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ANALYSIS AND COMPUTATIONS OF CHROMATIC INDEX FOR A CLASS OF INTEGRAL SUM GRAPHS BASED ON EDGE COLORING AND EDGE-SUM COLORING

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Abstract. We consider families of integral sum graphs $H_{m,\phi}^{-i,s}$ and $H_{\phi,j}^{-i,s}$, where $-i < 0 < s$, $1 \leq m < i$ and $1 \leq j < s$ for all $i, s, m, j \in \mathbb{N}$. Since the graphs contains parameters i and s, so it is difficult to compute theoretical and numerical results. We apply edge coloring and edge-sum coloring on $H_{\phi,j}^{-i,s}$ and $H_{m,\phi}^{-i,s}$. Here, we compute the general formula of chromatic index by considering the minimum number of independent color classes. Comparison between these two techniques can be done. Numerical results corroborate derived theoretical results.

Keywords: Edge coloring, edge sum coloring, integral sum graph, graph coloring.

AMS Subject Classification: 05C15; 05C35.

1. Introduction

Various real-world situations can be effectively depicted using diagrams that feature multiple distinct points, such as individuals or communication centers. The connection between these points can be done through mathematical abstraction. This type of scenario can be represented as a graph. A graph is a order pair of the vertices V and the edges E. Here, we consider the integral sum graphs ([3, 24, 23, 18]) as follows:

$$H_{m,\phi}^{-i,s} = G_{-i,s} \setminus (\{u_{-m}\} \cup [e_{-m}]), \quad (1)$$

$$H_{\phi,j}^{-i,s} = G_{-i,s} \setminus (\{u_j\} \cup [e_j]), \quad (2)$$

where $i, s, m, j \in \mathbb{N}$ such that $1 \leq m < i$ and $1 \leq j < s$, $G_{-i,s}$ be the sum graphs, $[e_k]$ is the edge-sum color class and u_k be the vertex for each $k \in [-i, s]$. The concept of an integral sum graph ([3, 24, 23, 18]) involves a sum graph associated with a range of integers $S \subset \mathbb{Z}$. When the range is defined as $S = [-i, s] = \{-i, -(i+1), \cdots, 0, \cdots, s-1, s\}$ for any $i,\ s \in \mathbb{N}$, the resulting integral sum graph is denoted as $G_{-i,s}$. We have obtained the integral sum graphs in (1) and (2) by removing one vertex and corresponding to its edge

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sum color class from the sum graphs $G_{-i,s}$. Hence, we can conclude that equations (1) and (2) are integral sum graphs having vertices $[-i,s]\setminus -m$ and $[-i,s]\setminus j$, respectively.

For instance, consider the sum graph $G_{-4,3}$, where $[e_{-4}]=\{[-4,0],\ [-3,-1]\}$, $[e_{-3}]=\{[-4,1],[-3,0],[-2,-1]\}$, $[e_{-2}]=\{[-4,2],[-2,0],[-3,1]\}$, $[e_{-1}]=\{[-3,2],[-1,0],[-2,1],[-4,3]\}$, $[e_0]=\{[-3,3],[-1,1],[-2,2]\}$, $[e_1]=\{[0,1],[-1,2],[-2,3]\}$, $[e_2]=\{[-1,3],[0,2]\}$, and $[e_3]=\{[0,3],[1,2]\}$. We can derive the edge-sum color classes of $H^{-4,3}_{1,\phi}$ as follows: $[e_{-4}]=\{[-4,0]\}$, $[e_{-3}]=\{[-3,0],[-4,1]\}$, $[e_{-2}]=\{[-4,2],[-2,0],[-3,1]\}$, $[e_0]=\{[-3,3],[-2,2]\}$, $[e_1]=\{[0,1],[-2,3]\}$, $[e_2]=\{[0,2]\}$, and $[e_3]=\{[0,3],[1,2]\}$. Similarly, we can obtain all the edge-sum color classes of $H^{-4,3}_{\phi,1}$.

The goal of our work is to perform graph coloring on the edges of integral sum graphs (1) and (2), subject to certain constraints. Graph coloring is a popular concept in graph theory and involves assigning colors to vertices or edges of a graph. In edge coloring, no two edges that share a vertex can have the same color. The study of graph coloring in integral sum graphs has applications in securely distributing information among authorized individuals who can then reconstruct it. The coloring problem is widely used in computer science, including in tasks such as time-table scheduling, sudoku, map coloring, networking, final exam timetabling, guarding, aircraft scheduling, storage and manipulation of relational databases, and art galleries. The coloring problem has been extensively studied in various books ([9]) and papers ([15]), which describe and solve problems and conjectures related to this field of research.

Based on above motivations, we employ edge-sum coloring and edge coloring techniques on a specific set of integral sum graphs (1) and (2), where $i,s,m,j\in\mathbb{N}$ satisfy $-i<0<s$, $1\leq m<i$ and $1\leq j<s$. Since the number of color classes required for these graphs varies depending on the values of i and s, therefore it is challenging to determine the minimum number of color classes for all cases. Here, we present a general formula for computing the minimum number of independent color classes and compare the results. We verify that numerical findings support theoretical results. As far as we know, no studies exist in the literature regarding the integral sum graphs (1) and (2) corresponding to edge coloroing as well as edge sum coloring.

In 1979, Baranyai ([1]) studied edge coloring of various types of hypergraphs, such as regular, r-partite, complete, and uniform hypergraphs with m vertices. Holyer ([11]) calculated the chromatic index of NP-complete problem using edge coloring. He verified the results for cubic graphs. Later, Goldberg ([7]) applied the edge coloring technique on multigraphs. He also obtained the lower and upper limits of the edge chromatic index. Subsequently, Hakimi ([8]) developed a generalization of edge coloring based on multigraphs. He counted the boundaries in such a way that each color appears in each vertex u at most $m(u)$ times, which leads to the development of some new edge coloring problems. Pancones et al. ([20]) developed fast randomized algorithms for real-life edge coloring problems. Kapoor ([16]) applied the edge coloring on bipartite graph with m edges and n nodes. Januario et al. ([14]) showed that some of the problems can be formulated and solved naturally and extremely efficiently using the basic ideas of graph theory. He also provided a solution using natural model edge coloring for sports plans. Problems breaking the symmetry of graphs have been one of the central subfields of distributed algorithms in recent years and have received a lot of attention ([6]). Dosa et al. ([5]) studied various coloring properties of mixed cycloids. Dara et al. ([4]) applied strong edge coloring to Cayley graphs and product graphs. Ruksasakchai ([21]) and Omoomi et al. ([19]) applied edge coloring to various types of graphs. Recently, Li et al. ([17]) applied strong edge coloring of class of sparse graphs. Hornak et al. ([12]) performed 3-facial edge-coloring of planar graphs. Hu et al. ([13]) and Zhu et al. ([26]) calculated the injective edge chromatic number of different types of graphs. To know more about edge coloring reader can read the references [25, 2] and the references therein.

We organize the remaining work in the following sections. In section 2, we derive the chromatic index of the graphs (1) and (2). We have placed the numerical results in section 3. Finally, in section 4 we conclude the work.

