} \right| \right\},$$



and

$$\left| \frac{a_3}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{3|\zeta| \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 \gamma \kappa_3}{B_1 \zeta \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2B_2}{B_1 \xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_0^2 \varrho_0^2 \gamma^2 \kappa_6}{\zeta^2 \kappa_1^3}$ , and  $\xi = \frac{3B_1 \varrho_0 \gamma \kappa_5}{2\zeta \kappa_1 \kappa_2}$ .

If we take  $\gamma = 1$ ,  $p = 1$  and  $q \rightarrow 1^-$  in Theorem 2.1 we obtain the next result:

**Corollary 2.4.** *Let  $\gamma \in \mathbb{C}^*$  and  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho S_\beta(\vartheta)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{B_1 \varrho_0}{|\zeta| \kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0}{2|\zeta| \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \kappa_3}{\zeta \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{B_1 \varrho_0}{2|\zeta| \kappa_2} \max \left\{ 1; \left| \frac{B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 [\kappa_3 - 2\mu \kappa_2]}{B_1 \zeta \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{B_1 \varrho_0}{3|\zeta| \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 \kappa_3}{B_1 \zeta \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2B_2}{B_1 \xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_0^2 \varrho_0^2 \kappa_6}{\zeta^2 \kappa_1^3}$ , and  $\xi = \frac{3B_1 \varrho_0 \kappa_5}{2\zeta \kappa_1 \kappa_2}$ .

If we take  $\varrho \equiv 1$  and  $\vartheta := \varpi$ , then Theorem 2.1 reduces to the next special case:

**Corollary 2.5.** *Let  $0 < q < p \leq 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \psi S_{\beta, \gamma}^{p, q}(\varpi)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{|\gamma|}{|\zeta| ([2]_{p, q} - 1) \kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{|\gamma|}{|\zeta| ([3]_{p, q} - 1) \kappa_2} \max \left\{ 1; \left| \frac{\gamma \kappa_3}{\zeta ([2]_{p, q} - 1) \kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{|\gamma|}{|\zeta| ([3]_{p, q} - 1) \kappa_2} \max \left\{ 1; \left| \frac{\gamma [([2]_{p, q} - 1) \kappa_3 - \mu ([3]_{p, q} - 1) \kappa_2]}{\zeta ([2]_{p, q} - 1)^2 \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{|\gamma|}{|\zeta| ([4]_{p, q} - 1) \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\gamma \kappa_3}{\zeta ([2]_{p, q} - 1) \kappa_1^2} - \frac{1}{\xi} \left( \eta + \frac{1}{12} \right) + 1 \right] \right| \right\},$$

where  $\zeta = 2(1 + i \tan \beta)$ ,  $\eta = \frac{\gamma^2 \kappa_6}{\zeta^2 ([2]_{p, q} - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{[(2]_{p, q} - 1) + ([3]_{p, q} - 1) \gamma \kappa_5}{\zeta ([2]_{p, q} - 1) ([3]_{p, q} - 1) \kappa_1 \kappa_2}$ .

**Remark 2.1.** *If we consider the function*

$$f_*(z) = z + \frac{\gamma w_1}{2(1 + i \tan \beta) ([2]_{p, q} - 1)} z^2,$$

then it is easy to check that  $f_* \in \psi S_{\beta, \gamma}^{p, q}(\varpi)$ , that is  $\psi S_{\beta, \gamma}^{p, q}(\varpi) \neq \emptyset$ .

Replacing  $p = 1$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$  in Theorem 2.1 we get:

**Corollary 2.6.** *Let  $0 < q < 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \psi S_{\beta, \gamma}^q(\varpi)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{|\gamma|}{|\zeta| ([2]_q - 1) \kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{|\gamma|}{|\zeta| ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{\gamma \kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{|\gamma|}{|\zeta| ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{\gamma [(2]_q - 1) \kappa_3 - \mu ([3]_q - 1) \kappa_2}{\zeta ([2]_q - 1)^2 \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{|\gamma|}{|\zeta| ([4]_q - 1) \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\gamma \kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} - \frac{1}{\xi} \left( \eta + \frac{1}{12} \right) + 1 \right] \right| \right\}.$$

where  $\zeta = 2(1 + i \tan \beta)$ ,  $\eta = \frac{\gamma^2 \kappa_6}{\zeta^2 ([2]_q - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{[(2]_q - 1) + ([3]_q - 1) \gamma \kappa_5}{\zeta ([2]_q - 1) ([3]_q - 1) \kappa_1 \kappa_2}$ .

