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FUZZY IDEALS IN MATRIX NEARRINGS

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Abstract. We introduce fuzzy ideal of a matrix nearring corresponding to a fuzzy ideal of a nearring. We prove properties relating to fuzzy ideals of a nearring and that of a matrix nearring. Finally, prove an order preserving one-one correspondence between the fuzzy ideals of R (over itself) and that of $M_n(R)$-group R^n.

Keywords: Nearring, fuzzy ideal, matrix nearring.

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1. Introduction

A nearring is a set R together with two binary operations $+$ and $\cdot$ such that: (1) $(R, +)$ is a group (not necessarily abelian), (2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, (3) $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$. In view of (iii), R satisfies the right distributive law, and so it is called as a right nearring. It is evident that $0 \cdot n = 0$ for all $n \in R$. However, $n \cdot 0$ need not be equal to 0, in general. We denote $R_0 = \{n \in R : n \cdot 0 = 0\}$, the zero-symmetric part of the right nearring. If $R = R_0$, then we say that the nearring R is zero-symmetric.

Let $(G, +)$ be a group. By an R-group, we mean a mapping $R \times G \to G$ (the image of $(n, g) \in R \times G$ is denoted by ng), satisfying the following conditions: (1) $(n + n^1)g = ng + n^1g$, and (2) $(nn^1)g = n(n^1g)$ for all $g \in G$ and $n, n^1 \in R$.

Throughout, we denote R for a right nearring and R-group by ${}_RG$ or (simply by G). If $R = G$ then we denote ${}_RR$. A subgroup $(H, +)$ of (G, +) with $RH \subseteq H$ is said to be an R-subgroup of G. A normal subgroup K of an R-group G is called an ideal if $n(x + a) - nx \in K$ for all $n \in R, x \in G$ and $a \in K$.

For preliminary definitions and results on Nearrings, R-groups and fuzzy aspects, we refer to [15, 3], for matrix nearrings, we refer to [5, 1, 2]. In section 3, we introduce fuzzy

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ideal of a matrix nearring corresponding to a fuzzy ideal of a nearring and prove an order preserving one-one correspondence between the fuzzy ideals of R (over itself) and those of $M_n(R)$-group R^n. In section 4, we prove properties of Insertion of Factors Property (IFP, in short) in matrix nearrings.

2. Preliminaries of matrix nearrings

Matrix nearrings over arbitrary nearrings were introduced in [14].

Definition 2.1. *Consider R with mutiplicative identity 1. R^n denotes the direct sum of n-copies of $(R,+)$. For any $r \in R, 1 \leq i \leq n$ and $1 \leq j \leq n$, define $f_{ij}^r : R^n \to R^n$ as $f_{ij}^r(a_1, a_2, \cdots, a_n) = (0, \cdots, ra_j, \cdots, 0)$ (here ra_j is in the i^{th} place). If $f^r : R \to R$ defined by $f^r(x) = rx$ for all $x \in R, i_i : R \to R^n$ is the canonical monomorphism; and $\pi_j : R^n \to R$ is the j^{th} projection map, then it is clear that $f_{ij}^r = i_i f^r \pi_j$ and $f_{ij}^r \in M(R^n)$ where $M(R^n)$ is the nearring of all mappings from $R^n \to R^n$. The sub-nearring $M_n(R)$ of $M(R^n)$ generated by $\left\{ f_{ij}^r : r \in R, 1 \leq i,j \leq n \right\}$ is called the matrix nearring over R, and R^n becomes an $M_n(R)-$group. The length of an expression is the number of f_{ij}^r in it. The weight $w(A)$ of a matrix A is the length of an expression of minimal length for A.*

Lemma 2.1. *(3.1(iii), (v), 2.3 of [14]):*

(1) *For any $r, s \in R$ we have*
$$f_{ij}^r f_{kl}^s = \begin{cases} f_{il}^{rs} & if\, j = k \\ f_{il}^{r0} & if\, j \neq k \end{cases}$$
where $i, j, k, l \in \{1, 2, \cdots, n\}$.