2. Main Results

Harary ([9, 10]) introduced families of sum graphs G_n and families of integral sum graphs $G_{-i,s}$ for all $i,\ s \in \mathbb{Z}$. Then, many authors ([22, 18, 23, 24]) studied various properties of integral sum graphs and sum graphs. We now present some useful proof-free ([22]) theorems that help us apply edge coloring and edge-sum coloring.

Theorem 2.1. *Let u_k denotes the vertex whose label is k in the sum graph G_n, then*

$$deg(k) = \begin{cases} n-k-1, & if\ 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ n-k, & if\ \left\lfloor \frac{n}{2} \right\rfloor < k \leq n. \end{cases} \tag{3}$$

Theorem 2.2. *Let $|r| + s \geq 3$, then the number of edges in the integral sum graph $G_{r,s}$ is*

$$E\left(G_{r,s}\right) = \frac{1}{4}\left(r^2 + s^2 - 3r + 3s - 4rs\right) - \frac{1}{2}\left(\left\lfloor \frac{|r|}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor\right). \tag{4}$$

Theorem 2.3. *Let $i,\ s \in \mathbb{N}$ be any constant such that $-i < 0 < s$. Then*

$$1.\ \left|E\left(H^{-i,s}_{m,\phi}\right)\right| = \left|E\left(G_{-i,s}\right)\right| - (i+s-m) - \left\lfloor \frac{m-1}{2} \right\rfloor - \min\{(i-m), s\}, \quad if\ 1 \leq m \leq \left\lfloor \frac{i}{2} \right\rfloor,$$

$$= \left|E\left(G_{-i,s}\right)\right| - (i+s+1-m) - \left\lfloor \frac{m-1}{2} \right\rfloor - \min\{(i-m), s\}, \quad if\ \left\lfloor \frac{i}{2} \right\rfloor < m \leq i,$$

$$2.\ \left|E\left(H^{-i,s}_{\phi,j}\right)\right| = \left|E\left(G_{-i,s}\right)\right| - (i+s-j) - \left\lfloor \frac{j-1}{2} \right\rfloor - \min\{i, s-j\}, \quad if\ 1 \leq j \leq \left\lfloor \frac{s}{2} \right\rfloor,$$

$$= \left|E\left(G_{-i,s}\right)\right| - (i+s+1-j) - \left\lfloor \frac{j-1}{2} \right\rfloor - \min\{i, s-j\}, \quad if\ \left\lfloor \frac{s}{2} \right\rfloor < j \leq s.$$

Definition 2.1. *([22]) Edge-sum chromatic number of a graph is the number of distinct edge-sum color classes of the given graph.*

We denote edge-sum chromatic number and edge chromatic number ([15]) of any graph $G(s)$ as $\chi'_{\mathbb{Z}-sum}\left(G(s)\right)$ and $\chi'\left(G(s)\right)$, respectively.

Definition 2.2. *A graph $G(s)$ is said to be perfect edge sum color graph if $\chi'_{\mathbb{Z}-sum}\left(G(s)\right)$ is equal to $\chi'\left(G(s)\right)$. Otherwise, $G(s)$ is said to be non-perfect edge sum color graph.*

Example 2.1. $\chi'_{Z-sum}\left(H^{-2,2}_{\phi,1}\right) = \chi'_{Z-sum}\left(H^{-2,2}_{1,\phi}\right) = 4$ *and* $\chi'\left(H^{-2,2}_{\phi,1}\right) = \chi'\left(H^{-2,2}_{1,\phi}\right) = 3$. *The integral sum graphs $H^{-2,2}_{\phi,1}$ and $H^{-2,2}_{1,\phi}$ are called non-perfect edge sum color graphs.*

2.1. Chromatic index of $H^{-i,s}_{\phi,j}$.

Here, we compute edge and edge-sum chromatic numbers of $H^{-i,s}_{\phi,j}$ corresponding to different values of $i, s, j \in \mathbb{N}$ such that $-i < 0 < s$.

Lemma 2.1. *Let $H^{-1,s}_{\phi,1}$ be the integral sum graphs for all $s = 2, 3, 4, 5$. Then the edge chromatic number is s.*

Proof. The independent edge color classes of $H_{\phi,1}^{-1,2}$ are $\{\{(0,-1)\},\{(0,2)\}\}$. Therefore, $\chi'\left(H_{\phi,1}^{-1,2}\right)=2$. Similarly, for $H_{\phi,1}^{-1,3}$, $H_{\phi,1}^{-1,4}$ and $H_{\phi,1}^{-1,5}$ we have

$$\{\{(0,-1)\},\{(0,3)\},\{(-1,3),(0,2)\}\}\,, \tag{5}$$
$$\{\{(0,-1)\},\{(0,4)\},\{(-1,3),(0,2)\},\{(-1,4),(0,3)\}\}\,, \tag{6}$$
$$\{\{(0,-1),(2,3)\},\{(0,5)\},\{(-1,k),(0,k-1)\},\ \forall k=3,4,5\}\,, \tag{7}$$

are independent edge color classes, respectively. . Hence, the proof is complete. □

By similar analysis as in Lemma 2.1, we can proof the following Lemma.

Lemma 2.2. *Let $\chi'(G(s))$ be the edge chromatic index of any graph $G(s)$, then*

1. $\chi'\left(H_{\phi,i-1}^{-1,s}\right)=s$ *for* $s=i-1,\cdots,2i+1$ *and* $i=3,4$,
2. $\chi'\left(H_{\phi,i-2}^{-2,s}\right)=s+1$ *for* $s=3,\cdots,2i$ *and* $i=3,4,5$,
3. $\chi'\left(H_{\phi,i-2}^{-3,s}\right)=s+2$ *for* $s=4,\cdots,2i+1$ *and* $i=3,4,5$.

Theorem 2.4. *Let $H_{\phi,j}^{-1,s}$ be the integral sum graphs for all $s\geq 2$ and $j=1,2,3$. Then these are non-perfect edge sum color graphs for all $s\geq 2$.*

Proof. The distinct edge sum-color classes of $H_{\phi,j}^{-1,s}$ are E_{-1}, E_0, $E_1,\cdots$, E_{j-1}, $E_{j+1},\cdots$, E_s. Therefore, $\chi'_{Z-sum}\left(H_{\phi,j}^{-1,s}\right)=s+1$. Now, by Vizing's theorem we get

$$\Delta\left(H_{\phi,j}^{-1,s}\right)\leq\chi'\left(H_{\phi,j}^{-1,s}\right)\leq\Delta\left(H_{\phi,j}^{-1,s}\right)+1. \tag{8}$$

Now, from Theorem 2.1, we have the maximum degree of $H_{\phi,j}^{-1,s}$, i.e., $\Delta\left(H_{\phi,j}^{-1,s}\right)=deg(0)=s$. Therefore, from inequality (8), we get $s\leq\chi'\left(H_{\phi,j}^{-1,s}\right)\leq s+1$. Now, we define

Case A: $j=1$ and $s\geq 6$:

$$\begin{aligned}&\left\{(0,-1),\ (2,L-2),\ (3,L-3),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor,\left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\&\{(0,L)\}\,;\text{where } L=s,\{(-1,L),\ (0,L-1)\};\ \ \text{where } L=3,4,5.\\&\left\{(-1,L),\ (0,L-1),\ (2,L-3),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor,\left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\};\ \text{where } L=6,\ \cdots,s.\end{aligned} \tag{9}$$

