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SEIDEL ENERGY OF NON-COMMUTING GRAPH FOR THE GROUP U_{6n}

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Abstract. The non-commuting graph for the group G, denoted by Γ_G, whose vertex set contains all group elements excluding central elements, where two distinct vertices v_i and v_j are adjacent whenever $v_iv_j \neq v_jv_i$. The non-diagonal entries of the Seidel matrix are -1 for two adjacent vertices, or one for non-adjacent vertices, whereas the diagonal entries are zero. This study presents the spectrum and energy of Γ_G for the group $G = U_{6n}$ associated with the Seidel matrix.

Keywords: Seidel matrix, energy of a graph, non-commuting graph, U_{6n}.

AMS Subject Classification: 05C25, 05C50, 15A18, 20D99

1. Introduction

Research which relates graph and group theories provides an analysis of a graph whose vertices are group elements, and whose edges link a pair of distinct vertices based on certain characteristics of the interactions between the elements. One of these graphs is the non-commuting graph for the group G, denoted by Γ_G which has $G\backslash Z(G)$ as the vertex set and $v_i, v_j \in G\backslash Z(G)$ are adjacent whenever $v_iv_j \neq v_jv_i$ [1].

In spectral graph theory, matrices are associated with graphs, commonly with a type of matrix called adjacency (A) matrices. Historically, Gutman in 1978 published an article showing that the adjacency energy of a finite graph is the sum of its absolute eigenvalues [5]. It is also evident in the spectra of the graph that emphasis is placed on the discussion of the spectral radius. According to Horn and Johnson [6], the spectral radius of Γ_G refers

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to the largest eigenvalue of a matrix associated with a graph. In addition, further research has proved that the adjacency energy of any graph is never an odd integer [2] or never the square root of an odd integer [12].

Apart from the adjacency matrix, Van Lint and Seidel [21] introduced the Seidel (S) matrix which is a symmetric $(0,-1,1)$-adjacency matrix for a graph as $S = J - 2A - I$, where J is a square matrix with all entries are equal to 1. Discussion on Seidel energy has been shown by Sarmin et al. [19] which focuses on the Cayley graph for dihedral groups. Recently, significant advances have been made in algebraic graph theory, particularly for commuting and non-commuting graphs. This is detailed in Bashir and Ahmadidelir [3] who examined the adjacency energy of Γ_G of Chein Maufang loops. The degree sum, degree exponent sum, degree subtraction, neighbors degree sum, maximum degree, and minimum degree energies of commuting and non-commuting graphs for dihedral groups can also be found in [14, 15, 16, 17, 18].

Furthermore, equienergetic properties of graphs corresponding to the Seidel matrix can be found in [20]. Ramane et al. [13] presented the characteristic polynomial of the Seidel Laplacian matrix of graphs as well as the Seidel signless Laplacian matrix. Furthermore, Mandal et al. [10] proved that -1 is always the eigenvalue of the Seidel matrix of the chain graphs. They also showed the Seidel energy bound of those graphs.

For the purpose of this paper, we focus on the finite and non-abelian group U_{6n} of order $6n$ for $n \geq 1$. It is defined as $U_{6n} = \langle a, b : a^{2n} = b^3 = e, a^{-1}ba = b^{-1} \rangle$ [7]. Hence, Γ_G for U_{6n} can be denoted by $\Gamma_{U_{6n}}$.

The paper is organized in the following manner. Several existing results are presented in the second section that is relevant to our research. Section 3 contains new main results on formulas for the spectrum and energy of $\Gamma_{U_{6n}}$ associated with the Seidel matrix.

2. Preliminaries

In this part, we describe some basic properties and previous results which are beneficial for the next section.

Definition 2.1. [21] *The Seidel matrix of order $n \times n$ associated with Γ_G is given by $S(\Gamma_G) = [s_{ij}]$ whose (i,j)-th entry*

$$s_{ij} = \begin{cases} -1, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 1, & \text{if } v_i \neq v_j \text{ and they are not adjacent} \\ 0, & \text{if } v_i = v_j. \end{cases}$$

The $S-$spectrum of Γ_G can be written as:

$$\sigma_S(\Gamma_G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ k_1 & k_2 & \dots & k_n \end{pmatrix},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues (not necessarily distinct) of $S(\Gamma_G)$ and $k_1, k_2, \dots, k_n$ are their respective multiplicities. Therefore, the $S-$energy of Γ_G can be defined as follows:

$$E_S(\Gamma_G) = \Sigma_{i=1}^{n} |\lambda_i|,$$

and $S-$spectral radius of Γ_G are defined as

$$\rho_S(\Gamma_G) = max\{|\lambda| : \lambda \in \sigma_S(\Gamma_G)\}.$$

Furthermore, $\Gamma_{U_{6n}}$ is a simple graph, therefore we can construct the Seidel matrix and compute the eigenvalues from the solution of the characteristic polynomial as a determinant form. The following theorems are the guidelines to simplify the determinant formula of a matrix which can be partitioned into four blocks.