(2) $f_{ij}^r(f_{1k_1}^1 + \cdots + f_{nk_n}^{r_n}) = f_{ij}^r(f_{jk_j})^{r_j} = f_{ik_j}^{rr_j}$.

(3) *Let $A \in M_n(R)$, $x \in R$, $1 \leq i, j \leq n$. Then there exist $a_1, a_2, \cdots, a_n \in R$ such that*
$$Af_{ij}^x = f_{1j}^{a_1} + \cdots + f_{nj}^{a_n}.$$

Result 2.1. *(Prop. 4.1 of [14]): If L is a left ideal of R then L^n is an ideal of the $M_n(R)$-group R^n.*

Notation: For an ideal I of $M_n(R)$,
$I_* = \{x \in R : x \in im(\pi_j A)\; for\; some\; A \in I\; and\; 1 \leq j \leq n\}$.

Result 2.2. *(Lemma 4.4 of [14]): If I is a two sided ideal of $M_n(R)$, then $a \in I_*$ if and only if $f_{11}^a \in I$.*

Result 2.3. *(Corollary 4.5 of [14]): If I is a two sided ideal of $M_n(R)$, then $a \in I_*$ if and only if $f_{ij}^a \in I$ for all $1 \leq i \leq n, 1 \leq j \leq n$.*

Theorem 2.1. *(Theorem 4.6 of [14]): If I is a two sided ideal of $M_n(R)$, then I_* is a two sided ideal of R.*

Definition 2.2. (1) *An element $A \in M_n(R)$ is said to be nilpotent if there exists a positive integer k such that $A^k = 0$.*

(2) *$M_n(R)$ is said to be reduced if $M_n(R)$ has no non-zero nilpotent elements.*

Definition 2.3. *Following the notation from ([2], Notation 1.1), for any ideal $\mathcal{I}$ of $M_n(R)$-group R^n, we write*
$$\mathcal{I}_{**} = \{a \in R : a = \pi_j A,\; for\; some\; A \in \mathcal{I}, 1 \leq j \leq n\}.$$
It can be seen that $\mathcal{I}_{**} = \{a \in R : (a, 0, \cdots, 0) \in \mathcal{I}\}$.

Lemma 2.2. *1.3, 1.4, 1.5 of* [2]

(1) I_{**} *is an ideal of* ${}_RR$.
(2) *If* L^n *is an ideal of* R^n, *then* $L = (L^n)_{**}$.
(3) L *is an ideal of* ${}_RR$, *then* $L = (L^n)_{**}$.
(4) $\mathcal{L}$ *is an ideal of* R^n, *then* $(\mathcal{L}_{**})^n = L$.

Theorem 2.2. *(Proposition 2.5 of* [6]*): Let* $S \subseteq R$. *Then* λ_S *is a fuzzy ideal of* R *if and only if* S *is an ideal of* R.

Definition 2.4. *(*[6]*): Let* μ *be a fuzzy subset of* R. *Then* $\mu_u = \{x \in R : \mu(x) \geq u\}$, *for all* $u \in [0,1]$, *is called the level subset of* u.

Theorem 2.3. *(*[6], [17]*): Let* μ *be a fuzzy subset of* R. *Then* μ_t, $t \in [0, \mu(0)]$ *is an ideal of* R *if and only if* μ *is a fuzzy ideal of* R.

Definition 2.5. *(*[6]*): A fuzzy ideal* μ *of* R *is called prime if for any two fuzzy ideals* σ *and* θ *of* R *such that* $\sigma \circ \theta \subseteq \mu$ *implies that* $\sigma \subseteq \mu$ *or* $\theta \subseteq \mu$.