Taking  $p = 1$ ,  $\gamma = 1$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$  Theorem 2.1 yields to the next result:

**Corollary 2.7.** *Let  $0 < q < 1$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \psi S_{\beta}^q(\varpi)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{1}{|\zeta| ([2]_q - 1) \kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{1}{|\zeta| ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{\kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{1}{|\zeta| ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{([2]_q - 1) \kappa_3 - \mu ([3]_q - 1) \kappa_2}{\zeta ([2]_q - 1)^2 \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{1}{|\zeta| ([4]_q - 1) \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} - \frac{1}{\xi} \left( \eta + \frac{1}{12} \right) + 1 \right] \right| \right\}$$

where  $\zeta = 2(1 + i \tan \beta)$ ,  $\eta = \frac{\kappa_6}{\zeta^2 ([2]_q - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{[(2]_q - 1) + ([3]_q - 1) \kappa_5}{\zeta ([2]_q - 1) ([3]_q - 1) \kappa_1 \kappa_2}$ .

For  $p = 1$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$  and  $q \rightarrow 1^-$  Theorem 2.1 reduces to the next special case:

**Corollary 2.8.** *Let  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \psi S_{\beta, \gamma}(\varpi)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{|\gamma|}{|\zeta| \kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{|\gamma|}{2|\zeta| \kappa_2} \max \left\{ 1; \left| \frac{\gamma \kappa_3}{\zeta \kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{|\gamma|}{2|\zeta| \kappa_2} \max \left\{ 1; \left| \frac{\gamma [\kappa_3 - 2\mu \kappa_2]}{\zeta \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{|\gamma|}{3|\zeta| \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\gamma \kappa_3}{\zeta \kappa_1^2} - \frac{1}{\xi} \left( \eta + \frac{1}{12} \right) + 1 \right] \right| \right\}.$$

where  $\zeta = 2(1 + i \tan \beta)$ ,  $\eta = \frac{\gamma^2 \kappa_6}{\zeta^2 \kappa_1^3}$ , and  $\xi = \frac{3\gamma \kappa_5}{2\zeta \kappa_1 \kappa_2}$ .

If we take  $\gamma = 1$ ,  $p = 1$ ,  $\varrho \equiv 1$ ,  $\vartheta := \varpi$  and  $q \rightarrow 1^-$  in Theorem 2.1 we obtain the next result:

**Corollary 2.9.** Let  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \psi S_\beta(\varpi)$  is given by (1), then

$$\left| \frac{a_1}{S} \right| \leq \frac{1}{|\zeta|\kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{1}{2|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{\kappa_3}{\zeta\kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{1}{2|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{\kappa_3 - 2\mu\kappa_2}{\zeta\kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{1}{3|\zeta|\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\kappa_3}{\zeta\kappa_1^2} - \frac{1}{\xi} \left( \eta + \frac{1}{12} \right) + 1 \right] \right| \right\}.$$

where  $\zeta = 2(1 + i \tan \beta)$ ,  $\eta = \frac{\kappa_6}{\zeta^2\kappa_1^3}$ , and  $\xi = \frac{3\kappa_5}{2\zeta\kappa_1\kappa_2}$ .

