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DIVISOR EQUITABLE DOMINATION IN FUZZY GRAPHS

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Abstract. The dominance of graph theory in a number of domains, including coding theory, facility location issues, biological network modelling and bus routing, has been credited with a wide range of applications. Finding the right group of representatives challenges, maintaining communication and electricity networks and land surveying all involve concepts from dominance. Hence, in this article, we introduced divisor equitable dominating sets in fuzzy graphs. We defined divisor equitable domination number and divisor minimal equitable dominating set. We also discussed the characterizations of minimal divisor equitable dominating sets. Further we studied the relation between divisor equitable independent sets and minimal divisor equitable dominating sets.

Keywords: divisor equitable domination, fuzzy equitable domination, divisor equitable degree, fuzzy independent set, minimal divisor equitable dominating set.

AMS Subject Classification: 05C72, 05C69

1. Introduction

Euler's[9] article where he found the solution to the Königsberg Bridge Problem gave rise to graph theory in 1736. He researched the issue and developed an approach known as an Eulerian graph. In essence, he established the first theorem in graph theory, making him the father of the discipline. The field did not advance for the following 100 years. Later, in 1857, Cayley [5] introduced trees while attempting to list the isomers of saturated carbons. It used to be thought of as a sub-field of combinatorics, graph theory. It is currently a very important area of applied mathematics.

Graphs are just relational models. It is an effective method of presenting data including relationships between objects. Vertices identify objects, whereas edges indicate connections. The study of dominance is one of the most swiftly developing areas in graph theory. One of the objectives is for specific chess pieces to cover or control particular chessboard movement sequences. Take into account the collection of cities that are linked together by roadways or other forms of connectivity. This is a case using a graph model. However,

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conditions in cities vary, and residents enjoy a variety of perks even in areas with little traffic. This incident also happens frequently on small highways. Therefore, the weighted graph model is what we first encounter. The privileges are ambiguous in nature rather than being clear. As a result, we lack a weighted graph model.

V. Swaminathan and K. M. Dharmalingam[33] was the one who first articulated the idea of equitable domination in fuzzy graphs. A. Anitha et al [1] established different types of equitable domination in graphs. The notions of connected equitable domination in graphs, neighbourhood connected equitable domination in graphs and neighbourhood connected 2-equitable domination in graphs were first developed by S. Sivakumar et al [31]. I. S. Hamid [11] introduced the concept of independent transversal domination in graphs. The split equitable domination in graphs was proposed by K. B. Murthy [18] and he pioneered the notion of a graph's split equitable domination number.

Sampath Kumar and Walikar [29] established the idea of connected domination in graphs. Total domination in graphs was first proposed by Cockayane and Hedetniemi [6] in 1977. Interest in the study of domination in graphs was sparked by Cockayane and Hedetniemi's survey paper [6], Kulli [16, 15] adopted inverse domination and Split dominance. Additionally, he developed the idea of strong non-split domination in graphs. Arumugam and Sivagnanam [2] first defined the idea of a neighbourhood connected dominating set before Kulli investigated it. Sampathkumar and Pushpalatha [28] first proposed the idea of strong domination.

When there is uncertainty in the descriptions of the items or in their relationships, or in both occurrences, we naturally have to construct a "Fuzzy Graph Model." A symmetric binary fuzzy relation on a fuzzy subset is a fuzzy graph. The concept of fuzzy sets and fuzzy relations were introduced by L. A. Zadeh [37] in 1965 and further studied. The basic idea of fuzzy graph was introduced by Kauffmann [14] in 1973. In 1975, Fuzzy graphs were introduced by A. Rosenfeld [26]. Rosenfeld considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. A fuzzy relation accurately depicts the relationship between the objects in a collection by describing the strength of the association between them. Additionally, he developed other fuzzy related graph theoretical notions, including the bridge, cut vertex and tree. K. R. Bhutani and A. Rosenfeld [4] studied about strong arcs in fuzzy graphs. The concept of fuzzy trees, blocks, bridges and cut nodes in fuzzy graph has been studied by M. S. Sunitha and A. Vijayakumar [32].

Fuzzy graphs have many more uses in the modelling of real-time systems where the degree of information inherent in the system varies with changing levels of precision. The theory of dominance in fuzzy graphs utilizing effective edges was first presented by S. Somasundaram [27]. A. Nagoorgani and V. T. Chandrasekaran [19] studied dominance in fuzzy graphs using strong arc. Strong (weak) domination in fuzzy graphs is a concept that was first developed by C. Natarajan and S. K. Ayyaswamy [21] as an extension of strong (weak) domination in crisp graphs. O. T. Manjusha and M. S. Sunitha [17] established the idea of 1-strong dominance in fuzzy graphs as an extension of the concept of domination in fuzzy graphs with strong edges. As an extension of 2-domination in graphs, the idea of 2-domination in fuzzy graphs was developed in 2015 by A. Nagoor Gani and K. Prasanna Devi [20]. In order to extract the traditional results and reduce the value of the odd dominance number, O. T. Manjusha and M. S. Sunitha [17] established the idea of strong domination in fuzzy graphs in 2015. Many author studied the concept of domination in fuzzy graphs [3, 12, 23, 35, 36].