Example 2.1. *Let* $R = \{p, q, r, s\}$ *be a set with two binary operations* $+$ *and* $\cdot$ *is defined as follows.*

$+$	p	q	r	s
p	p	q	r	s
q	q	p	s	r
r	r	s	p	q
s	s	r	q	p

TABLE 1

$$x \cdot y = \begin{cases} p, & \text{if } (x \in \{p,q\}) \text{ or } (x \in \{r,s\},\ y \neq s) \\ q, & \text{if } x \in \{r,s\}, y = s \end{cases}$$

Then $(R,+)$ *is an* $(R,+,\cdot)$*-group. Define a fuzzy subset* $\mu : R \to [0,1]$ *by* $\mu(r) = \mu(s) < \mu(q) < \mu(p)$. *Then* μ *is a fuzzy ideal of* R.

3. Fuzzy ideals of matrix nearrings

The concept of fuzzy subset was initiated in [21]. Later the authors [6, 17] studied the concept fuzzy in different algebriac systems, particularly in the theory of rings and nearrings. A mapping $\mu : X \to [0,1]$, where X be a non-empty set, is called the fuzzy subset of X.

Definition 3.1. *For any two fuzzy subsets* σ *and* θ *of* R, *we define the fuzzy subset* $\sigma \circ \theta$ *of* R *as follows:*

$$(\sigma \circ \theta)(x) = \begin{cases} \sup_{x=yz} \{\min(\sigma(y), \theta(z))\}, & \text{if } x = yz; \\ 0, & \text{otherwise}. \end{cases}$$

Further, σ *and* θ *of* R, $\sigma \subseteq \theta$, *we mean* $\sigma(x) \leq \theta(x)$ *for all* $x \in R$.

Definition 3.2. *Let* μ *and* σ *be fuzzy subsets of* X *and* Y *respectively, and* f *a function of* X *into* Y. *The image of* μ, *under* f, *is a fuzzy subset of* Y, *defined by*

$$(f(\mu)) = \begin{cases} \sup_{f(a)=b} \mu(a), & \text{if } f^{-1}(b) \neq \phi; \\ 0, & \text{if } f^{-1}(b) = \phi. \end{cases}$$

and $(f^{-1}(\sigma))(x) = \sigma(f(x))$ *for all* $x \in X$.

Definition 3.3. *Let* μ *be a non-empty fuzzy subset of a nearring* R *(that is* $\mu(u) \neq 0$ *for some* $u \in R$*). Then* μ *is said to be a fuzzy ideal of* R *if it satisfies the following conditions: (1)* $\mu(u+v) \geq \min\{\mu(u), \mu(v)\}$*, (2)* $\mu(-u) = \mu(u)$*, (3)* $\mu(u) = \mu(v+u-v)$*, (4)* $\mu(uv) \geq \mu(u)$*, (5)* $\mu\{u(v+i) - uv\} \geq \mu(i)$ *for all* $u, v, i \in R$.

Note 3.1. *If* μ *is a fuzzy ideal of* R*, then* $\mu(u+v) = \mu(v+u)$*, and* $\mu(0) \geq \mu(u)$*, for all* $u, v \in R$.

Definition 3.4. *Let* I *be an ideal of* R*. We define the characteristic function on* I *as* $\lambda_I : R \to [0,1]$*, where*

$$\lambda_I(u) = \begin{cases} 1 & \text{if } u \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We introduce fuzzy ideal of a matrix nearring corresponding to a fuzzy ideal of a nearring.

Definition 3.5. *Let* μ *be fuzzy ideal of* R*. We define* $\mu^\star : R^n \to [0,1]$ *by* $\mu^\star(u_1, \cdots, u_n) = \min\{\mu(u_1), \cdots, \mu(u_n)\}$.

Note 3.2. $\mu^\star(0, \cdots, u_j, \cdots, 0) \geq \mu^\star(u_1, \cdots, u_n)$*, for any* u_i*,* $1 \leq i \leq n$ *in* R*. Since* μ *is a fuzzy ideal of* R*,* $\mu(0) \geq \mu(x), \forall x \in R$*. Therefore,* $\min\{\mu(0), \cdots, \mu(u_j), \cdots, \mu(0)\} = \mu(u_j) \geq \min\{\mu(u_1), \mu(u_2), \cdots, \mu(u_n)\}$.

Lemma 3.1. *If* μ *is a fuzzy subgroup of* R*, then* $\mu^\star$ *is a fuzzy subgroup of* $M_n(R)$*-group* R^n.