Case B: $j=2$ and $s\geq 8$:

$$\begin{aligned}&\left\{(0,-1),\ (1,L-1),\ (3,L-3),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor,\left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\&\{(0,L),\ (-1,1)\}\,;\ \text{where } L=s,\{(0,1)\}\,,\ \{(-1,4),\ (0,3)\}\,,\\&\{(-1,L),\ (0,L-1),\ (1,L-2)\}\,;\ \text{where } L=5,6,7,\\&\left\{(-1,L),\ (0,L-1),\ (1,L-2),\ (3,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor,\left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\};\\&\qquad\text{where } L=8,9,\ \cdots,s.\end{aligned} \tag{10}$$

Case C: $j = 3$ and $s \geq 10$:

$$\begin{aligned}
&\left\{(0,-1),\ (1,L-1),\ (2,L-2),\ (4,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor, \left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\
&\{(0,L),\ (-1,1)\};\ \text{where } L = s.\\
&\{(-1,2),\ (0,1)\},\ \{\ (0,2)\},\ \{(-1,5),\ (0,4)\},\ \{(-1,6),\ (0,5),\ (1,4)\},\\
&\{(-1,L),\ (0,L-1),\ (1,L-2),\ (2,L-3)\};\ \text{where } L = 7,8,9,\\
&\left\{(-1,L),\ (0,L-1),\ (1,L-2),\ (2,L-3),\ (4,L-5),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor, \left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\};\\
&\qquad\qquad \text{where } L = 10, 11,\ \cdots, s.
\end{aligned} \tag{11}$$

By using equations (9), (10), (11) and Lemma 2.2, we have

$$\chi'\left(H_{\phi,j}^{-1,s}\right) = s \text{ for all } s \geq 2 \text{ and } j = 1,2,3.$$

From Definition 2.2, we can conclude $H_{\phi,j}^{-1,s}$ is non-perfect edge-sum color graphs for all $s \geq 2$ and $j = 1, 2, 3$. □

Theorem 2.5. *Let $H_{\phi,j}^{-2,s}$ be the integral sum graphs for all $s \geq 3$ and $j = 1, 2, 3$. Then these are non-perfect edge sum color graphs for all $s \geq 3$.*

Proof. By similar analysis, we have $\chi'_{Z-sum}\left(H_{\phi,j}^{-2,s}\right) = s+2$. Now, we define the following independent edge color classes for different values of j:
Case A: $j = 1$ and $s \geq 7$:

$$\begin{aligned}
&\left\{(0,-2),\ (2,L-3),\ (3,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor, \left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-1),\ (2,L-2),\ (3,L-3),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor, \left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\
&\{(0,L),\ (-2,2)\},\ \{(-1,L),\ (0,L-1)\};\ \text{where } L = s,\\
&\{(-2,L),\ (-1,L-1),\ (0,L-2)\};\ \text{where } L = 4,5,6,\\
&\left\{(-2,L),\ (-1,L-1),\ (0,L-2),\ (2,L-4),\ (3,L-5),\ \cdots,\ \left(\left\lfloor\frac{L-3}{2}\right\rfloor, \left\lfloor\frac{L}{2}\right\rfloor\right)\right\};\\
&\qquad \text{where } L = 7,8,9\ \cdots, s.
\end{aligned} \tag{12}$$

Case B: $j = 2$ and $s \geq 9$:

$$\begin{aligned}
&\left\{(0,-2),\ (1,L-2),\ (3,L-4),\ (4,L-5),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor, \left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-1),\ (1,L-1),\ (3,L-3),\ (4,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor, \left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\
&\{(0,L),\ (-1,1)\},\ \{(-2,1),\ (-1,L),\ (0,L-1)\};\ \text{where } L = s,\\
&\{(-2,3),\ (0,1)\},\ \{(-2,5),\ (-1,4),\ (0,3)\},\\
&\{(-2,L),\ (-1,L-1),\ (0,L-2),\ (1,L-3)\};\ \text{where } L = 6,7,8,\\
&\{(-2,L),\ (-1,L-1),\ (0,L-2),\ (1,L-3),\ (3,L-5),\ (4,L-6),\ (5,L-7),\ \cdots\\
&\qquad \cdots,\ \left(\left\lfloor\frac{L-3}{2}\right\rfloor, \left\lfloor\frac{L}{2}\right\rfloor\right)\right\};\ \text{where } L = 9, 10,\ \cdots,\ s.
\end{aligned} \tag{13}$$

Case C: $j = 3$ and $s \geq 11$:

$$\begin{aligned}
&\left\{(0,-2),\ (1,L-2),\ (2,L-3),\ (4,L-5),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor, \left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-1),\ (1,L-1),\ (2,L-2),\ (4,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor, \left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\
&\{(0,L),\ (-1,1),\ (-2,2)\},\ \{(-2,1),\ (-1,L),\ (0,L-1)\};\ \text{where } L=s,\\
&\{(-1,2),\ (0,1)\},\ \{(-2,4),\ (0,2)\},\ \{(-2,6),\ (-1,5),\ (0,4)\},\\
&\{(-2,7),\ (-1,6),\ (0,5),\ (1,4)\},\\
&\{(-2,L),\ (-1,L-1),\ (0,L-2),\ (1,L-3),\ (2,L-4)\};\ \text{where } L=8,9,10,\\
&\{(-2,L),\ (-1,L-1),\ (0,L-2),\ (1,L-3),\ (2,L-4),\ (4,L-6),\ (5,L-7),\ \cdots\\
&\qquad\cdots,\ \left(\left\lfloor\frac{L-3}{2}\right\rfloor, \left\lfloor\frac{L}{2}\right\rfloor\right)\right\};\text{where } L=11,12,\ \cdots,s.
\end{aligned} \tag{14}$$

Hence, from equations (12), (13), (14) and Lemma 2.2, we get

$$\chi'\left(H_{\phi,j}^{-2,s}\right) = s+1 \text{ for all } s\geq 3 \text{ and } j=1,2,3,$$

which also satisfies the Vizing's theorem. Hence, the proof is complete. □

Theorem 2.6. *Let $H_{\phi,j}^{-3,s}$ be the integral sum graphs for all $s \geq 4$ and $j = 1, 2, 3$. Then these are non-perfect edge sum color graphs for all $s \geq 4$.*

Proof. By similar calculation as in Theorem 2.4, we have $\chi'_{Z-sum}\left(H_{\phi,j}^{-3,s}\right) = s+3$. Now, we calculate the independent color classes as follows:

Case A: $j = 1$ and $s \geq 8$:

$$\begin{aligned}
&\left\{(-1,-2),(0,-3),\ (2,L-4),\ (3,L-5),\ \cdots,\ \left(\left\lfloor\frac{L-3}{2}\right\rfloor, \left\lfloor\frac{L}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-2),\ (2,L-3),\ (3,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor, \left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-1),\ (2,L-2),\ (3,L-3),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor, \left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\
&\{(0,L),\ (-2,2),\ (-3,3)\},\ \{\ (-2,L),\ (-1,L-1),\ (0,L-2)\},\\
&\{(-3,2),\ (-1,L),\ (0,L-1)\};\ \text{where } L=s,\\
&\{(-3,L),\ (-2,L-1),\ (-1,L-2),\ (0,L-3)\};\text{where } L=5,6,7.\\
&\{(-3,L),\ (-2,L-1),\ (-1,L-2),\ (0,L-3),\ (2,L-5),\ (3,L-6),\\
&\cdots,\ \left(\left\lfloor\frac{L-4}{2}\right\rfloor, \left\lfloor\frac{L-1}{2}\right\rfloor\right)\right\};\ \text{where } L=8,9,\ \cdots,s.
\end{aligned} \tag{15}$$

Case B: $j = 2$ and $s \geq 10$:

$$\begin{aligned}
&\left\{(-1,-2),\ (0,-3),\ (1,L-3),\ (3,L-5),\ \cdots,\ \left(\left\lfloor\frac{L-3}{2}\right\rfloor,\left\lfloor\frac{L}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-2),\ (1,L-2),\ (3,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor,\left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-1),\ (1,L-1),\ (3,L-3),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor,\left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\
&\{(0,L),\ (-1,1),\ (-3,3)\},\ \{\ (-3,1),\ (-2,L),\ (-1,L-1),\ (0,L-2)\},\\
&\{(-2,1),\ (-1,L),\ (0,L-1)\},\ \text{where } L=s,\\
&\{(-3,4),\ (-2,3),\ (0,1)\},\ \{(-3,6),\ (-2,5),\ (-1,4),\ (0,3)\},\\
&\{(-3,L),\ (-2,L-1),\ (-1,L-2),\ (0,L-3),\ (1,L-4)\};\ \text{where } L=7,8,9,\\
&\{(-3,L),\ (-2,L-1),\ (-1,L-2),\ (0,L-3),\ (1,L-4),\ (3,L-6),\\
&\qquad\cdots,\ \left(\left\lfloor\frac{L-4}{2}\right\rfloor,\left\lfloor\frac{L-1}{2}\right\rfloor\right)\right\};\ \text{where } L=10,11,\ \cdots,s.
\end{aligned} \tag{16}$$

Case C: $j = 3$ and $s \geq 12$:

$$\begin{aligned}
&\left\{(-1,-2),\ (0,-3),\ (1,L-3),\ (2,L-4),\ (4,L-6),\ \cdots,\ \left(\left\lfloor\frac{L-3}{2}\right\rfloor,\left\lfloor\frac{L}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-2),\ (1,L-2),\ (2,L-3),\ (4,L-5),\ \cdots,\ \left(\left\lfloor\frac{L-2}{2}\right\rfloor,\left\lfloor\frac{L+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,-1),\ (1,L-1),\ (2,L-2),\ (4,L-4),\ \cdots,\ \left(\left\lfloor\frac{L-1}{2}\right\rfloor,\left\lfloor\frac{L+2}{2}\right\rfloor\right)\right\},\\
&\{(0,L),\ (-1,1),\ (-2,2)\},\ \{\ (-3,1),\ (-2,L),\ (-1,L-1),\ (0,L-2)\},\\
&\{(-3,2),\ (-2,1),\ (-1,L),\ (0,L-1)\};\ \text{where } L=s,\\
&\{(-3,4),\ (-1,2),\ (0,1)\},\ \{(-3,5),\ (-2,4),\ (0,2)\},\\
&\{(-3,7),\ (-2,6),\ (-1,5),\ (0,4)\},\ \{(-3,8),\ (-2,7),\ (-1,6),\ (0,5),\ (1,4)\},\\
&\{(-3,L),\ (-2,L-1),\ (-1,L-2),\ (0,L-3),\ (1,L-4),\ (2,L-5)\};\\
&\qquad\qquad \text{where } L=9,10,11,\\
&\{(-3,L),\ (-2,L-1),\ (-1,L-2),\ (0,L-3),\ (1,L-4),\ (2,L-5),\ (4,L-7),\ \cdots\\
&\qquad\cdots,\left(\left\lfloor\frac{L-4}{2}\right\rfloor,\left\lfloor\frac{L-1}{2}\right\rfloor\right)\right\}\ \text{where } L=12,13,\ \cdots,s.
\end{aligned} \tag{17}$$

Hence, by using Lemma 2.2, equations (15), 16 and (17), we have

$$\chi'\left(H_{\phi,j}^{-3,s}\right) = s+2 \text{ for all } s \geq 4 \text{ and } j = 1,2,3.$$

Hence, the proof is complete. □

2.2. **Chromatic index of $H_{m,\phi}^{-i,s}$.** Here, we present results regarding edge sum chromatic number and edge chromatic number of $H_{m,\phi}^{-i,s}$ for $i, s, m \in \mathbb{N}$.

Lemma 2.3. *Let $H_{1,\phi}^{-i,1}$ be the integral sum graphs for all $i = 2,3,4,5$. Then $\chi'\left(H_{1,\phi}^{-i,1}\right) = i$ for all $i = 2,3,4,5$.*

Proof. $\{\{(0,1)\},\{(0,-2)\}\}$ are the independent edge color classes of $H^{-2,1}_{1,\phi}$. Therefore, $\chi'\left(H^{-2,1}_{1,\phi}\right)=2$. By similar analysis, we have

$$\{\{(0,1)\},\{(0,-3)\},\{(1,-3),(0,-2)\}\}\,, \tag{18}$$
$$\{\{(0,1)\},\{(0,-4)\},\{(1,-3),(0,-2)\},\{(1,-4),(0,-3)\}\}\,, \tag{19}$$
$$\{\{(0,1),(-2,-3)\},\{(0,-5)\},\{(1,-k),(0,-(k-1))\},\ \forall k=3,4,5\}\,, \tag{20}$$

are the independent color classes of $H^{-3,1}_{1,\phi}$, $H^{-4,1}_{1,\phi}$ and $H^{-5,1}_{1,\phi}$, respectively. Hence, we get the results. □

By similar analysis as in Lemma 2.3, we can prove the following Lemma.

Lemma 2.4. *Let $\chi'(G(s))$ be the edge chromatic index of any graph $G(s)$, then*

1. $\chi'\left(H^{-i,1}_{k-1,\phi}\right)=i$ *for* $i=k-1,\cdots,2k+1$ *and* $k=3,4$,
2. $\chi'\left(H^{-i,2}_{k-2,\phi}\right)=i+1$ *for* $i=3,\cdots,2k$ *and* $k=3,4,5$,
3. $\chi'\left(H^{-i,3}_{k-2,\phi}\right)=i+2$ *for* $i=4,\cdots,2k+1$ *and* $k=3,4,5$.