The concept of equitable domination in fuzzy graphs was studied by S. Revathi and C. V. R. Harinarayanan [24]. S. Revathi et.al [25] developed the idea in regular equitable domination number in fuzzy graph. The concept of effective edge domination in fuzzy graph is introduced by A. Selvam et.al [30]. R. Sumathi et.al [34] are taken the equitable domination concepts to trees.

In this paper, the concept of divisor equitable domination in the fuzzy graph is introduced. The minimal divisor domination number of a fuzzy graph is characterized.

2. Preliminaries

Definition 2.1. [1] *A fuzzy graph* $\mathscr{Q} = (\chi, \upsilon)$ *on a graph* $\mathscr{Q}^* = (W, E)$ *is a pair of functions* $\chi : W \to [0,1]$ *and* $\upsilon : W \times W \to [0,1]$*, where* χ *is a fuzzy subset of non empty set* W *and* υ *is a symmetric relation on* χ *such that* $\forall$ m, w *in* W *the relation* $\upsilon(m,w) \leqslant \chi(m) \wedge \chi(w)$ *is satisfied.*

Definition 2.2. [13] *Let* $\mathscr{Q} = (\chi, \upsilon)$ *be a fuzzy graph on* $\mathscr{Q}^* = (W, E)$*. The neighbourhood of* w *is the set* $\mathbb{N}_{\mathscr{Q}}(w) = \mathbb{N}(w) = \{m \in W(\mathscr{Q}) : mw \in E(\mathscr{Q})\}$*. If* $T \subseteq W(\mathscr{Q})$*, then the open neighbourhood of* T *is the set* $\mathbb{N}_{\mathscr{Q}}(T) = \mathbb{N}(T) = \bigcup\limits_{w \in T} \mathbb{N}_{\mathscr{Q}}(w)$*. The closed neighbourhood of* T *is* $\mathbb{N}_{\mathscr{Q}}[T] = T \cup \mathbb{N}(T)$*.*

Definition 2.3. [8] *The order* p *and size* q *of a fuzzy graph* $\mathscr{Q} = (\chi, \upsilon)$ *are defined as* $p = \sum\limits_{m \in W} \chi(m)$ *and* $q = \sum\limits_{m,w \in E} \upsilon(m,w)$*.*

Definition 2.4. [24] *Let* $\mathscr{Q} = (\chi, \upsilon)$ *be a fuzzy graph on* $\mathscr{Q}^* = (W, E)$*. The degree of a vertex* t *is* $\mathrm{d}_{\mathscr{Q}}(\mathrm{t}) = \sum\limits_{\mathrm{m} \neq \mathrm{t}} \upsilon(\mathrm{m},\mathrm{t})$*. Since* $\upsilon(\mathrm{m},\mathrm{t}) > 0$ *for* $\mathrm{mt} \in \mathrm{E}$ *and* $\upsilon(\mathrm{mt}) = 0$ *for* $\mathrm{mt} \notin \mathrm{E}$ *this is equivalent to* $\mathrm{d}_{\mathscr{Q}}(\mathrm{t}) = \sum\limits_{\mathrm{mt} \in \mathrm{E}} \upsilon(\mathrm{mt})$*. The minimum degree of* $\mathscr{Q}$ *is* $\delta(\mathscr{Q}) = \wedge\{\mathrm{d}_{\mathscr{Q}}(\mathrm{t}) : \mathrm{t} \in \mathscr{Q}\}$*. The maximum degree of* $\mathscr{Q}$ *is* $\Delta(\mathscr{Q}) = \vee\{\mathrm{d}_{\mathscr{Q}}(\mathrm{t}) : \mathrm{t} \in \mathscr{Q}\}$*.*

Definition 2.5. [10] *The neighbourhood degree of a vertex* m *is defined to be the sum of the weights of the vertices adjacent to* m *and is denoted by* $d_{\mathbb{N}}(m)$*, the minimum neighbourhood degree of* $\mathscr{Q}$ *is* $\delta_{\mathbb{N}}(\mathscr{Q}) = \min\{d_{\mathbb{N}}(m) : m \in W\}$ *and the maximum neighbourhood degree of* $\mathscr{Q}$ *is* $\Delta_{\mathbb{N}}(\mathscr{Q}) = \max\{d_{\mathbb{N}}(m) : m \in W\}$*.*

Definition 2.6. [24] *The strength of the connectedness between two nodes* m, w *in a fuzzy graph* $\mathscr{Q}$ *is* $\upsilon^{\infty}(m,w) = sup\left\{\upsilon^{k}(m,w) : k = 1,2,3...\right\}$ *where*
$\upsilon^{k}(m,w) = sup\{\upsilon(m,m_1) \wedge \upsilon(m_1,m_2) \wedge \upsilon(m_2,m_3) \wedge ...\upsilon(m_{k-1},w)\}$*.*

Definition 2.7. [10] *An arc* (m,w) *in a fuzzy graph* $\mathscr{Q} = (\chi, \upsilon)$ *is said to be strong if* $\upsilon^{\infty}(m,w) = \upsilon(m,w)$ *then* m, w *are called strong neighbours.*