Proof. Suppose μ is a fuzzy subgroup of R. Take $\rho_1, \rho_2 \in R^n$. Then $\rho_1 = (u_1, \cdots, u_n)$ and $\rho_2 = (v_1, \cdots, v_n)$ for some $u_i, v_i \in R, 1 \leq i \leq n$. Now

$$\begin{aligned}
\mu^\star(\rho_1 + \rho_2) &= \mu^\star((u_1, \cdots, u_n) + (v_1, \cdots, v_n)) \\
&= \mu^\star(u_1 + v_1, \cdots, u_n + y_n) \\
&= \min\{\mu(u_1 + v_1), \cdots, \mu(u_n + v_n)\} \\
&\geq \min\{\mu(u_1), \mu(v_1), , , , \mu(u_n), \mu(v_n)\} \\
&= \min\{\mu(u_1), \cdots, \mu(u_n), \mu(v_1), \cdots, \mu(v_n)\} \\
&= \min\{\min\{(u_1, \cdots, u_n)\}, \min\{(v_1, \cdots, v_n)\}\} \\
&= \min\{\mu^\star(\rho_1), \mu^\star(\rho_2)\}.
\end{aligned}$$

□

Lemma 3.2. *If* μ *is a fuzzy ideal of* R*, then*

$$\mu^\star(f_{ij}^r(u_1, \cdots, u_n)) \geq \mu^\star(u_1, \cdots, u_n),$$

for all $r \in R$ *and* $1 \leq i, j \leq n$.

Proof. We have $\mu^\star(f_{ij}^r(\mu_1, \mu_2, \cdots, \mu_n))$
$= \mu^\star(0, \cdots, ru_j, \cdots, 0)$
$= \min\{\mu(0), \cdots, \mu(ru_j), \cdots, \mu(0)\}$ (since $\mu(0) = 0$)
$= \mu(ru_j)$ (since $\mu(0) \geq \mu(x)$, for all $x \in R$) $\geq \mu(ru_j)$ (since μ is a fuzzy ideal of R)
$= \min\{\mu(0), \cdots, \mu(u_j), \cdots, \mu(0)\}$
$= \mu^\star(0, \cdots, u_j, \cdots, 0) \geq \mu^\star(u_1, \cdots, u_j, \cdots, u_n)$. □

Lemma 3.3. *μ is a fuzzy ideal of R, $\mu^\star : R^n \to [0,1]$ satisfies $\mu^\star(X+Y-X) = \mu^\star(Y)$, for all $X, Y \in R^n$.*

Proof. Let $X = (u_1, \cdots, u_n)$ and $Y = (v_1, \cdots, v_n)$ be elements of R^n. Now $\mu^\star((u_1, \cdots, u_n)+(v_1, \cdots, v_n) - (u_1, \cdots, u_n)) = \mu^\star((u_1 + v_1 - u_1), \cdots, (u_n + v_n - u_n))$
$= \min\{\mu(u_1 + v_1 - u_1), \cdots, \mu(u_n + v_n - u_n)\}$. Since μ is a fuzzy ideal of R, $\min\{\mu(u_1 + v_1 - u_1), \cdots, \mu(u_n + v_n - u_n)\} \geq \min\{\mu(v_1), \cdots, \mu(v_n)\} = \mu^\star(v_1, \cdots, v_n)$. □

Lemma 3.4. *If μ is a fuzzy ideal of R (over itself) then $\mu^\star$ is a left ideal of $M_n(R)$-group R^n.*