Theorem 2.7. *Let $H^{-i,1}_{m,\phi}$ be the integral sum graphs for all $i\geq 2$ and $m=1,2,3$. Then these are non-perfect edge sum color graph for all $i\geq 2$.*

Proof. By similar analysis as in Theorem 2.4, We get $\chi'_{Z-sum}\left(H^{-i,1}_{m,\phi}\right)=i+1$. Now, we define
Case A: $m=1$ and $i\geq 6$:

$$\begin{aligned}
&\left\{(0,1),\ (-2,-(p-2)),\ (-3,-(p-3)),\ \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor,-\left\lfloor\frac{p+2}{2}\right\rfloor\right)\right\},\\
&\{(0,-p)\}\,;\ \text{where } p=i,\\
&\{(1,-p),\ (0,-(p-1))\}\,;\ \text{where } p=3,4,5,\\
&\{(1,-p),\ (0,-(p-1)),\ (-2,-(p-3)),\ (-3,-(p-4)),\\
&\qquad\cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor,-\left\lfloor\frac{p+1}{2}\right\rfloor\right)\right\},\text{where } p=6,7,\ \cdots,i.
\end{aligned} \tag{21}$$

Case B: $m=2$ and $i\geq 8$:

$$\begin{aligned}
&\left\{(0,1),\ (-1,-(p-1)),\ (-3,-(p-3)),\ \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor,-\left\lfloor\frac{p+2}{2}\right\rfloor\right)\right\},\\
&\{(0,-p),\ (-1,1)\}\,;\ \text{where } p=i,\\
&\{(0,-1)\}\,,\ \{(1,-4),\ (0,-3)\}\,,\\
&\{(1,-p),\ (0,-(p-1)),\ (-1,-(p-2))\}\,;\text{where } p=5,6,7,\\
&\{(1,-p),\ (0,-(p-1)),\ (-1,-(p-2)),\ (-3,-(p-4)),\\
&\qquad\cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor,-\left\lfloor\frac{p+1}{2}\right\rfloor\right)\right\};\ \text{where } p=8,9,\ \cdots,i.
\end{aligned} \tag{22}$$

Case C: $m = 3$ and $i \geq 10$:

$$\begin{gathered}\{(0,1),\ (-1,-(p-1)),\ (-2,-(p-2)),\ (-3,-(p-3)),\ (-4,-(p-4)),\ \cdots \\ \cdots,\left(-\left\lfloor\frac{p-1}{2}\right\rfloor,-\left\lfloor\frac{p+2}{2}\right\rfloor\right)\Big\},\\ \{(0,-p),\ (-1,1)\}\,;\ \text{where } p=i,\\ \{(1,-2),(0,-1)\}\,,\ \{(0,-2)\}\,,\ \{(1,-5),(0,-4)\}\,,\ \{(1,-6),(0,-5),(-1,-4)\}\,, \\ \{(1,-p),\ (0,-(p-1)),\ (-1,-(p-2)),\ (-2,-(p-3))\}\,;\text{where } p=7,8,9,\\ \{(1,-p),\ (0,-(p-1)),\ (-1,-(p-2)),\ (-2,-(p-3)),\ (-4,-(p-5)),\\ (-5,-(p-6)),\ \cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor,-\left\lfloor\frac{p+1}{2}\right\rfloor\right)\Big\};\ \text{where } p=10,11,\ \cdots,i.\end{gathered} \tag{23}$$

So, from equations (21), (22), (23) and Lemma 2.4, we get

$$\chi'\left(H^{-i,1}_{m,\phi}\right)=i \text{ for all } i\geq 2 \text{ and } m=1,2,3.$$

This completes the proof. □

Theorem 2.8. *Let $H^{-i,2}_{m,\phi}$ be the integral sum-graphs for all $i \geq 3$ and $m = 1, 2, 3$. Then these are non-perfect edge sum color graphs for all $i \geq 3$.*

Proof. By similar analysis, we have $\chi'_{Z-sum}\left(H^{-i,2}_{m,\phi}\right) = i + 2$. Now, we have
Case A: $m = 1$ and $i \geq 7$:

$$\begin{gathered}\Big\{(0,2),\ (-2,-(p-3)),\ (-3,-(p-4)),\ \cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor,-\left\lfloor\frac{p+1}{2}\right\rfloor\right)\Big\},\\ \Big\{(0,1),\ (-2,-(p-2)),\ (-3,-(p-3)),\ \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor,-\left\lfloor\frac{p+2}{2}\right\rfloor\right)\Big\},\\ \{(0,-p),\ (-2,2)\}\,;\ \{(1,-p),\ (0,-(p-1))\}\,;\ \text{where } p=i,\\ \{(2,-p),\ (1,-(p-1)),\ (0,-(p-2))\}\,;\ \text{where } p=4,5,6,\\ \{(2,-p),\ (1,-(p-1)),\ (0,-(p-2)),\ (-2,-(p-4)),\ (-3,-(p-5)),\ \cdots\\ \cdots,\left(-\left\lfloor\frac{p-3}{2}\right\rfloor,-\left\lfloor\frac{p}{2}\right\rfloor\right)\Big\}\,;\text{where } p=7,8,9\ \cdots,i.\end{gathered} \tag{24}$$

Case B: $m = 2$ and $i \geq 9$:

$$\begin{gathered}\{(0,2),\ (-1,-(p-2)),\ (-3,-(p-4)),\ (-4,-(p-5)),\\ \cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor,-\left\lfloor\frac{p+1}{2}\right\rfloor\right)\Big\},\\ \{(0,1),\ (-1,-(p-1)),\ (-3,-(p-3)),\ (-4,-(p-4)),\\ \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor,-\left\lfloor\frac{p+2}{2}\right\rfloor\right)\Big\},\\ \{(0,-p),\ (-1,1)\}\,,\ \{(2,-1),\ (1,-p),\ (0,-(p-1))\}\,;\text{where } p=i,\\ \{(2,-3),\ (0,-1)\}\,,\ \{(2,-5),\ (1,-4),\ (0,-3)\}\,,\\ \{(2,-p),\ (1,-(p-1)),\ (0,-(p-2)),\ (-1,-(p-3))\}\,;\text{where } p=6,7,8,\\ \{(2,-p),\ (1,-(p-1)),\ (0,-(p-2)),\ (-1,-(p-3)),\ (-3,-(p-5)),\\ (-4,-(p-6)),\ (-5,-(p-7)),\ \cdots,\ \left(-\left\lfloor\frac{p-3}{2}\right\rfloor,-\left\lfloor\frac{p}{2}\right\rfloor\right)\Big\};\\ \text{where } p=9,10,11,12,\ \cdots,i.\end{gathered} \tag{25}$$

Case C: $m = 3$ and $i \geq 11$:

$$\begin{aligned}
&\{(0,2),\ (-1,-(p-2)),\ (-2,-(p-3)),\ (-4,-(p-5)),\\
&\qquad\qquad \cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor, -\left\lfloor\frac{p+1}{2}\right\rfloor\right)\right\},\\
&\{(0,1),\ (-1,-(p-1)),\ (-2,-(p-2)),\ (-4,-(p-4)),\\
&\qquad\qquad \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor, -\left\lfloor\frac{p+2}{2}\right\rfloor\right)\right\},\\
&\{(0,-p),\ (-1,1),\ (-2,2)\},\ \{(2,-1),\ (1,-p),\ (0,-(p-1))\};\ \text{where } p = i, \qquad (26)\\
&\{(1,-2),\ (0,-1)\},\ \{(2,-4),\ (0,-2)\},\ \{(2,-6),\ (1,-5),\ (0,-4)\},\\
&\{(2,-7),\ (1,-6),\ (0,-5),\ (-1,-4),\},\\
&\{(2,-p),\ (1,-(p-1)),\ (0,-(p-2)),\ (-1,-(p-3)),\ (-2,-(p-4))\};\\
&\qquad\qquad \text{where } p = 8, 9, 10.\\
&\{(2,-p),\ (1,-(p-1)),\ (0,-(p-2)),\ (-1,-(p-3)),\ (-2,-(p-4)),\\
&(-4,-(p-6)), (-5,-(p-7)),\ \cdots,\ \left(-\left\lfloor\frac{p-3}{2}\right\rfloor, -\left\lfloor\frac{p}{2}\right\rfloor\right)\right\};\\
&\qquad\qquad \text{where } p = 11, 12,\ \cdots, i.
\end{aligned}$$