Definition 2.8. [10] *The strong neighbourhood of the vertex* m *is characterised as* $\mathbb{N}_S(m) = \{w \in W \mid (m,w)$ *is a strong arc* $\}$*.*

Definition 2.9. [24] *A fuzzy graph* $\mathscr{Q} = (\chi, \upsilon)$ *on a graph* $\mathscr{Q}^* = (W, E)$*. A subset* F *of* W *is called a dominating set in* $\mathscr{Q}$ *if every vertex in* $\mathrm{W} \backslash \mathrm{F}$, $\exists$ $\mathrm{m} \in \mathrm{F}$ *such that* m *dominates* w*. The domination number of* $\mathscr{Q}$ *is the minimum cardinality taken over all dominating sets in* $\mathscr{Q}$ *and is denoted by* $\gamma(\mathscr{Q})$ *or simply* γ_{f}*. A fuzzy dominating set* F *of a fuzzy graph* $\mathscr{Q}$ *is called minimal fuzzy dominating set of* $\mathscr{Q}$*, if for every node* $\mathrm{w} \in \mathrm{F}, \mathrm{F} \backslash \{\mathrm{w}\}$ *is not a fuzzy dominating set.*

Example 2.1. *A fuzzy graph* $\mathscr{Q} = (\chi, \upsilon)$ *on a graph* $\mathscr{Q}^* = (W, E)$ *defined as follows.*

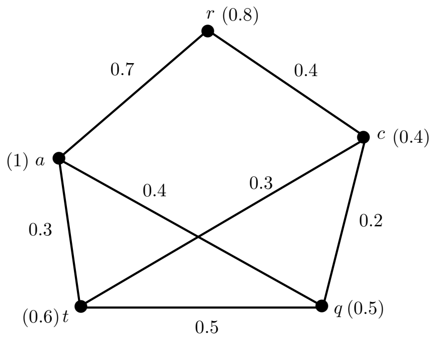

FIGURE 1. Example of fuzzy dominating set.

Here $F = \{a, c\}$ *is a dominating set with* $\gamma(\mathscr{Q}) = 1.4$. *Also* $\mathrm{d}_{\mathscr{Q}}(a) = 1.4, \mathrm{d}_{\mathscr{Q}}(r) = 1.1, \mathrm{d}_{\mathscr{Q}}(c) = 0.9, \mathrm{d}_{\mathscr{Q}}(t) = 1.1, \mathrm{d}_{\mathscr{Q}}(q) = 1.1$. *Minimum degree of* $\mathscr{Q} = \delta(\mathscr{Q}) = 0.9$ *and maximum degree of* $\mathscr{Q} = \Delta(\mathscr{Q}) = 1.4$. □

Definition 2.10. [24] *Let* $\mathscr{Q} = (\chi, \upsilon)$ *be a fuzzy graph on a graph* $\mathscr{Q}^* = (W, E)$. *If* $\mathrm{d}_{\mathscr{Q}}(\mathrm{w}) = \mathrm{K}$ *for all* $\mathrm{w} \in \mathrm{W}$. *Then* $\mathscr{Q}$ *is said to be a regular fuzzy graph of degree* K *or a* $\mathrm{K}-$ *regular fuzzy graph.*

Definition 2.11. [24] *Let* $\mathscr{Q} = (\chi, \upsilon)$ *be a fuzzy graph on a graph* $\mathscr{Q}^* = (W, E)$. *A subset* F *of* W *is called a fuzzy equitable dominating set if for every* $\mathrm{w} \in \mathrm{W} \backslash \mathrm{F}$ *there exists a vertex* $\mathrm{m} \in \mathrm{F}$ *such that* $\mathrm{mw} \in \mathrm{E}(\mathscr{Q})$ *and* $|d_{\mathscr{Q}}(m) - d_{\mathscr{Q}}(w)| \leq 1$. *The minimum cardinality of such a dominating set is denoted by* γ_{fe} *and is termed as the equitable domination number of* $\mathscr{Q}$.

Definition 2.12. [24] *If a vertex* $m \in W$ *be such that* $|(d_{\mathscr{Q}}(m), d_{\mathscr{Q}}(w)| \geq 2$ *for all* $\mathrm{w} \in \mathbb{N}(\mathrm{m})$, *then* m *is in every fuzzy equitable dominating set and the points are said to be fuzzy equitable isolates. The collection of all fuzzy equitable isolates is identified as* I_{fe}.

Definition 2.13. [7] *Let* $m \in W$. *The fuzzy equitable neighbourhood of* m *denoted by* $\mathbb{N}^{ef}(m)$ *is defined as* $\mathbb{N}^{ef}(m) = \{w \in W : w \in \mathbb{N}(m), mw$ *is a strong arc and* $|d_{\mathscr{Q}}(m) - d_{\mathscr{Q}}(w)\ | \leq 1\}$ *and* $m \in I_{fe} \Longleftrightarrow \mathbb{N}^{ef}(m) = \phi$. *The cardinality of* $\mathbb{N}^{ef}(m)$ *is termed as fuzzy equitable degree of* m *and it is indicated as* $\mathrm{d}^{ef}_{\mathscr{Q}}(m)$.