**Theorem 2.2.** Let  $0 < q < p \leq 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . Set

$$\kappa_1 = \frac{2(1 + \sigma)}{3k} \left( -\frac{c}{4} \right), \quad \kappa_2 = \frac{4(1 + 2\sigma)}{15k(k + 1)} \left( -\frac{c}{4} \right)^2,$$

$$\kappa_3 = \frac{4(1 + \sigma)^2}{9k^2} \left( -\frac{c}{4} \right)^2, \quad \kappa_4 = \frac{8(1 + 3\sigma)}{105k(k^2 + 3k + 2)} \left( -\frac{c}{4} \right)^3,$$

$$\kappa_5 = \frac{8(1 + \sigma)(1 + 2\sigma)}{45k^2(k + 1)} \left( -\frac{c}{4} \right)^3, \quad \kappa_6 = \frac{8(1 + \sigma)^3}{27k^3} \left( -\frac{c}{4} \right)^3.$$

If  $f \in \psi_\varrho C_{\beta,\gamma}^{p,q}(\vartheta)$  is given by (1), then

$$\left| \frac{a_1}{S} \right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[2]_{p,q}([2]_{p,q} - 1)\kappa_1},$$

$$\left| \frac{a_2}{S} \right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[3]_{p,q}([3]_{p,q} - 1)\kappa_2} \max \left\{ 1; \left| \frac{B_1\varrho_0\gamma\kappa_3}{\zeta([2]_{p,q} - 1)\kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[3]_{p,q}([3]_{p,q} - 1)\kappa_2}$$

$$\times \max \left\{ 1; \left| \frac{B_2\zeta[2]_{p,q}^2([2]_{p,q} - 1)^2\kappa_1^2 + B_1^2\varrho_0\gamma \{ [2]_{p,q}^2([2]_{p,q} - 1)\kappa_3 - \mu[3]_{p,q}([3]_{p,q} - 1)\kappa_2 \}}{B_1\zeta[2]_{p,q}^2([2]_{p,q} - 1)^2\kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[4]_{p,q}([4]_{p,q} - 1)\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{B_2\zeta([2]_{p,q} - 1)\kappa_1^2 + B_1^2\varrho_0\gamma\kappa_3}{B_1\zeta([2]_{p,q} - 1)\kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1\xi} \right. \right. \right.$$

$$\left. \left. + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1\xi} \right) + \left( 1 + \frac{2}{\xi} \right) + \frac{1}{\varrho_0\xi} (\varrho_1 + \varrho_2) \right] \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_1^2\varrho_0^2\gamma^2\kappa_6}{\zeta^3([2]_{p,q} - 1)^2\kappa_1^3}$ , and  $\xi = \frac{B_1\varrho_0\gamma [(2]_{p,q} - 1) + ([3]_{p,q} - 1)\kappa_5}{\zeta^2([2]_{p,q} - 1)([3]_{p,q} - 1)\kappa_1\kappa_2}$ .

*Proof.* Proceeding as in the proof of Theorem 2.1 excepting that instead of using (9) we will use the subordination

$$\frac{1}{\gamma} \left( (1 + i \tan \beta) \frac{z\mathfrak{D}_{p,q}(\mathfrak{D}_{p,q}\mathcal{F}(z))}{\mathfrak{D}_{p,q}\mathcal{F}(z)} - i \tan \beta - 1 \right) \prec_\rho \vartheta(z) - 1,$$

our result follows easily. □

If we take  $p = 1$  in Theorem 2.2 we obtain the following result:

**Corollary 2.10.** *Let  $0 < q < 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho C_{\beta, \gamma}^q(\vartheta)$  is given by (1), then*

$$\begin{aligned} \left| \frac{a_1}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| [2]_q ([2]_q - 1) \kappa_1}, \\ \left| \frac{a_2}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| [3]_q ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\}, \\ \left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| [3]_q ([3]_q - 1) \kappa_2} \\ &\times \max \left\{ 1; \left| \frac{B_2 \zeta [2]_q^2 ([2]_q - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma \{ [2]_q^2 ([2]_q - 1) \kappa_3 - \mu [3]_q ([3]_q - 1) \kappa_2 \}}{B_1 \zeta [2]_q^2 ([2]_q^2 - 1)^2 \kappa_1^2} \right| \right\}, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{a_3}{S} \right| &\leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| [4]_q ([4]_q - 1) \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta ([2]_q - 1) \kappa_1^2 + B_1^2 \varrho_0 \gamma \kappa_3}{B_1 \zeta ([2]_q - 1) \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2}{\xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\}, \end{aligned}$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_1^2 \varrho_0^2 \gamma^2 \kappa_6}{\zeta^3 ([2]_q - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{B_1 \varrho_0 \gamma [(2]_q - 1) + ([3]_q - 1) \kappa_5}{\zeta^2 ([2]_q - 1) ([3]_q - 1) \kappa_1 \kappa_2}$ .