Theorem 2.1. [4] *If a square matrix* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *can be partitioned into four blocks where* $|A| \neq 0$*, then the determinant of* M *is*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A| \left|D - CA^{-1}B\right|.$$

Now we define the sets $G_1 = \left\{a^{2r+1} : 0 \leq r \leq n-1\right\}$, $G_2 = \left\{a^{2r+1}b : 0 \leq r \leq n-1\right\}$, $G_3 = \left\{a^{2r+1}b^2 : 0 \leq r \leq n-1\right\}$, $G_4 = \left\{a^{2r}b : 0 \leq r \leq n-1\right\}$, and $G_5 = \left\{a^{2r}b^2 : 0 \leq r \leq n-1\right\}$. It is clear that $|G_1| = |G_2| = |G_3| = |G_4| = |G_5| = n$.

Lemma 2.1. [11] *For the group* U_{6n} *and* $0 \leq r \leq n-1$*, we have the following:*

1. *The center of* U_{6n} *is* $Z(U_{2n}) = \langle a^2 \rangle$.
2. *The centralizer of element* a^{2r+1} *is* $C_{U_{6n}}\left(a^{2r+1}\right) = \langle a \rangle$.
3. $C_{U_{6n}}\left(a^{2r+1}b\right) = \langle a^2 \rangle \cdot \langle \left\{a^{2s+1}b : 0 \leq s \leq n-1\right\} \rangle$.
4. $C_{U_{6n}}\left(a^{2r+1}b^2\right) = \langle a^2 \rangle \cdot \langle \left\{a^{2s+1}b^2 : 0 \leq s \leq n-1\right\} \rangle$.
5. $C_{U_{6n}}\left(a^{2r}b\right) = \langle a^2 \rangle \cdot \langle \left\{a^{2s}b, a^{2s}b^2 : 0 \leq s \leq n-1\right\} \rangle$.

Corollary 2.1. [8] *Let* $\Gamma_{U_{6n}}$ *be the non-commuting graph for* U_{6n} *where* $n \geq 1$*. If* d_{v_i} *is the degree of vertex* v_i*, which is the number of vertex that is adjacent to* v_i*, then for* $0 \leq r \leq n-1$*,*

1. $d_{a^{2r+1}} = 4n$,
2. $d_{a^{2r+1}b} = 4n$,
3. $d_{a^{2r+1}b^2} = 4n$,
4. $d_{a^{2r}b} = 3n$,
5. $d_{a^{2r}b^2} = 3n$.

Afterward, graphs with n vertices can be hyperenergetic, i.e. they have a larger energy than a complete graph K_n with n vertices. In light of the fact that U_{6n} has $5n$ vertices, we have the following definition.

Definition 2.2. [9] *A* $5n-$*vertex graph* $\Gamma_{U_{6n}}$ *is hyperenergetic if* $E(\Gamma_{U_{6n}}) > 2(5n-1)$.

3. Main Results

In this part, we determine spectral properties for $\Gamma_{U_{6n}}$ with respect to the Seidel matrix. We begin with the characteristic polynomial of the Seidel matrix of $\Gamma_{U_{6n}}$.