Definition 2.14. [22] *A vertex* $m \in W$ *is termed as degree equitable in fuzzy graph with a vertex* $w \in W$ *if* $|d_{\mathscr{Q}}(m) - d_{\mathscr{Q}}(w)| \leq 1$ *and* $\upsilon(mw) \leq \chi(m) \wedge \chi(w)$.

Definition 2.15. [7] *A subset* F *of* W *is called a fuzzy equitable independent set if for any* $\mathrm{m} \in \mathrm{F}, \mathrm{w} \notin \mathbb{N}^{\mathrm{fe}}(\mathrm{m})$ *for all* $\mathrm{w} \in \mathrm{F} \backslash \{\mathrm{m}\}$.

3. MAIN RESULTS

Somasundaram et.al [27] introduced the concept of domination in fuzzy graphs. In this section, we defined a new type of domination, divisor domination in fuzzy graphs and studied their properties.

Definition 3.1. *Let* $\mathscr{Q} = (\chi, \upsilon)$ *be a fuzzy graph on a graph* $\mathscr{Q}^* = (W, E)$. *A subset* F *of* W *is called a divisor equitable dominating set if* $\forall\ \mathrm{w} \in \mathrm{W} \backslash \mathrm{F}\ \exists$ *a vertex* $m \in \mathrm{F} \ni \mathrm{mw} \in \mathrm{E}(\mathscr{Q})$ *and* $gcd(d_{\mathscr{Q}}(m), d_{\mathscr{Q}}(w)) \leq 1$. *The minimum cardinality of such a dominating set is denoted by* γ_{de} *and is called the divisor equitable domination number of* $\mathscr{Q}$.

The below examples show that the notions of fuzzy equitable domination set and divisor equitable domination set vary in fuzzy graph.

Example 3.1. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

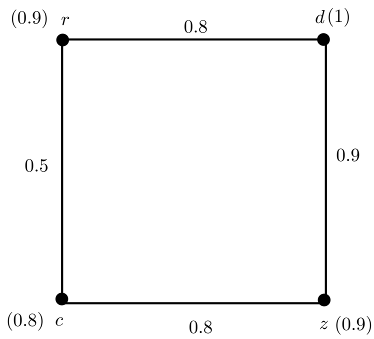

FIGURE 2. Example of fuzzy equitable dominating set but not divisor equitable dominating set.

Here $F = \{r, z\}$ is a fuzzy equitable domination set, but not a divisor equitable domination set as $gcd(d_{\mathscr{Q}}(d), d_{\mathscr{Q}}(z)) = 1.7 > 1$. □

Example 3.2. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

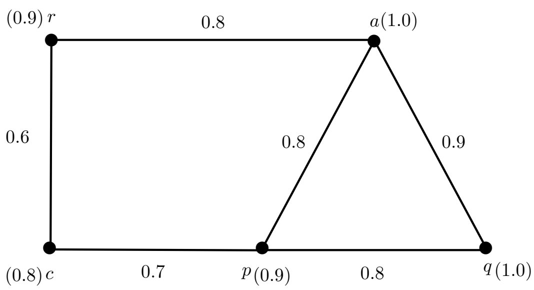

FIGURE 3. Example of divisor equitable dominating set, but not fuzzy equitable dominating set.

Here $\{a, c\}, \{r, p\}, \{c, q\}, \{a, r\}$ are both fuzzy equitable dominating set and divisor equitable dominating set. But $\{a, p\}$ is a divisor equitable dominating set but not a fuzzy equitable dominating set as $|d_{\mathscr{Q}}(a) - d_{\mathscr{Q}}(r)| = 1.1 > 1$. □

Definition 3.2. *A vertex $m \in W$ of a fuzzy graph is said to be degree divisor equitable with a vertex $w \in W$ if $gcd(d_{\mathscr{Q}}(m) - d_{\mathscr{Q}}(w)) \leq 1$ and $\upsilon(mw) \leq \chi(m) \wedge \chi(w)$.*

Example 3.3. *In Example 3.1, the vertex r is degree equitable with d.*
In Example 3.2, the vertex r is degree equitable with the vertices a and c. □

Theorem 3.1. *A divisor equitable dominating set* F *is minimal if and only if for every vertex* $\mathrm{m} \in \mathrm{F}$ *one of the following holds.*
(i) Either $\mathbb{N}(m) \cap F = \emptyset$.
(ii) There exists a vertex $w \in \mathrm{W} \backslash \mathrm{F} \ni \mathbb{N}(\mathrm{w}) \cap \mathrm{F} = \{\mathrm{m}\}$ *and* $gcd(d_{\mathscr{Q}}(\mathrm{w}), d_{\mathscr{Q}}(\mathrm{m})) \leq 1$.

Proof. Suppose that F is a minimal divisor equitable dominating set with (i) and (ii) does not hold. Then for some $m \in \mathrm{F}\ \exists\ w \in \mathbb{N}(\mathrm{m}) \cap \mathrm{F} \ni gcd(d_{\mathscr{Q}}(\mathrm{w}), d_{\mathscr{Q}}(\mathrm{m})) \leq 1$ and for every $\mathrm{w} \in \mathrm{W}\backslash\mathrm{F}$ either $\mathbb{N}(\mathrm{w}) \cap \mathrm{F} \neq \{\mathrm{m}\}$ or $gcd(d_{\mathscr{Q}}(\mathrm{w}), (d_{\mathscr{Q}}(\mathrm{m})) \geq 2$ or both. Hence $F\backslash\{\mathrm{m}\}$ is an divisor equitable dominating set, which is a contradiction to the minimality of F. Therefore (i) and (ii) holds.