Proof. Let $X = (u_1, \cdots, u_n)$ and $Y = (v_1, \cdots, v_n)$ be elements of R^n. We prove this by induction on weight of a matrix. We prove for weight of $A \in M_n(R)$ is 1. Then $A = f^r_{ij}$. Now $\mu^\star(f^r_{ij}(X+Y) - f^r_{ij}(X)) = \mu^\star(f^r_{ij}(u_1+v_1, \cdots, u_n+v_n) - f^r_{ij}(u_1, \cdots, u_n)) = \mu^\star(0, \cdots, r(x_j + y_j) - rx_j, \cdots, 0) = \mu^\star((0, \cdots, r(u_j + v_j), \cdots, 0) - (0, \cdots, rx_j, \cdots, 0)) = \mu(r(u_j + v_j) - rx_j) \geq \mu(v_j)$ (since μ is a fuzzy ideal) $= \min\{\mu(0), \cdots, \mu(v_j), \cdots, \mu(0)\}$
$= \mu^\star(0, \cdots, v_j, \cdots, 0) \geq \mu^\star(v_1, \cdots, v_n)$. By induction on weight of $A \in M_n(R)$, we can prove for the cases either $A = B + C$ or $A = BC$, where $w(B) \leq w(A)$ and $w(C) \leq w(A)$, so $\mu^\star$ is a left ideal of $M_n(R)$-group R^n. □

Now we summarize the following one-one correspondence theorem as follows.

Theorem 3.1. *There is an order preserving one-one correspondence between $FI(R)$, the set of the fuzzy ideals of R-group R and $FI(M_n(R))$, the set of the fuzzy ideals of $M_n(R)$-group R^n.*

Proof. Define $\psi : FI(R) \to FI(M_n(R))$ by $\mu \to \mu^*$. To prove ψ is one-one:
Suppose $\mu_1 \neq \mu_2$. Then there exists $u_1 \in R$ such that $\mu_1(u_1) \neq \mu_2(u_1)$.
Take $A = f'_{11} \in M_n(R)$. $\mu_1^*(A) = \mu_1^*(f'_{11}(u_1, \cdots, u_n))$
$= \mu_1^*(u_1, 0, 0 \cdots, 0)$. By definition of μ_1^*, we get $\mu_1^*(u_1, 0, 0 \cdots, 0) = \min\{\mu(u_1), \cdots, \mu(0)\}$.
By the supposition $\min\{\mu(u_1), \cdots, \mu(0)\} = \mu_1(u_1) \neq \mu_2(u_1)$.
By definition of μ_2^*, we have
$\mu_2(u_1) = \min\{\mu_2(u_1), \mu_2(0), \cdots\} = \mu_2^*(u_1, \cdots, 0)$
$= \mu_2^*(f'_{11}(u_1, \cdots, u_n)) = \mu_2^*(A)$. Therefore ψ is one-one.
Now we prove the order preserving property.
Suppose μ_1 and μ_2 be two fuzzy ideals of R such that $\mu_1 \subseteq \mu_2$. Now for any $f^r_{ij} \in M_n(R)$, $\psi(\mu_2)(f^r_{ij})$
$= (\mu_2^* f^r_{ij})(u_1, u_2, \cdots, u_n) = \mu_2^*(0, \cdots, ru_j, \cdots, 0)$
$= \min\{\mu_2(0), \cdots, \mu_2(ru_j), \cdots, \mu_2(0)\}$ (by definition of μ_2^*) $= \mu_2(ru_j)$ (since μ_2 is a fuzzy ideal of R) $\geq \mu_1(ru_j)$ (by the supposition) $= \min\{\mu_1(0), \cdots, \mu_1(ru_j), \cdots, \mu_1(0)\}$
$= \mu_1^*(0, \cdots, ru_j, \cdots 0)$ (by definition of μ_1^*) $= \psi(\mu_1)(f^r_{ij})$. Therefore $\psi(\mu_1) \subseteq \psi(\mu_2)$. We show that ψ is onto: Let δ be a fuzzy ideal of R^n. Define $\overline{\delta}(x) = \delta(f^x_{11})$.
It can be verified that $\overline{\delta}$ is a fuzzy ideal of R.
Clearly $\overline{\delta}(y + x - y) = \overline{\delta}(x)$.
Now $\overline{\delta}(nx) = \delta(nx, 0, 0, \cdots, 0) \geq \delta(f^n_{11}(x, 0, \cdots, 0))$
$\geq \delta(x, 0, \cdots, 0) = \overline{\delta}(x)$, and $\overline{\delta}(n(n' + x) - nn')$
$= \delta(n(n' + x) - nn', 0, 0, \cdots, 0)$
$= \delta((n(n' + x), \cdots, 0) - (nn', \cdots, 0))$