By using equations (24), (25), (26) and Lemma 2.4, we have

$$\chi'\left(H^{-i,2}_{m,\phi}\right) = i + 1 \text{ for all } i \geq 3 \text{ and } m = 1, 2, 3.$$

Also, the edge chromatic index of $H^{-i,2}_{m,\phi}$ is verified by Vizing's theorem. Hence, we can conclude the proof. □

Theorem 2.9. *Let $H^{-i,3}_{m,\phi}$ be the integral sum graphs for all $i \geq 4$ and $m = 1, 2, 3$. Then these are non-perfect edge sum color graph for all $i \geq 4$.*

Proof. By similar analysis, we have $\chi'_{Z-sum}\left(H^{-i,3}_{m,\phi}\right) = i + 3$. Now, we compute the independent edge color classes as follows:
Case A: $m = 1$ and $i \geq 8$:

$$\begin{aligned}
&\left\{(1,2),\ (0,3),\ (-2,-(p-4)),\ (-3,-(p-5)),\ \cdots,\ \left(-\left\lfloor\frac{p-3}{2}\right\rfloor, -\left\lfloor\frac{p}{2}\right\rfloor\right)\right\},\\
&\left\{(0,2),\ (-2,-(p-3)),\ (-3,-(p-4)),\ \cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor, -\left\lfloor\frac{p+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,1),\ (-2,-(p-2)),\ (-3,-(p-3)),\ \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor, -\left\lfloor\frac{p+2}{2}\right\rfloor\right)\right\},\\
&\{(0,-p),\ (-2,2),\ (-3,3)\};\ \{\ (2,-p),\ (1,-(p-1)),\ (0,-(p-2))\}, \qquad (27)\\
&\{(3,-2),\ (1,-p),\ (0,-(p-1))\};\ \text{where } p = i,\\
&\{(3,-p),\ (2,-(p-1)),\ (1,-(p-2)),\ (0,-(p-3)\};\ \text{where } p = 5, 6, 7,\\
&\{(3,-p),\ (2,-(p-1)),\ (1,-(p-2)),\ (0,-(p-3)),\ (-2,-(p-5)),\\
&\qquad (-3,-(p-6)),\ \cdots,\ \left(-\left\lfloor\frac{p-4}{2}\right\rfloor, -\left\lfloor\frac{p-1}{2}\right\rfloor\right)\right\};\\
&\qquad\qquad \text{where } p = 8, 9,\ \cdots, i.
\end{aligned}$$

Case B: $m = 2$ and $i \geq 10$:

$$\begin{aligned}
&\left\{(1,2),(0,3),\ (-1,-(p-3)),\ (-3,-(p-5)),\ \cdots,\ \left(-\left\lfloor\frac{p-3}{2}\right\rfloor,-\left\lfloor\frac{p}{2}\right\rfloor\right)\right\},\\
&\left\{(0,2),\ (-1,-(p-2)),\ (-3,-(p-4)),\ \cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor,-\left\lfloor\frac{p+1}{2}\right\rfloor\right)\right\},\\
&\left\{(0,1),\ (-1,-(p-1)),\ (-3,-(p-3)),\ \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor,-\left\lfloor\frac{p+2}{2}\right\rfloor\right)\right\},\\
&\{(0,-p),\ (-1,1),\ (-3,3)\};\ \{\ (3,-1),\ (2,-p),\ (1,-(p-1)),\ (0,-(p-2))\},\\
&\{(2,-1),\ (1,-p),\ (0,-(p-1))\};\ \text{where } p=i,\\
&\{(3,-4),\ (2,-3),\ (0,-1)\},\ \{(3,-6),\ (2,-5),\ (1,-4),\ (0,-3)\},\\
&\{(3,-p),\ (2,-(p-1)),\ (1,-(p-2)),\ (0,-(p-3)),\ (-1,-(p-4))\};\\
&\qquad\qquad \text{where } p=7,8,9,\\
&\{(3,-p),\ (2,-(p-1)),\ (1,-(p-2)),\ (0,-(p-3)),\ (-1,-(p-4)),\ (-3,-(p-6)),\\
&\qquad\qquad \cdots,\ \left(-\left\lfloor\frac{p-4}{2}\right\rfloor,-\left\lfloor\frac{p-1}{2}\right\rfloor\right)\right\};\\
&\qquad\qquad\qquad \text{where } p=10,11,\ \cdots,i.
\end{aligned} \tag{28}$$

Case C: $m = 3$ and $i \geq 12$:

$$\begin{aligned}
&\{(1,2),(0,3),\ (-1,-(p-3)),\ (-2,-(p-4)),\ (-4,-(p-6)),\\
&\qquad\qquad \cdots,\ \left(-\left\lfloor\frac{p-3}{2}\right\rfloor,-\left\lfloor\frac{p}{2}\right\rfloor\right)\right\},\\
&\{(0,2),\ (-1,-(p-2)),\ (-2,-(p-3)),\ (-4,-(p-5)),\\
&\qquad\qquad \cdots,\ \left(-\left\lfloor\frac{p-2}{2}\right\rfloor,\left\lfloor\frac{p+1}{2}\right\rfloor\right)\right\},\\
&\{(0,1),\ (-1,-(p-1)),\ (-2,-(p-2)),\ (-4,-(p-4)),\\
&\qquad\qquad \cdots,\ \left(-\left\lfloor\frac{p-1}{2}\right\rfloor,-\left\lfloor\frac{p+2}{2}\right\rfloor\right)\right\},\\
&\{(0,-p),\ (-1,1),\ (-2,2)\},\ \{\ (3,-1),\ (2,-p),\ (1,-(p-1)),\ (0,-(p-2))\},\\
&\{(3,-2),\ (2,-1),\ (1,-p),\ (0,-(p-1))\};\ \text{where } p=i,\\
&\{(3,-4),\ (1,-2),\ (0,-1)\},\ \{(3,-5),\ (2,-4),\ (0,-2)\},\\
&\{(3,-7),\ (2,-6),\ (1,-5),\ (0,-4)\},\\
&\{(3,-8),\ (2,-7),\ (1,-6),\ (0,-5),\ (-1,-4)\},\\
&\{(3,-p),\ (2,-(p-1)),\ (1,-(p-2)),\ (0,-(p-3)),\ (-1,-(p-4)),\ (-2,-(p-5))\};\\
&\qquad\qquad\qquad \text{where } p=9,10,11,\\
&\{(3,-p),\ (2,-(p-1)),\ (1,-(p-2)),\ (0,-(p-3)),\ (-1,-(p-4)),\ (-2,-(p-5)),\\
&\qquad\qquad (-4,-(p-7)),\ \cdots,\ \left(-\left\lfloor\frac{p-4}{2}\right\rfloor,-\left\lfloor\frac{p-1}{2}\right\rfloor\right)\right\};\\
&\qquad\qquad\qquad \text{where } p=12,13,\ \cdots,i.
\end{aligned} \tag{29}$$