Conversely, for every $m \in \mathrm{F}$, one of the assertions (i) or (ii) is true. Suppose F is not minimal. Then there exists $m \in F$ such that $F\backslash\{m\}$ is a divisor equitable dominating set. So, there exists $\mathrm{w} \in \mathrm{F}\backslash\{\mathrm{m}\}$ such that w divisor equitably dominates m, that is $w \in \mathbb{N}(m)$ and $gcd(d_{\mathscr{Q}}(\mathrm{w}), (d_{\mathscr{Q}}(\mathrm{m})) \leq 1$. Therefore m does not satisfy (i). Then m must fulfil (ii). Then there exists a $\mathrm{w} \in \mathrm{W}\backslash\mathrm{F}$ such that $\mathbb{N}(\mathrm{w}) \cap \mathrm{F} = \{\mathrm{m}\}$ and $gcd(d_{\mathscr{Q}}(\mathrm{w}), (d_{\mathscr{Q}}(\mathrm{m})) \leq 1$. Since $\mathrm{F}\backslash\{\mathrm{m}\}$ is a divisor equitable dominating set, there exists $\mathrm{s} \in \mathrm{F}\backslash\{m\}$ such that s is adjacent to w and s is degree divisor equitable with w.

So, $\mathrm{s} \in \mathbb{N}(\mathrm{w}) \cap \mathrm{F}, gcd(d_{\mathscr{Q}}(\mathrm{s}), (d_{\mathscr{Q}}(\mathrm{w})) \leq 1$ and $\mathrm{s} \neq \mathrm{m}$ which is a contradiction to $\mathbb{N}(\mathrm{w}) \cap F = \{\mathrm{m}\}$.

Hence F is minimal divisor equitable dominating set. □

Example 3.4. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

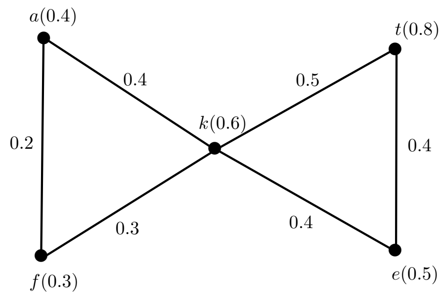

FIGURE 4. Example of minimal divisor equitable dominating set.

Here $F_1 = \{a, k\}, F_2 = \{f, e\}, F_3 = \{a, t\}, F_4 = \{k\}, F_5 = \{e, a\}, F_6 = \{f, t\}, F_7 = \{f, k\}, F_8 = \{t, k\}, F_9 = \{e, k\}$ are divisor equitable dominating sets of $\mathscr{Q}$ with the cardinality $|F_1| = 1.0, |F_2| = 0.8, |F_3| = 1.2, |F_4| = 0.6, |F_5| = 0.9, |F_6| = 1.1, |F_7| = 0.9, |F_8| = 1.4, |F_9| = 1.1$. Then, the minimal divisor cardinality number is

$\gamma_{de} = min\{|F_1|, |F_2|, |F_3|, |F_4|, |F_5|, |F_6|, |F_7|, |F_8|, |F_9|\}$

$\gamma_{de} = min\{1.0, 0.8, 1.2, 0.6, 0.9, 1.1, 0.9, 1.4, 1.1\}$

$\gamma_{de} = 0.6.$

Hence, the minimal divisor equitable dominating set is F_4. □

Definition 3.3. *For a set F, every $u \in W\backslash F$ there exist a vertex $v \in F$ such that $uv \in E(\mathscr{Q})$ and one of the vertex u or v is with degree k and 0 then is with degree $k+1$ and in the case $\mathscr{Q}$ is called bi-regular fuzzy graph.*

Theorem 3.2. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$. If $\mathscr{Q}$ is r-regular fuzzy graph or $(\mathrm{r}, \mathrm{r}+1)$ bi-regular fuzzy graph, for some $r \leq 1$, then $\gamma_{\mathrm{de}} = \gamma_{\mathrm{f}}$.*

Proof. Suppose $\mathscr{Q}$ is a regular fuzzy graph on $\mathscr{Q}^*$. Then every vertex of $\mathscr{Q}$ has the same degree say r. Let F be a minimum dominating set of $\mathscr{Q}$. Then $|F| = \gamma(\mathscr{Q}) = \gamma_{\mathrm{f}}$. As F is a dominating set for $m \in W\backslash F$, there exists $\mathrm{w} \in \mathrm{F}$ and $mw \in E(\mathscr{Q})$ such that $d_{\mathscr{Q}}(m) = d_{\mathscr{Q}}(w) = r$ and $gcd(d_{\mathscr{Q}}(m), d_{\mathscr{Q}}(w)) = r \leq 1$ imply that F is a divisor equitable dominating set of $\mathscr{Q}$ so that $\gamma_{\mathrm{de}}(\mathscr{Q}) \leq |\mathrm{F}| = \gamma_{\mathrm{f}}$. But $\gamma_{\mathrm{f}} \leq \gamma_{\mathrm{de}}$. Hence $\lambda_{\mathrm{f}} = \gamma_{\mathrm{de}}$.