Theorem 3.1. *Let* $\Gamma_{U_{6n}}$ *be the non-commuting graph for* U_{6n}*, then the characteristic polynomial of* $S(\Gamma_{U_{6n}})$ *is*

$$P_{S(\Gamma_{U_{6n}})}(\lambda) = (\lambda+1)^{5n-3}(\lambda-1)^2\left(\lambda^2 + (2-n)\lambda + 1 - n - 8n^2\right).$$

Proof. We know that $|U_{6n} \backslash Z(U_{6n})| = 5n$ which implies $S(\Gamma_{U_{6n}})$ to have $5n$ vertices, and they are the members of G_1, G_2, G_3, G_4 and G_5. By Lemma 2.1 and Corollary 2.1, we can provide $5n \times 5n$ matrix $S(\Gamma_{U_{6n}})$ whose entries are:

1. $s_{ij} = -1$, for $1+k \leq i \neq j \leq n+k$ and $k = 0, n, 2n, 3n, 4n$;
2. $s_{ij} = -1$, for $3n+1 \leq i \leq 4n$ and $4n+1 \leq j \leq 5n$;
3. $s_{ij} = -1$, for $4n+1 \leq i \leq 5n$ and $3n+1 \leq j \leq 4n$;
4. $s_{ij} = 1$, otherwise.

Now $S(\Gamma_{U_{6n}})$ can be presented as follows:

$$
\begin{array}{c} \\ a \\ \vdots \\ a^{2n-1} \\ ab \\ \vdots \\ a^{2n-1}b \\ ab^2 \\ \vdots \\ a^{2n-1}b^2 \\ b \\ \vdots \\ a^{2(n-1)}b \\ b^2 \\ \vdots \\ a^{2(n-1)}b^2 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccc} a & \dots & a^{2n-1} & ab & \dots & a^{2n-1}b & ab^2 & \dots & a^{2n-1}b^2 & b & \dots & a^{2(n-1)}b & b^2 & \dots & a^{2(n-1)}b^2 \end{array} \\
\begin{pmatrix}
0 & \dots & 1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \dots & 0 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 \\
-1 & \dots & -1 & 0 & \dots & 1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & \dots & -1 & 1 & \dots & 0 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 \\
-1 & \dots & -1 & -1 & \dots & -1 & 0 & \dots & 1 & -1 & \dots & -1 & -1 & \dots & -1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & \dots & -1 & -1 & \dots & -1 & 1 & \dots & 0 & -1 & \dots & -1 & -1 & \dots & -1 \\
-1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & 0 & \dots & 1 & 1 & \dots & 1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & 1 & \dots & 0 & 1 & \dots & 1 \\
-1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & 1 & \dots & 1 & 0 & \dots & 1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & 1 & \dots & 1 & 1 & \dots & 0
\end{pmatrix}
\end{array}
$$

Clearly, the matrix $S(\Gamma_{U_{6n}})$ can be partitioned into 16 blocks as given below:

$$
S(\Gamma_{U_{6n}}) = \begin{bmatrix} (J-I)_n & -J_n & -J_n & -J_{n\times(2n)} \\ -J_n & (J-I)_n & -J_n & -J_{n\times(2n)} \\ -J_n & -J_n & (J-I)_n & -J_{n\times(2n)} \\ -J_{(2n)\times n} & -J_{(2n)\times n} & -J_{(2n)\times n} & (J-I)_{2n} \end{bmatrix}.
$$

Consequently, the characteristic polynomial of $S(\Gamma_{U_{6n}})$, $P_{S(\Gamma_{U_{6n}})}(\lambda)$ is given as follows:

$$
P_{S(\Gamma_{U_{6n}})}(\lambda) = \begin{vmatrix} \lambda I_n - (J-I)_n & J_n & J_n & J_{n\times(2n)} \\ J_n & \lambda I_n - (J-I)_n & J_n & J_{n\times(2n)} \\ J_n & J_n & \lambda I_n - (J-I)_n & J_{n\times(2n)} \\ J_{(2n)\times n} & J_{(2n)\times n} & J_{(2n)\times n} & \lambda I_{2n} - (J-I)_{2n} \end{vmatrix}.
$$

In order to get the formula of $P_{S(\Gamma_{U_{6n}})}(\lambda)$, row and column operations need to be performed. Let R_i and C_i be the $i-$th row and column of $P_{S(\Gamma_{U_{6n}})}(\lambda)$, respectively. We apply the following steps:

(1) $R_{4n+i} \longrightarrow R_{4n+i} - R_{3n+i}$, for $i = 1, 2, \dots, n$.
(2) $R_{3n+1+i} \longrightarrow R_{3n+1+i} - R_{3n+1}$, for $i = 1, 2, \dots, n-1$.
(3) $R_{j-i} \longrightarrow R_{j-i} - R_{3n-i}$, for $i = 1, 2, \dots, n$ and $j = n, 2n$.
(4) $R_{3n+1-i} \longrightarrow R_{3n+1-i} - R_{3n+1}$, for $i = 1, 2, \dots, n-1$.
(5) $C_{2n+i} \longrightarrow C_{2n+i} + C_{j+i}$, for $i = 1, 2, \dots, n$ and $j = 0, n$.
(6) $C_{3n+i} \longrightarrow C_{3n+i} + C_{4n+i}$, for $i = 1, 2, \dots, n$.
(7) $R_{j+i} \longrightarrow R_{j+i} - R_{2n}$, for $i = 1, 2, \dots, n-1$ and $j = 0, n$.
(8) $C_j \longrightarrow C_j + C_{j-1} + C_{j-2} + \dots + C_{j-(n-1)}$, $j = n, 2n, 3n$.
(9) $C_{3n+1} \longrightarrow C_{3n+1} + C_{3n+2} + C_{3n+3} + \dots + C_{4n-1}$,

and $P_{S(\Gamma_{U_{6n}})}(\lambda)$ can be rewritten as the following determinant

$$
\begin{vmatrix}
(\lambda+1)I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
-2J_{1\times(n-1)} & \lambda - 1 & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} & 0 & 0 & 0_{1\times(n-1)} & 0_{1\times n} \\
0_{n-1} & 0_{(n-1)\times 1} & (\lambda+1)I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
0_{1\times(n-1)} & 0 & -2J_{1\times(n-1)} & \lambda - 1 & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda+1)I_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
J_{1\times(n-1)} & n & J_{1\times(n-1)} & n & J_{1\times(n-1)} & \lambda + n + 1 & 2n & 2J_{1\times(n-1)} & J_{1\times n} \\
J_{1\times(n-1)} & n & J_{1\times(n-1)} & n & 3J_{1\times(n-1)} & 3n & \lambda - 2n + 1 & -2J_{1\times(n-1)} & -J_{1\times n} \\
0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & (\lambda+1)I_{n-1} & 0_{(n-1)\times n} \\
0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & (\lambda+1)I_n
\end{vmatrix}.
$$

By Theorem 2.1, $P_{S(\Gamma_{U_{6n}})}(\lambda)$ can be simplified as follows:

$$P_{S(\Gamma_{U_{6n}})}(\lambda) = (\lambda+1)^{5n-4}(\lambda-1)^2\left(\lambda^2+(2-n)\lambda+1-n-8n^2\right).$$

□

Consequently, the Seidel spectrum of the non-commuting graph for U_{6n} can be expressed as described in the following theorem.

Theorem 3.2. *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for U_{6n}, then the $S-$spectrum of $\Gamma_{U_{6n}}$ is*

$$\begin{pmatrix} \frac{n-2}{2}+n\frac{\sqrt{33}}{2} & 1 & -1 & \frac{n-2}{2}-n\frac{\sqrt{33}}{2} \\ 1 & 2 & 5n-4 & 1 \end{pmatrix}.$$

Proof. The four eigenvalues of $\Gamma_{U_{6n}}$ is given by the roots of $P_{S(\Gamma_{U_{6n}})}(\lambda) = 0$ which is obtained from Theorem 3.1. The eigenvalues are $\lambda_{1,2} = \frac{n-2}{2} \pm n\frac{\sqrt{33}}{2}$, each of multiplicity 1, $\lambda_3 = 1$ of multiplicity 2, and $\lambda_4 = -1$ with multiplicity $5n-4$. Therefore, we get the spectrum of $\Gamma_{U_{6n}}$ associated with Seidel matrix. □

The followings are the results of the Seidel spectral radius and energy of the non-commuting graph for U_{6n}

Theorem 3.3. *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for U_{6n}, then the $S-$spectral radius of $\Gamma_{U_{6n}}$ is*

$$\rho_S(\Gamma_{U_{6n}}) = \frac{n-2}{2} + n\frac{\sqrt{33}}{2}.$$

Proof. It is clear by Theorem 3.2, the maximum absolute value of λ_i for $i = 1,2,3,4$ is $\frac{n-2}{2} + n\frac{\sqrt{33}}{2}$. □