$= \delta(f_{11}^{n}((n',0,\cdots,0)+(x,\cdots,0)) - f_{11}^{n}(n',0,\cdots,0))$
$\geq \delta(x,0,\cdots,0) = \overline{\delta}(x)$.
$(\overline{\delta})^{*}(u_1,u_2,\cdots,u_n) = \min\{\overline{\delta}(u_1),\cdots,\overline{\delta}(u_n)\}$
$= \min\{\delta(u_1,0,\cdots,0),\cdots,\delta(u_n,\cdots,0)\}$
$= \min\{\delta(f_{11}^{'}(u_1,\cdots u_n)),\cdots,\delta(f_{1n}^{'}(u_1,\cdots,u_n))\}$
$\geq \min\{\delta(u_1,\cdots,u_n),\delta(u_1,\cdots,u_n),\cdots,\delta(u_1,\cdots,u_n)\}$,
$= \delta(u_1,\cdots,u_n)$. Therefore $(\overline{\delta})^{*} \supseteq \delta$. Also $\delta(u_1,\cdots,u_n) = \delta((u_1,0,0,\cdots,0)+\cdots+(0,\cdots,u_n))$
$\geq \min\{\delta(u_1,0,0,\cdots,0),\cdots,\delta(0,\cdots,u_n)\}$
$= \min\{\delta(f_{11}^{'}(u_1,\cdots,u_n)),\cdots,\delta(f_{nn}^{'}(u_1,\cdots,u_n))\}$
$= \min\{\delta(u_1,0,0,\cdots,0),\cdots,\delta(u_n,0,0,\cdots,0)\}$
$= (\overline{\delta})^{*}(u_1,\cdots,u_n)$, and the proof is complete. □

4. More results on matrix nearrings

Definition 4.1. [5] *Let I be a two sided ideal of R. Then I^{+} is the ideal generated by $\{f_{kl}^{s} : s \in I,\ 1 \leq k,l \leq n\}$ in $M_n(R)$.*

Result 4.1. *Let R be a zero symmetric right nearring. Then an ideal I satisfies Insertion of Factors Property (abbr. IFP) if and only if I^{+} satisfies IFP.*

Proof. Suppose I satisfies IFP. To show I^{+} satisfies IFP, let $f_{ij}^{a}, f_{kl}^{b} \in M_n(R)$ such that $f_{ij}^{a}f_{kl}^{b} \in I^{+}$ where
$a,b \in I, 1 \leq i,j,k,l \leq n$. On a contrary, suppose there exists $f_{pq}^{c} \in M_n(R)$ such that $f_{ij}^{a}f_{pq}^{c}f_{kl}^{b} \notin I^{+}$.
Case (i) : Suppose $j = p, q = k$. Then $f_{ij}^{a}f_{pq}^{c}f_{kl}^{b} \notin I^{+}$. This implies $f_{iq}^{ac}f_{kl}^{b} \notin I^{+}$ (by Lemma 2.1), and so $f_{il}^{acb} \notin I^{+}$, a contradiction, since I satisfies IFP.
Case (ii) : Suppose $j \neq p, q \neq b$. Then $f_{ij}^{a}f_{pq}^{c}f_{kl}^{b} \notin I^{+}$.. This implies $f_{iq}^{ac}f_{kl}^{b} \notin I^{+}$, implies $f_{il}^{acb} \notin I^{+}$, and so $f_{il}^{ac} \notin I^{+}$, a contradiction. Therefore I^{+} satisfies IFP.
Conversely suppose that I^{+} satisfies IFP. On a contrary way, suppose that I does not satisfy IFP. Then for all $a,b \in R$ such that $ab \in I$ and $acb \notin I$ for some$C \in R$. Now we have $f_{14}^{ab} \in I^{+}$, and so $f_{12}^{a}f_{24}^{b} \in I^{+}$.. Since I^{+} satisfies IFP, $f_{12}^{a}Af_{23}^{b} \in I^{+}\ \forall A \in M_n(R)$. Take $A = f_{22}^{c}$, then $f_{12}^{a}f_{22}^{c}f_{23}^{b} = f_{13}^{acb} \notin I^{+}$, a contradiction to I^{+} satisfies IFP. □