Suppose $\mathscr{Q}$ is a bi-regular fuzzy graph. Then for every vertex of $\mathscr{Q}$ has degree either r (or) r+1. Let F be a minimum dominating set of $\mathscr{Q}$. Then $|F| = \gamma_f$. As F is a dominating set, for m $\in$ W\F there exists w $\in$ F and $mw \in E(\mathscr{Q})$ such that $d_{\mathscr{Q}}(m) = r$ (or) r + 1 and $d_{\mathscr{Q}}(w)$ = r (or) r + 1. Therefore $gcd(d_{\mathscr{Q}}(m), d_{\mathscr{Q}}(w)) = 1$ which gives F is a divisor equitable dominating set of $\mathscr{Q}$ such that $\gamma_{de} \leq |F| = \gamma_f$. But $\gamma_f \leq \gamma_{de}$. Hence $\gamma_f = \gamma_{de}$. □

Remark 3.1. *From the above Theorem 3.2, we have the following results.*
1. If $r > 0$, then r-regular fuzzy graph and $(r, r+1)$ bi-regular have $\gamma_{fe} = \gamma_f$.
2. If $0 < r < 1$, then r-regular fuzzy graph and $(r, r+1)$ bi-regular fuzzy graph, we have $\gamma_{de} = \gamma_f$.
3. If $r > 1$ and r is integer, then $(r, r+1)$ bi-regular fuzzy graph we have $\gamma_{de} = \gamma_f$ but r-regular fuzzy graph we have $\gamma_{de} \neq \gamma_f$.

Example 3.5. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

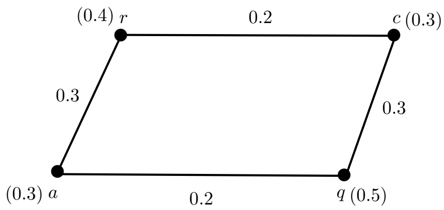

FIGURE 5. Example of regular divisor fuzzy graph.

Here $\mathscr{Q}$ is a regular fuzzy graph and $F = \{a, c\}$ is a divisor equitable dominating set with $\gamma_{de} = \gamma_f = 0.6$. □

Example 3.6. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

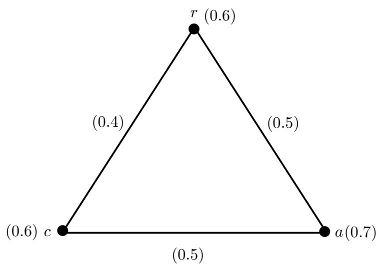

FIGURE 6. Example of bi-regular divisor fuzzy graph.

Here $\mathscr{Q}$ is a bi-regular fuzzy graph $\gamma_{de} = \gamma_f = 0.6$. □

Definition 3.4. *If a vertex $m \in W$ be such that $gcd(d_{\mathscr{Q}}(m), d_{\mathscr{Q}}(w)) \geq 2$ for all* w $\in$ $\mathbb{N}$(m), *then* m *is in every divisor equitable dominating set and the points are called divisor equitable isolates. The collection of all divisor equitable isolates defined as I_{de}.*

Example 3.7. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

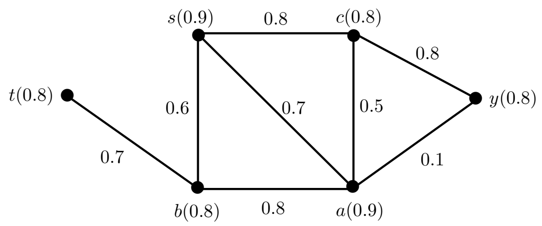

Figure 7. Example of divisor Equitable isolates.

Here the vertex s is a divisor equitable isolates. It is in every divisor equitable dominating set. □

Theorem 3.3. *A fuzzy graph $\mathscr{Q}$ has an unique minimal divisor equitable dominating set if and only if the set of all divisor equitable isolates forms an equitable dominating set.*

Proof. Sufficient condition is obvious. Let $\mathscr{Q}$ have a unique minimal divisor equitable dominating set F and D = {m ∈ W : m is a divisor equitable isolate}. Then D ⊆ F. We shall prove that D = F.