Hence, by using Lemma 2.4 and equations (27), (28) and (29), we have the edge chromatic index is

$$\chi'\left(H_{m,\phi}^{-i,3}\right) = i+2 \text{ for all } i \geq 4 \text{ and } m = 1,2,3.$$

Hence, the proof is complete. □

3. Numerical Results

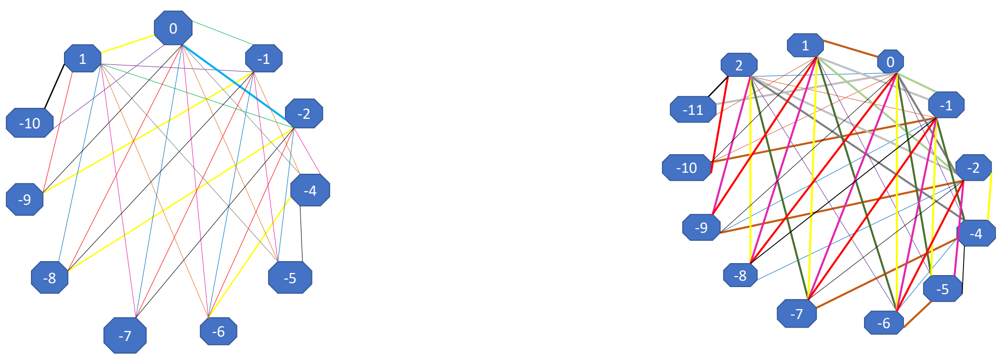
Here, we place numerical results to verify our derived theoretical results. We put $i = 10$ and $m = 3$ in $H^{-i,1}_{m,\phi}$. From Theorem 2.2, we have total number of edges of $G_{-10,1}$ is 41. By using Theorem 2.3, we have computed the edges of $H^{-10,1}_{3,\phi}$ is 31. Hence, we have to color these 31 edges by considering the minimum number of colors. Now, by using Theorem 2.7 and equation (23), we have the following independent edge color classes of $H^{-10,1}_{3,\phi}$:

$\{(0,1),\ (-1,-9),\ (-2,-8),\ (-4,-6)\} \to$ Yellow, $\{(0,-10),\ (-1,1)\} \to$ Purple,
$\{(1,-2),(0,-1)\} \to$ Green, $\{(0,-2)\} \to$ Cyan, $\{(1,-5),(0,-4)\} \to$ Grey
$\{(1,-7),(0,-6),(-1,-5),(-2,-4)\} \to$ Pink, $\{(1,-6),(0,-5),(-1,-4)\} \to$ Orange,
$\{(1,-8),(0,-7),(-1,-6),(-2,-5)\} \to$ Blue,
$\{(1,-9),(0,-8),(-1,-7),(-2,-6)\} \to$ Red,
and $\{(1,-10),(0,-9),(-1,-8),(-2,-7),(-4,-5)\} \to$ Black.

Therefore, we get $\chi'\left(H^{-10,1}_{3,\phi}\right) = 10$. Again, we consider $i = 11$ and $m = 3$. So, by using Theorem 2.8 and equation (26), we have the independent edge color classes of $H^{-11,2}_{3,\phi}$ are as follows:

$\{(0,2),\ (-1,-9),\ (-2,-8),\ (-4,-6)\} \to$ Blue, $\{(0,-11),\ (-1,1),\ (-2,2)\} \to$ Ivory,
$\{(0,1),(-1,-10),(-2,-9),(-4,-7),\ (-5,-6)\} \to$ Brown,
$\{(-1,2),(-11,1),(0,-10)\} \to$ Orange, $\{(1,-2),(0,-1)\} \to$ Olive green,
$\{(2,-4),(0,-2)\} \to$ Grey, $\{(2,-6),(1,-5),(0,-4)\} \to$ Purple,
$\{(2,-7),(1,-6),(0,-5),(-1,-4)\} \to$ Green,
$\{(2,-8),(1,-7),(0,-6),(-1,-5),(-2,-4)\} \to$ Yellow,
$\{(2,-9),(1,-8),(0,-7),(-1,-6),(-2,-5)\} \to$ Pink,
$\{(2,-10),(1,-9),(0,-8),(-1,-7),(-2,-6)\} \to$ Red,
$\{(2,-11),(1,-10),(0,-9),(-1,-8),(-2,-7),(-4,-5)\} \to$ Black.

So, $\chi'\left(H^{-11,2}_{3,\phi}\right) = 12$. Below, we have placed the edge coloring (See: figure 1) of integral sum graphs.



(A) $H^{-10,1}_{3,\phi}$ (B) $H^{-11,2}_{3,\phi}$

Figure 1. Edge coloring

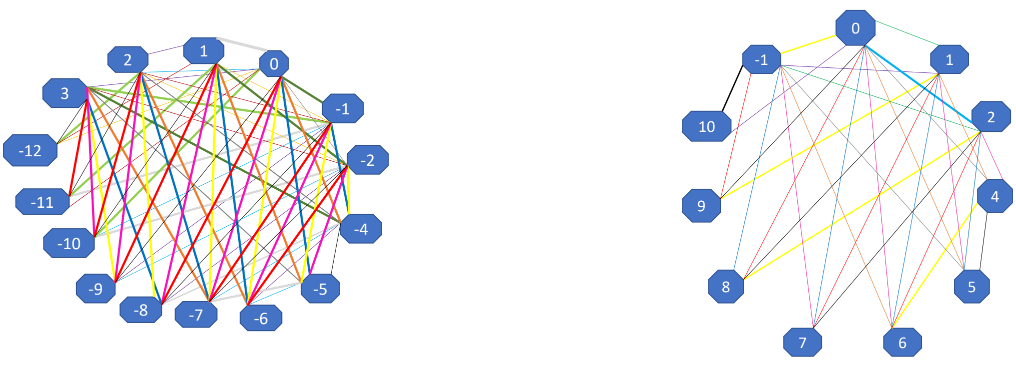
Here, we put $i = 12$ and $m = 3$ to compute the independent edge color classes of $H^{-i,3}_{m,\phi}$. Therefore, from Theorem 2.9, we have

$\{(1,2),(0,3),\ (-1,-9),\ (-2,-8),\ (-4,-6)\} \to$ Purple,
$\{(0,2),(-1,-10),(-2,-9),(-4,-7),\ (-5,-6)\} \to$ Cyan,
$\{(0,1),(-1,-11),(-2,-10),(-4,-8),(-5,-7)\} \to$ ivory,
$\{(3,-1),(2,-12),(1,-11),(0,-10)\} \to$ Light green,
$\{(3,-2),(2,-1),(1,-12),(0,-11)\} \to$ Maroon,
$\{(3,-4),(1,-2),(0,-1)\} \to$ Olive green, $\{(0,-12),\ (-1,1),\ (-2,2)\} \to$ Musturd,
$\{(3,-5),(2,-4),(0,-2)\} \to$ Grey, $\{(3,-7),(2,-6),(1,-5),(0,-4)\} \to$ Orange,
$\{(3,-8),(2,-7),(1,-6),(0,-5),(-1,-4)\} \to$ Blue,
$\{(3,-9),(2,-8),(1,-7),(0,-6),(-1,-5),(-2,-4)\} \to$ Yellow,
$\{(3,-10),(2,-9),(1,-8),(0,-7),(-1,-6),(-2,-5)\} \to$ Pink,
$\{(3,-11),(2,-10),(1,-9),(0,-8),(-1,-7),(-2,-6)\} \to$ Red,
$\{(3,-12),(2,-11),(1,-10),(0,-9),(-1,-8),(-2,-7)\} \to$ Black.