Theorem 3.4. *Let $\Gamma_{U_{6n}}$ be the non-commuting graph for U_{6n}, then the $S-$energy of $\Gamma_{U_{6n}}$ is*

$$E_S(\Gamma_{U_{6n}}) = 5n-2+n\sqrt{33}.$$

Proof. Based on Theorem 3.2, we can calculate Seidel energy of $\Gamma_{U_{6n}}$ in the following manner:

$$E_S(\Gamma_{U_{6n}}) = \left|\frac{n-2}{2} \pm n\frac{\sqrt{33}}{2}\right| + (5n-4)|-1| + (2)\,|1| = 5n-2+n\sqrt{33}.$$

□

According to Theorem 3.4 and Definition 2.2, the following can be concluded:

Corollary 3.1. *$\Gamma_{U_{6n}}$ is hyperenergetic with respect to the Seidel energy.*

Proof. From Theorem 3.4, we know that

$$E_S(\Gamma_{U_{6n}}) = 5n-2+n\sqrt{33} = (5+\sqrt{33})n-2 > 10n-2.$$

Based on Definition 2.2, $\Gamma_{U_{6n}}$ is hyperenergetic. □

Corollary 3.2. *$E_S(\Gamma_{U_{6n}})$ is never an odd integer.*

Proof. Since n is a natural number, then $E_S\Gamma_{U_{6n}}) = 5n-2+n\sqrt{33}$ is never an odd integer. □

Corollaries 3.1 and 3.2 comply with the well-known fact from [2] and [12] that the energy of a graph is never an odd integer as well as never the square root of an odd integer.

We end this section with Example 3.1 to serve as an example of computation when $n = 1$.

Example 3.1. *Let* $U_6 = \{e, a, b, ab, b^2, ab^2\}$ *and* $Z(U_6) = \{e\}$*, where* $C_{U_6}(a) = \{e, a\}$, $C_{U_6}(b) = \{e, b, b^2\} = C_{U6}(b^2)$, $C_{U_6}(ab) = \{e, ab\}$, $C_{U_6}(ab^2) = \{e, ab^2\}$*. For* $G = U_6\backslash Z(U_6)$*, then the non-commuting graph* Γ_{U_6}*, whose set of vertices is* $U_6\backslash Z(U_6)$*, is a simple graph of order five, as illustrated in Figure 1.*

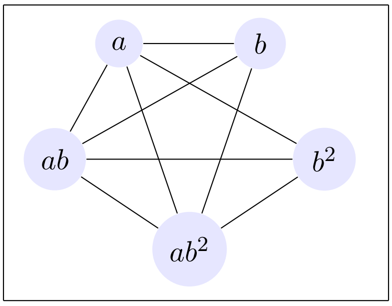

FIGURE 1. The non-commuting graph for the group U_6, Γ_{U_6}

Now we construct 5×5 *Seidel matrix of* Γ_{U_6} *as follows:*

$$S(\Gamma_{U_6}) = \begin{array}{c} \\ a \\ ab \\ ab^2 \\ b \\ b^2 \end{array}\begin{array}{c} \begin{array}{ccccc} a & ab & ab^2 & b & b^2 \end{array} \\ \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 0 \end{pmatrix} \end{array}$$

Hence, the characteristic polynomial of $S(\Gamma_{U_6})$ *is*

$$P_{S(\Gamma_{U_6})}(\lambda) = (\lambda + 1)(\lambda - 1)^2(\lambda^2 + \lambda - 8).$$

By using Maple, we have confirmed that the S*-spectrum of* Γ_{U_6} *is*

$$\sigma_S(\Gamma_{U_6}) = \begin{pmatrix} \frac{-1}{2} + \frac{\sqrt{33}}{2} & 1 & -1 & \frac{-1}{2} - \frac{\sqrt{33}}{2} \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

and the S*-spectral radius of* Γ_{U_6} *is*

$$\rho_S(\Gamma_{U_6}) = \frac{-1}{2} + \frac{\sqrt{33}}{2}.$$

Therefore, the $S-$*energy of* Γ_{U_6} *is*

$$E_S(\Gamma_{U_6}) = (1)|-1| + (2)|1| + \left|\frac{-1}{2} \pm \frac{\sqrt{33}}{2}\right| = 3 + \sqrt{33}.$$

4. CONCLUSIONS

Seidel energy of the non-commuting graph for the group U_{6n}, $\Gamma_{U_{6n}}$, is certainly not an odd integer. Moreover, $\Gamma_{U_{6n}}$ is a hyperenergetic graph with respect to Seidel energy.

As a future view of these methods, we recommend combining them with [22, 23], which is essentially an extension of the graph matrix based on Q-NSS matrix.

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