Suppose F − D ≠ ∅. Then for $w \in$ F\D, w is not a divisor equitable isolate and W\{w} is a divisor equitable dominating set. Hence there exists a minimal divisor equitable dominating set. $F_1 \subseteq$ W\{w} and $F_1 \neq F$, a contradicts to the fact that $\mathscr{Q}$ has a unique divisor equitable dominating set. Therefore D = F. □

Definition 3.5. *For $m \in$ W, the divisor equitable neighborhood of m indicated by $\mathbb{N}^{de}(u)$ and it is defined as $\mathbb{N}^{de}(m) = \{w \in W : w \in \mathbb{N}(m), (m, w)$ is a strong arc and $gcd(d_{\mathscr{Q}}(m), d_{\mathscr{Q}}(w)) \leq 1\}$ and $m \in I_{de} \Leftrightarrow \mathbb{N}^{de}(m) = \phi$. The cardinality of $\mathbb{N}^{de}(m)$ is denoted by $d^{de}_{\mathscr{Q}}(m)$.*

Remark 3.2. *In Example 3.2, neighborhood of a is $\mathbb{N}(a) = \{r, p, q\}$; divisor equitable neighborhood of a is $\mathbb{N}^{de}(a) = \{r, p, q\}$; fuzzy equitable neighborhood of a is $\mathbb{N}^{fe}(a) = \{p, q\}$.*

Theorem 3.4. *A dominating set F of $\mathscr{Q}$ on the graph $\mathscr{Q}^* = (W, E)$ is a minimal divisor equitable dominating set if and only if for each $r \in F$ one of the following two conditions holds:*
i) $\mathbb{N}^{de}(r) \cap F = \emptyset$
ii) ∃ vertex $c \in W \backslash F$ such that $\mathbb{N}^{de}(c) \cap F = \{r\}$.

Proof. $(i) \implies (ii)$ Let F be a minimal divisor equitable dominating set and $r \in F$. Then $F_r = F\backslash\{r\}$ is not a divisor equitable dominating set and hence there exists $l \in W\backslash F_r$ such that l is not dominated by any element of F_r.
Case (1): If $l = r$ we have (i)
Case (2): If $l \neq r$ we have (ii).

For each $r \in F$ one of the following two conditions holds:
i) $\mathbb{N}^{de}(r) \cap F = \emptyset$
ii) ∃ vertex $c \in W\backslash F \ni \mathbb{N}^{de}(c) \cap F = \{r\}$.

Suppose F is not minimal. Then there exists $k \in F$ such that $F\backslash\{k\}$ is a divisor equitable dominating set. So there exists $w \in F\backslash\{k\}$ such that w is divisor equitable domination k, hence $w \in \mathbb{N}(k)$ and $gcd(d_{\mathscr{Q}}(w), d_{\mathscr{Q}}(k)) \leq 1$. Therefore k does not satisfy (i). Then k must satisfy (ii), there exists $w \in W\backslash F$ such that $\mathbb{N}^{de}(w) \cap F = \{k\}$ and $gcd(d_{\mathscr{Q}}(w), d_{\mathscr{Q}}(k)) \leq 1$. As $F\backslash\{k\}$ is a divisor equitable dominating set, there exists $l \in F\backslash\{k\}$ such that l is adjacent to w and l is degree divisor equitable with w. So $l \in \mathbb{N}^{de}(w) \cap F, gcd(d_{\mathscr{Q}}(l), d_{\mathscr{Q}}(w)) \leq 1$ and $l = k$ which is a contradiction to (ii).

Hence F is a minimal divisor equitable dominating set of $\mathscr{Q}$. □

Remark 3.3. *The maximum and minimum divisor equitable degree of a point in $\mathscr{Q}$ are denoted by $\Delta^{de}(\mathscr{Q})$ and $\delta^{de}(\mathscr{Q})$ and it is defined as $\Delta^{de}(\mathscr{Q}) = \max_{m \in W(\mathscr{Q})} \left|\mathbb{N}^{de}(m)\right|$ and $\delta^{de}(\mathscr{Q}) = \min_{m \in W(\mathscr{Q})} \left|\mathbb{N}^{de}(m)\right|$.*

Definition 3.6. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph $\mathscr{Q}^* = (W, E)$. Then $F \subseteq W$ is called a strong (weak) divisor equitable dominating set of $\mathscr{Q}$ if every vertex $w \in W \backslash F$ is strongly (weakly) dominated by some vertex m in F. We denote a strong (weak) divisor equitable dominating set by sded-set (wded-set).*

The minimum scalar cardinality of a sded-set (wded-set) is called the strong (weak) divisor equitable domination number of $\mathscr{Q}$ and it is indicated as $\gamma^{\mathrm{sde}}(\mathscr{Q})$ $\left(\gamma^{wde}(\mathscr{Q})\right)$.

Theorem 3.5. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on the graph $\mathscr{Q}^* = (W, E)$ of order p, then:*
i) $\gamma^{de}(\mathscr{Q}) \leq \gamma^{sde}(\mathscr{Q}) \leq p - \Delta^{de}(\mathscr{Q})$,
ii) $\gamma^{de}(\mathscr{Q}) \leq \gamma^{wde}(\mathscr{Q}) \leq p - \delta^{de}(\mathscr{Q})$.

Proof. Every strong divisor equitable dominating set is a equitable dominating set of $\mathscr{Q}, \gamma^{\mathrm{de}}(\mathscr{Q}) \leq \gamma^{\mathrm{sde}}(\mathscr{Q})$ and every weak divisor equitable dominating set is a equitable dominating set of $\mathscr{Q}, \gamma^{de}(\mathscr{Q}) \leq \gamma^{\mathrm{wde}}(\mathscr{Q})$.