Hence, $\chi'\left(H^{-12,3}_{3,\phi}\right) = 14$. Now, we consider $s = 10$ and $j = 3$ to determine the edge chromatic number of $H^{-1,s}_{\phi,3}$. By using Theorem 2.4, we have

$\{(0,-1),\ (1,9),\ (2,8),\ (4,6)\} \to$ Yellow, $\{(0,10),\ (-1,1)\} \to$ Purple,
$\{(-1,2),(0,1)\} \to$ Green, $\{(0,2)\} \to$ Cyan,
$\{(-1,5),(0,4)\} \to$ Grey, $\{(-1,6),(0,5),(1,4)\} \to$ Orange,
$\{(-1,7),(0,6),(1,5),(2,4)\} \to$ Pink, $\{(-1,8),(0,7),(1,6),(2,5)\} \to$ Blue,
$\{(-1,9),(0,8),(1,7),(2,6)\} \to$ Red,
and $\{(-1,10),(0,9),(1,8),(2,7),(4,5)\} \to$ Black.

So, $\chi'\left(H^{-1,10}_{\phi,3}\right) = 10$. In figure 2, we have plotted the graphs.



(A) $H^{-12,3}_{3,\phi}$ (B) $H^{-1,10}_{\phi,3}$

FIGURE 2. Edge coloring

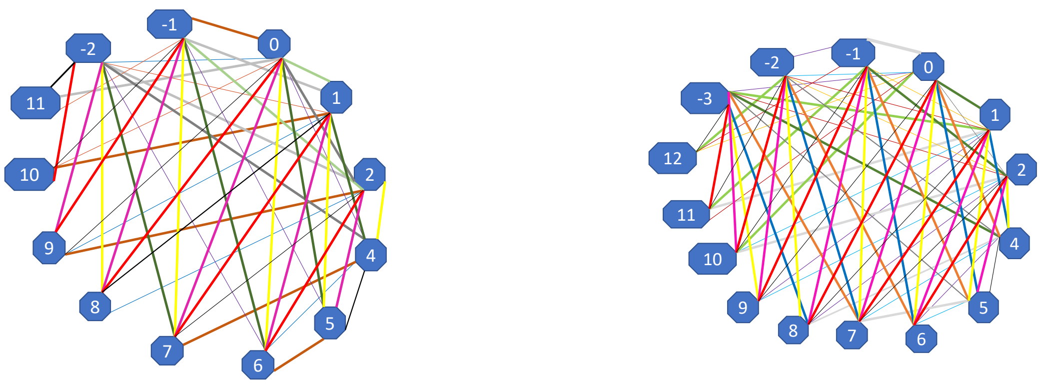
By using $s = 11$ and $j = 3$, from Theorem 2.5, we can get the edge color classes of the $H_{\phi,3}^{-2,s}$ as follows:

$\{(0,-2),\ (1,9),\ (2,8),\ (4,6)\} \to$ Blue, $\{(0,11),\ (-1,1),\ (-2,2)\} \to$ Ivory,
$\{(0,-1),(1,10),(2,9),(4,7),\ (5,6)\} \to$ Brown,
$\{(1,-2),(11,-1),(0,10)\} \to$ Orange, $\{(-1,2),(0,1)\} \to$ Olive green,
$\{(-2,4),(0,2)\} \to$ Grey, $\{(-2,6),(-1,5),(0,4)\} \to$ Purple,
$\{(-2,7),(-1,6),(0,5),(1,4)\} \to$ Green,
$\{(-2,8),(-1,7),(0,6),(1,5),(2,4)\} \to$ Yellow,
$\{(-2,9),(-1,8),(0,7),(1,6),(2,5)\} \to$ Pink,
$\{(-2,10),(-1,9),(0,8),(1,7),(2,6)\} \to$ Red,
$\{(-2,11),(-1,10),(0,9),(1,8),(2,7),(4,5)\} \to$ Black.

By similar analysis, for $s = 12$ and $j = 3$, from Theorem 2.6, we have

$\{(-1,-2),(0,-3),\ (1,9),\ (2,8),\ (4,6)\} \to$ Purple,
$\{(0,-2),(1,10),(2,9),(4,7),\ (5,6)\} \to$ Cyan,
$\{(0,-1),(1,11),(2,10),(4,8),(5,7)\} \to$ ivory,
$\{(-3,1),(-2,12),(-1,11),(0,10)\} \to$ Light green,
$\{(-3,4),(-1,2),(0,1)\} \to$ Olive green, $\{(0,12),\ (-1,1),\ (-2,2)\} \to$ Musturd,
$\{(-3,5),(-2,4),(0,2)\} \to$ Grey, $\{(-3,8),(-2,7),(-1,6),(0,5),(1,4)\} \to$ Blue,
$\{(-3,7),(-2,6),(-1,5),(0,4)\} \to$ Orange,
$\{(-3,2),(-2,1),(-1,12),(0,11)\} \to$ Maroon,
$\{(-3,9),(-2,8),(-1,7),(0,6),(1,5),(2,4)\} \to$ Yellow,
$\{(-3,10),(-2,9),(-1,8),(0,7),(1,6),(2,5)\} \to$ Pink,
$\{(-3,11),(-2,10),(-1,9),(0,8),(1,7),(2,6)\} \to$ Red,
$\{(-3,12),(-2,11),(-1,10),(0,9),(1,8),(2,7),(4,5)\} \to$ Black,

Therefore, the edge chromatic number of the graphs $H_{\phi,3}^{-2,11}$ and $H_{\phi,3}^{-3,12}$ are 12 and 14, respectively. In figure 3, we have shown the edge coloring of these two graphs.



(A) $H_{\phi,3}^{-2,11}$ (B) $H_{\phi,3}^{-3,12}$

FIGURE 3. Edge coloring

4. Conclusions

We have successfuly studied the properties of a class of integral sum graph $H_{m,\phi}^{-i,s}$ and $H_{\phi,j}^{-i,s}$ for all $i, s, m, k \in \mathbb{N}$ subject to the condition $-i < 0 < s$. We have performed edge coloring and edge-sum coloring. We have derived the general formula for computing of chromatic number. From the figures, we have noticed that no adjacent edges get the same color. We have observed that edge coloring is more effective than edge sum coloring. Numerical results guaranteed that our derived theoretical results are valid and reliable. Finally, we conclude that the results are promising and can play an important role in the existing literature.

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