For  $p = 1$  and  $\gamma = 1$  in Theorem 2.2 yields to the next result:

**Corollary 2.11.** *Let  $0 < q < 1$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho C_\beta^q(\vartheta)$  is given by (1), then*

$$\begin{aligned} \left| \frac{a_1}{S} \right| &\leq \frac{B_1 \varrho_0}{|\zeta| [2]_q ([2]_q - 1) \kappa_1}, \\ \left| \frac{a_2}{S} \right| &\leq \frac{B_1 \varrho_0}{|\zeta| [3]_q ([3]_q - 1) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \kappa_3}{\zeta ([2]_q - 1) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\}, \\ \left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| &\leq \frac{B_1 \varrho_0}{|\zeta| [3]_q ([3]_q - 1) \kappa_2} \\ &\times \max \left\{ 1; \left| \frac{B_2 \zeta [2]_q^2 ([2]_q - 1)^2 \kappa_1^2 + B_1^2 \varrho_0 \{ [2]_q^2 ([2]_q - 1) \kappa_3 - \mu [3]_q ([3]_q - 1) \kappa_2 \}}{B_1 \zeta [2]_q^2 ([2]_q^2 - 1)^2 \kappa_1^2} \right| \right\}, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{a_3}{S} \right| &\leq \frac{B_1 \varrho_0}{|\zeta| [4]_q ([4]_q - 1) \kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta ([2]_q - 1) \kappa_1^2 + B_1^2 \varrho_0 \kappa_3}{B_1 \zeta ([2]_q - 1) \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2}{\xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\}, \end{aligned}$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_1^2 \varrho_0^2 \kappa_6}{\zeta^3 ([2]_q - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{B_1 \varrho_0 [(2]_q - 1) + ([3]_q - 1) \kappa_5}{\zeta^2 ([2]_q - 1) ([3]_q - 1) \kappa_1 \kappa_2}$ .

Considering the special case  $p = 1$  and  $q \rightarrow 1^-$  in Theorem 2.2 we obtain:

**Corollary 2.12.** Let  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho C_{\beta, \gamma}(\vartheta)$  is given by (1), then

$$\left| \frac{a_1}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{2|\zeta|\kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{6|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{6|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{4B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 \gamma (4\kappa_3 - 6\mu\kappa_2)}{36B_1 \zeta \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{12|\zeta|\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 \gamma \kappa_3}{B_1 \zeta \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2}{\xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_1^2 \varrho_0^2 \gamma^2 \kappa_6}{\zeta^3 \kappa_1^3}$ , and  $\xi = \frac{3B_1 \varrho_0 \gamma \kappa_5}{2\zeta^2 \kappa_1 \kappa_2}$ .

If we take  $\gamma = 1$ ,  $p = 1$  and  $q \rightarrow 1^-$  in Theorem 2.2 we obtain the next result:

**Corollary 2.13.** Let  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ . If  $f \in \psi_\varrho C_\beta(\vartheta)$  is given by (1), then

$$\left| \frac{a_1}{S} \right| \leq \frac{B_1 \varrho_0}{2|\zeta|\kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0}{6|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \kappa_3}{\zeta \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{B_1 \varrho_0}{6|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{4B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 (4\kappa_3 - 6\mu\kappa_2)}{36B_1 \zeta \kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{B_1 \varrho_0}{12|\zeta|\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{B_2 \zeta \kappa_1^2 + B_1^2 \varrho_0 \kappa_3}{B_1 \zeta \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} + \frac{\varrho_1}{\varrho_0} \left( 1 + \frac{B_2}{B_1 \xi} \right) + \left( 1 + \frac{2}{\xi} \right) + \frac{1}{\varrho_0 \xi} (\varrho_1 + \varrho_2) \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{B_1^2 \varrho_0^2 \kappa_6}{\zeta^3 \kappa_1^3}$ , and  $\xi = \frac{3B_1 \varrho_0 \kappa_5}{2\zeta^2 \kappa_1 \kappa_2}$ .