Result 4.2. *$\{f_{11}^{a} : a \in R\}$ are central idempotent in $M_n(R)$ if and only if $\{a \in R : a$ is a central idempotent in $R\}$.*

Proof. Suppose f_{11}^{a} is central. We show a is a central element. Now $f_{11}^{ax} = f_{11}^{a}f_{11}^{x} = f_{11}^{x}f_{11}^{a}$ (since f_{11}^{a} is central) $= f_{11}^{xa}$. Therefore $f_{11}^{ax}(1,1,\cdots,1) = f_{11}^{xa}(1,1,\cdots,1) \Rightarrow (ax,0,0,\cdots) = (xa,0,0\cdots,0) \Rightarrow ax = xa$. Next we suppose that f_{11}^{a} is an idempotent. Now
$f_{11}^{a^2} = f_{11}^{a.a} = f_{11}^{a}f_{11}^{a} = f_{11}^{a}$ (since f_{11}^{a} is an idempotent). Therefore $f_{11}^{a^2}(1,0,\cdots,0) = f_{11}^{a}(1,0,\cdots 0)$. This implies $(a^2,0,\cdots,0) = (a,0,\cdots,0)$. Therefore $a^2 = a$, shows that a is an idempotent. □

Corollary 4.1. *$a \in R$ is an idempotent in R if and only if f_{ii}^{a} is an idempotent in $M_n(R)$, for all $\leq i \leq n$.*

Proof. $(f_{ii}^{a} \cdot f_{ii}^{a})(x_1,x_2,\cdots,x_n)$
$= f_{ii}^{a}(0\cdots,0,ax_i,0,\cdots 0)$

$= (0\cdots,0,aax_i,0,\cdots 0) = (0\cdots,0,ax_i,0,\cdots 0)$ (since $a^2 = a$) $= f^a_{ii}(x_1,x_2,\cdots,x_n)$. Conversely, take $i = 1$ and the result follows from the Result 4.2. Let I be an ideal of R. $I^* = (I^n : R^n) = \{A \in M_n(R) : A\rho \in I^n$ for all $\rho \in R^n\}$. □

Theorem 4.1. *Let I be an ideal of a zero symmetric nearring R. Then I is nilpotent in R if I^* is nilpotent in $M_n(R)$.*

Proof. Suppose $(I^*)^k = \{0\}$ for some $k \in \mathbb{Z}^+$. Take $p_1,p_2,\cdots,p_k \in I$. Then $f^{p_1}_{11},\cdots,f^{p_k}_{11} \in I^*$. Now $(f^{p_1}_{11},\cdots,f^{p_k}_{11})(1,1,\cdots,1)$
$= f^{p_1}_{11},\cdots,f^{p_{k-1}}_{11}(f^{p_k}_{11}(1,1,\cdots,1))$
$= f^{p_1}_{11},\cdots,f^{p_{k-1}}_{11}(p_k,0,\cdots 0)$
$= f^{p_1}_{11},\cdots,f^{p_{k-2}}_{11}(p_{k-1}p_k,0,\cdots 0)\cdots$
$= (p_1p_2p_3\cdots p_{k-1}p_k,0,\cdots,0) = (0,0,\cdots,0)$ (by supposition).

This implies that $p_1p_2p_3\cdots p_{k-1}p_k = 0$, and so $I^k = 0$ where $k \in \mathbb{Z}^+$. Hence, I is a nilpotent ideal in R. □

5. Conclusions

This paper established an order preserving correspondence between the fuzzy ideals of nearring module R (over itself) and that of R^n over matrix nearring $M_n(R)$. The concept further can be extended to study various prime ideal notions and related radicals in both the structures.

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