Let $m, w \in W$ and $d^{de}_{\mathscr{Q}}(m) = \Delta^{de}(\mathscr{Q})$ and $d^{de}_{\mathscr{Q}}(w) = \delta^{de}(\mathscr{Q})$. Then $W \backslash \mathbb{N}^{de}(m)$ is a strong divisor equitable dominating set and $W \backslash \mathbb{N}^{de}(w)$ is a weak divisor equitable dominating set. Therefore $\gamma^{\mathrm{sde}}(\mathscr{Q}) \leq \left|W \backslash \mathbb{N}^{de}(m)\right|^{\mathrm{de}}$ and $\gamma^{\mathrm{wde}}(\mathscr{Q}) \leq \left|W \backslash \mathbb{N}^{de}(w)\right|^{\mathrm{de}}$ which imply $\gamma^{\mathrm{sde}}(\mathscr{Q}) \leq p - \Delta^{de}(\mathscr{Q})$ and $\gamma^{\mathrm{wde}}(\mathscr{Q}) \leq p - \delta^{de}(\mathscr{Q})$. □

Definition 3.7. [7] *Two nodes of a fuzzy graph are called fuzzy independent if there is no strong arc between them. A subset F of $\mathbb{N}$ is said to be a fuzzy independent set of $\mathscr{Q}$ if any two nodes of F are fuzzy independent.*

Example 3.8. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

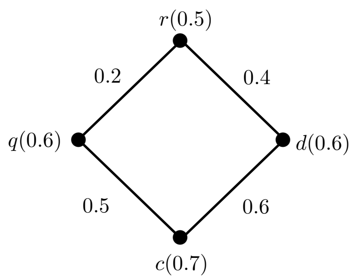

FIGURE 8. Example of fuzzy independent.

Here $\upsilon(q, r) \neq \upsilon^{\infty}(q, r)$. So, the vertices q and r are fuzzy independent. □

Definition 3.8. *A subset F of W is called a divisor equitable independent set if for any $m \in \mathrm{F}, w \notin \mathbb{N}^{\mathrm{de}}(m)$ for all $w \in F \backslash \{m\}$.*

Example 3.9. *Let $\mathscr{Q} = (\chi, \upsilon)$ be a fuzzy graph on a graph $\mathscr{Q}^* = (W, E)$ and described as follows.*

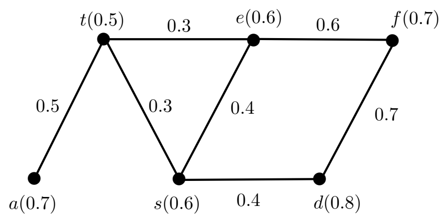

FIGURE 9. Example of divisor equitable independent.

Let $F = \{t, f\}$. *Then* $\mathbb{N}^{de}(t) = \{a, e\}, \mathbb{N}^{de}(f) = \{d\}$ *and for any* $a \in F, b \notin \mathbb{N}^{de}(a)$ *for all* $b \in F\backslash\{a\}$. *Hence* F *is a divisor equitable independent set.* □

Theorem 3.6. *Let* F *be a maximal divisor equitable independent set. Then* F *is a minimal divisor equitable dominating set.*

Proof. Let F be a maximal divisor equitable independent set and $m \in W\backslash F$ and $m \notin \mathbb{N}^{de}(w)$ for every $w \in F$. Then $F \cup \{m\}$ is an divisor equitable independent set, a contradiction to the maximality of F. So, $m \in \mathbb{N}^{de}(w)$ for some $w \in F$ and F is a divisor equitable dominating set. Since for any $m \in F, m \notin \mathbb{N}^{de}(w)$ for every $w \in F\backslash\{m\}$, either $\mathbb{N}(m) \cap F = \phi$ or $gcd(d_{\mathscr{Q}}(w), d_{\mathscr{Q}}(m)) \geq 2 \ \forall \ w \in \mathbb{N}(m) \cap F$. Therefore F is a minimal divisor equitable dominating set. □

4. CONCLUSIONS

One of the most popular areas of research right now in both graph theory and combinatorics is called "dominance theory". In recent years, graph theory research has centred on the topic of domination. The study of domination and associated subset problems like independence, covering, matching, decomposition and labelling is the field of graph theory that is expanding the fastest. Applications to social network theory, land surveying, game theory, interconnection networks, parallel computing, image processing, and more can be found in Domination. Equitability among citizens in terms of services, health and education benefits, etc. is important in a democratic country. These practical concepts are modeled by equitability in graphs. An equitable domination has interesting application in the context of social networks. Learning the concept of GCD can be quite beneficial in planning, estimation and dividing things. The real-life applications of these concepts enhance problem-solving and critical thinking in research. It is also applied computer security. Some applications like generating modular multiplicative inverse are used in various cryptographic algorithms, reducing public keys' disclosure within closed groups, the importance of GCD in cryptographic algorithms, key refreshment message authentication, and peer validation. Studying these ideas will be useful in instances where issues must be resolved cryptographically and equitably.

In this article, we investigated divisor equitable domination in fuzzy graphs. We looked at the minimal divisor equitable domination set's attributes. Additionally, we discovered the connections between equitable independence and minimal divisor domination. These findings may be applied to Intuitionistic fuzzy graphs and Pythagorean fuzzy graphs using the methodology outlined in this article.

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