For  $\varrho \equiv 1$  and  $\vartheta := \varpi$  Theorem 2.2 yields to the next special case:

**Corollary 2.14.** Let  $0 < q < p \leq 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \hat{\psi}C_{\beta,\gamma}^{p,q}(\varpi)$  is given by (1), then

$$\begin{aligned} \left| \frac{a_1}{S} \right| &\leq \frac{|\gamma|}{2|\zeta|[2]_{p,q}([2]_{p,q} - 1)\kappa_1}, \\ \left| \frac{a_2}{S} \right| &\leq \frac{|\gamma|}{2|\zeta|[3]_{p,q}([3]_{p,q} - 1)\kappa_2} \max \left\{ 1; \left| \frac{\gamma\kappa_3}{2\zeta([2]_{p,q} - 1)\kappa_1^2} \right| \right\}, \\ \left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| &\leq \frac{|\gamma|}{2|\zeta|[3]_{p,q}([3]_{p,q} - 1)\kappa_2} \\ &\quad \times \max \left\{ 1; \left| \frac{\gamma([2]_{p,q}^2([2]_{p,q} - 1)\kappa_3 - \mu[3]_{p,q}([3]_{p,q} - 1)\kappa_2)}{2\zeta[2]_{p,q}^2([2]_{p,q} - 1)^2\kappa_1^2} \right| \right\}, \end{aligned}$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{|\gamma|}{2|\zeta|[4]_{p,q}([4]_{p,q} - 1)\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\gamma\kappa_3}{\zeta([2]_{p,q} - 1)\kappa_1^2} + \frac{2\eta}{\xi} + \frac{23}{12\xi} \right] \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{\gamma^3 \kappa_6}{8\zeta^3([2]_{p,q} - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{(([2]_{p,q} - 1) + ([3]_{p,q} - 1)) \gamma^2 \kappa_5}{4\zeta^2([2]_{p,q} - 1)([3]_{p,q} - 1)\kappa_1 \kappa_2}$ .

**Remark 2.2.** If we consider the function

$$\hat{f}_*(z) = z + \frac{\gamma w_1}{(1 + i \tan \beta)[2]_{p,q}([2]_{p,q} - 1)} z^2, z \in \mathbb{D},$$

then it is easy to check that  $\hat{f}_* \in \hat{\psi}C_{\beta,\gamma}^{p,q}(\varpi)$ , that is  $\hat{\psi}C_{\beta,\gamma}^{p,q}(\varpi) \neq \emptyset$ .

For  $p = 1$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$  Theorem 2.2 yields to the next special case:

**Corollary 2.15.** Let  $0 < q < 1$ ,  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \hat{\psi}C_{\beta,\gamma}^q(\varpi)$  is given by (1), then

$$\begin{aligned} \left| \frac{a_1}{S} \right| &\leq \frac{|\gamma|}{2|\zeta|[2]_q([2]_q - 1)\kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{|\gamma|}{2|\zeta|[3]_q([3]_q - 1)\kappa_2} \max \left\{ 1; \left| \frac{\gamma\kappa_3}{2\zeta([2]_q - 1)\kappa_1^2} \right| \right\}, \\ \left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| &\leq \frac{|\gamma|}{2|\zeta|[3]_q([3]_q - 1)\kappa_2} \max \left\{ 1; \left| \frac{\gamma([2]_q^2([2]_q - 1)\kappa_3 - \mu[3]_q([3]_q - 1)\kappa_2)}{2\zeta[2]_q^2([2]_q - 1)^2\kappa_1^2} \right| \right\}, \end{aligned}$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{|\gamma|}{2|\zeta|[4]_q([4]_q - 1)\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\gamma\kappa_3}{\zeta([2]_q - 1)\kappa_1^2} + \frac{2\eta}{\xi} + \frac{23}{12\xi} \right] \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{\gamma^3 \kappa_6}{8\zeta^3([2]_q - 1)^2 \kappa_1^3}$ , and  $\xi = \frac{(([2]_q - 1) + ([3]_q - 1)) \gamma^2 \kappa_5}{4\zeta^2([2]_q - 1)([3]_q - 1)\kappa_1 \kappa_2}$ .

Putting  $p = 1$ ,  $\gamma = 1$ ,  $\varrho \equiv 1$  and  $\vartheta := \varpi$  in Theorem 2.2 we get:

**Corollary 2.16.** *Let  $0 < q < 1$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \hat{\psi}C_{\beta}^q(\varpi)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{1}{2|\zeta|[2]_q([2]_q - 1)\kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{1}{2|\zeta|[3]_q([3]_q - 1)\kappa_2} \max \left\{ 1; \left| \frac{\kappa_3}{2\zeta([2]_q - 1)\kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{1}{2|\zeta|[3]_q([3]_q - 1)\kappa_2} \max \left\{ 1; \left| \frac{([2]_q^2([2]_q - 1)\kappa_3 - \mu[3]_q([3]_q - 1)\kappa_2)}{2\zeta[2]_q^2([2]_q - 1)^2\kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{1}{2|\zeta|[4]_q([4]_q - 1)\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\kappa_3}{\zeta([2]_q - 1)\kappa_1^2} + \frac{2\eta}{\xi} + \frac{23}{12\xi} \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{\kappa_6}{8\zeta^3([2]_q - 1)^2\kappa_1^3}$ , and  $\xi = \frac{([2]_q - 1) + ([3]_q - 1)\kappa_5}{4\zeta^2([2]_q - 1)([3]_q - 1)\kappa_1\kappa_2}$ .

Considering the special case  $p = 1$ ,  $\rho \equiv 1$ ,  $\vartheta := \varpi$  and  $q \rightarrow 1^-$  in Theorem 2.2 we obtain:

**Corollary 2.17.** *Let  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \hat{\psi}C_{\beta,\gamma}(\varpi)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{|\gamma|}{4|\zeta|\kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{|\gamma|}{12|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{\gamma\kappa_3}{2\zeta\kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{|\gamma|}{12|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{2\gamma\kappa_3 - 3\mu\kappa_2}{4\zeta\kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{|\gamma|}{24|\zeta|\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\gamma\kappa_3}{\zeta\kappa_1^2} + \frac{2\eta}{\xi} + \frac{23}{12\xi} \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{\gamma^3\kappa_6}{8\zeta^3\kappa_1^3}$ , and  $\xi = \frac{3\gamma^2\kappa_5}{8\zeta^2\kappa_1\kappa_2}$ .

If we take  $\gamma = 1$ ,  $p = 1$ ,  $\rho \equiv 1$ ,  $\vartheta := \varpi$  and  $q \rightarrow 1^-$  in Theorem 2.2 we get the next special case:

**Corollary 2.18.** *Let  $\gamma \in \mathbb{C}^*$ ,  $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ , and  $\varpi$  be defined by (8). If  $f \in \hat{\psi}C_{\beta}(\varpi)$  is given by (1), then*

$$\left| \frac{a_1}{S} \right| \leq \frac{1}{4|\zeta|\kappa_1}, \quad \left| \frac{a_2}{S} \right| \leq \frac{1}{12|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{\kappa_3}{2\zeta\kappa_1^2} \right| \right\},$$

$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{1}{12|\zeta|\kappa_2} \max \left\{ 1; \left| \frac{2\kappa_3 - 3\mu\kappa_2}{4\zeta\kappa_1^2} \right| \right\},$$

and

$$\left| \frac{a_3}{S} \right| \leq \frac{1}{24|\zeta|\kappa_4} \max \left\{ 1; \left| \xi \left[ \frac{\kappa_3}{\zeta\kappa_1^2} + \frac{2\eta}{\xi} + \frac{23}{12\xi} \right] \right| \right\},$$

where  $\zeta = 1 + i \tan \beta$ ,  $\eta = \frac{\kappa_6}{8\zeta^3\kappa_1^3}$ , and  $\xi = \frac{3\kappa_5}{8\zeta^2\kappa_1\kappa_2}$ .

## CONCLUDING REMARKS

In our present investigation we have made use of the  $(p, q)$ -Jackson derivative to introduce and investigate the new classes  $\psi_{\rho}S_{\beta, \gamma}^{p, q}(\varpi)$  and  $\hat{\psi}_{\rho}C_{\beta, \gamma}^{p, q}(\varpi)$  of analytic functions in the open unit disk  $\mathbb{D}$ . Using the subordination principle we have obtained the bounds of the three first coefficients for the functions belonging to these classes.

## REFERENCES

- [1] Ahuja, O. P., Çetinkaya, A. and Ravichandran, V., (2019), Harmonic univalent functions defined by post quantum calculus operators, *Acta Univ. Sapientiae Math.*, 11(1), pp. 5–17.
- [2] Altıntaş, O. and Owa, S., (1992), Majorizations and quasi-subordinations for certain analytic functions, *Proc. Japan. Acad. Ser. A Math. Sci.*, 68(7), pp. 181–185.
- [3] Alexander, J. W., (1915), Functions which map the interior of the unit circle upon simple regions, *Ann. of Math. (2)*, 17(1), pp. 12–22.
- [4] Altinkaya, Ş. and Olatunji, S. O., (2020), Generalized distribution for analytic function classes associated with error functions and Bell numbers, *Bol. Soc. Mat. Mex. (3)*, 26, pp. 377–384. <https://doi.org/10.1007/s40590-019-00265-z>.
- [5] Altinkaya, Ş. and Yalçın, S., (2017), Faber polynomial coefficient estimates for bi-univalent functions of complex order based on subordinate conditions involving the Jackson  $(p, q)$ -derivative, *Miskolc Math. Notes*, 18(2), pp. 555–572.
- [6] Brown, R. K., (1960), Univalence of Bessel Functions, *Proc. Amer. Math. Soc.*, 11(2), pp. 278–283.
- [7] Baricz, Á., Kupán, P. A. and Szász, R., (2014), The radius of starlikeness of normalized Bessel functions of the first kind, *Proc. Amer. Math. Soc.*, 142(6), pp. 2019–2025.
- [8] Baricz, Á. and Szász, R., (2015), The radius of convexity of normalized Bessel functions, *Anal. Math.*, 41, pp. 141–151.
- [9] Bohra, N. and Ravichandran, V., (2018), Radii problems for normalized Bessel functions of first kind, *Comput. Methods Funct. Theory*, 18, pp. 99–123. <https://doi.org/10.1007/s40315-017-0216-0>.
- [10] Dziok, J. and Srivastava, H. M., (1999), Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, 103(1), pp. 1–13.
- [11] Din, M. U., Raza, M., Hussain, S. and Darus, M., (2018), Certain Geometric Properties of Generalized Dini Functions, *J. Funct. Spaces 2018*, Article ID 2684023, 9 pages.
- [12] Ebadian, A., Cho, N. E., Adegani, E. A., Bulut, S. and Bulboacă, T., (2020), Radii problems for some classes of analytic functions associated with Legendre polynomials of odd degree, *J. Inequal. Appl.* 2020, 178. <https://doi.org/10.1186/s13660-020-02443-4>.
- [13] El-Ashwah, R. and Kanas, S., (2015), Fekete-Szegő inequalities for quasi-subordination functions classes of complex order, *Kyungpook Math. J.*, 55(3), pp. 679–688.
- [14] Frasin, B. A., Al-Hawary, T. and Yousef, F., (2019), Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, *Afr. Mat.*, 30, pp. 223–230.
- [15] Fadipe-Joseph, O. A., Oladipo, A. T. and Ezeafulukwe, U. A., (2013), Modified sigmoid function in univalent theory, *Int. J. Math. Sci. Eng. Appl.*, 7(5), pp. 313–317.
- [16] Janowski, W., (1970), Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.*, 23, pp. 159–177.
- [17] Jahangiri, J. M., Ramachandran, C. and Annamalai, S., (2018), Fekete-Szegő problem for certain analytic functions defined by hypergeometric functions and Jacobi polynomial, *J. Fract. Calc. Appl.*, 9(1), pp. 1–7.
- [18] Keogh, F. R. and Merkes, E. P., (1969), A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, 20, pp. 8–12.
- [19] Kumar, V., Cho, N. E., Ravichandran, V. and Srivastava, H. M., (2019), Sharp coefficient bounds for starlike functions associated with the Bell numbers, *Math. Slovaca*, 69(5), pp. 1053–1064.
- [20] Mondal, S. R. and Swaminathan, A., (2012), Geometric properties of generalized Bessel functions, *Bull. Malays. Math. Sci. Soc.*, 35(1), pp. 179–194.
- [21] Ma, W. and Minda, D., (1994), A unified treatment of some special classes of univalent functions, *Conf. Proc. Lecture Notes Anal.*, I, Int. Press, Cambridge, MA, 1994.
- [22] Oladipo, A. T., (2016), Coefficient inequality for subclass of analytic univalent functions related to simple logistic activation functions, *Stud. Univ. Babeş-Bolyai Math.*, 61(1), pp. 45–52.
- [23] Oladipo, A. T., (2019), Generalized distribution associated with univalent functions in conical domain, *An. Univ. Oradea Fasc. Mat.*, 26(1), pp. 163–169.



- [24] Oladipo, A. T., (2019), Bounds for probabilities of the generalized distribution defined by generalized polylogarithm, Punjab Univ. J. Math. (Lahore), 51(7), pp. 19-26.
- [25] Olatunji, S. O. and Dutta, H., (2019), Sigmoid function in the space of univalent  $\lambda$ -pseudo-( $p, q$ )-derivative operators related to shell-like curves connected with Fibonacci numbers of Sakaguchi type functions, Malays. J. Math. Sci., 13(1), pp. 95–106.
- [26] Owa, S. and Srivastava, H. M., (1987), Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 39(5), pp. 1057–1077.
- [27] Porwal, S., (2018), Generalized distribution and its geometric properties associated with univalent functions, J. Complex Anal. 2018, Art. ID 8654506, pp. 1–5.
- [28] Robertson, M. S., (1970), Quasi-subordination and coefficient conjectures, Bull. Amer. Math. Soc., 76, pp. 1–9.
- [29] Ren, F. Y., Owa, S. and Fukui, S., (1991), Some inequalities on quasi-subordinate functions, Bull. Aust. Math. Soc., 43(2), pp. 317–324.
- [30] Silverman, H., (1993), Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl., 172(2), pp. 574–581.
- [31] Srivastava, H. M., Murugusundaramoorthy, G. and Janani, T., (2014), Uniformly Starlike Functions and Uniformly Convex Functions Associated with the Struve Function, J. Appl. Computat. Math., 3:6.
- [32] Srivastava, H. M., Chaudhry, M. A. and Agarwal, R. P., (2012), The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, Integral Transforms Spec. Funct., 23(9), pp. 659–683.
- [33] Sim, Y. J., Kwon, O. S., Cho, N. E. and Srivastava, H. M., (2012), Some classes of analytic functions associated with conic regions, Taiwanese J. Math., 16(1), pp. 387–408.
- [34] Watson, G. N., (1921), The zeros of Lommel's polynomials, Proc. Lond. Math. Soc. (3), 19, pp. 266–272.
- [35] Yağmur, N. and Orhan, H., (2013), Starlikeness and Convexity of Generalized Struve Functions, Abstr. Appl. Anal., Article ID954513, 6pp.



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